

A FORMULA FOR THE R -MATRIX USING A SYSTEM OF WEIGHT PRESERVING ENDOMORPHISMS

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ABSTRACT. We give a formula for the universal R -matrix of the quantized universal enveloping algebra $U_q(\mathfrak{g})$. This is similar to a previous formula due to Kirillov-Reshetikhin and Levendorskii-Soibelman, except that where they use the action of the braid group element T_{w_0} on each representation V , we show that one can instead use a system of weight preserving endomorphisms. One advantage of our construction is that it is well defined for all symmetrizable Kac-Moody algebras. However we have only established that the result is equal to the universal R -matrix in finite type.

1. INTRODUCTION

Let \mathfrak{g} be a finite type complex simple Lie algebra and $U_q(\mathfrak{g})$ the corresponding quantized universal enveloping algebra. In [KR] and [LS], Kirillov-Reshetikhin and Levendorskii-Soibelman developed a formula for the universal R -matrix

$$(1) \quad R = (X^{-1} \otimes X^{-1})\Delta(X),$$

where X belongs to a completion of $U_q(\mathfrak{g})$. The element X is constructed using the braid group element T_{w_0} corresponding to the longest word of the braid group, and as such only makes sense when \mathfrak{g} is of finite type.

The element X in Equation (1) defines a vectors space endomorphism X_V on each representation V of $U_q(\mathfrak{g})$, and in fact X is defined by the system of endomorphisms $\{X_V\}$. Furthermore, any natural system of vector space endomorphisms $\{E_V\}$ can be represented as an element E in a certain completion of $U_q(\mathfrak{g})$ (see [KT]). The action of the coproduct $\Delta(E)$ on a tensor product $V \otimes W$ is then simply $E_{V \otimes W}$. Thus the right side of Equation 1 is well defined if X is replaced by $E = \{E_V\}$.

In this note we consider the case where \mathfrak{g} is a symmetrizable Kac-Moody algebra. We define a system of weight preserving endomorphisms $\Theta = \{\Theta_V\}$ of all integrable highest weight representations V of $U_q(\mathfrak{g})$. When \mathfrak{g} is of finite type, we show that

$$(2) \quad R = (\Theta^{-1} \otimes \Theta^{-1})\Delta(\Theta),$$

where the equality means that, for any V and W , the actions of the two sides of Equation (2) on $V \otimes W$ agree. We expect this remains true in other cases, although this has not been proven.

Our endomorphisms Θ_V are not linear over the field $\mathbb{C}(q)$, but are instead compatible with the automorphism which inverts q . For this reason, Θ cannot be realized using an element in a completion of $U_q(\mathfrak{g})$, and it is crucial to work with

2000 Mathematics Subject Classification. Primary: 17B37. Secondary: 16W30.
This work was supported by the RTG grant DMS-0354321.

systems of endomorphisms. There is a further technicality in that Θ_V actually depends on a choice of global basis for V . Nonetheless, we give a precise meaning to Equation (2).

This note is organized as follows. In Section 2 we fix notation and conventions. In Section 3 we review the universal R -matrix. In Section 4 we review a method developed by Henriques and Kamnitzer [HK] to construct isomorphisms $V \otimes W \rightarrow W \otimes V$. In Section 5 we state some background results on crystal bases and global bases. In Section 6 we construct our endomorphism Θ . In Section 7 we prove our main theorem (Theorem 7.13), which establishes Equation (2) when \mathfrak{g} is of finite type. In Section 8 we briefly discuss future directions for this work.

1.1. Acknowledgements. We thank Joel Kamnitzer, Noah Snyder, and Nicolai Reshetikhin for many helpful discussions.

2. CONVENTIONS

We must first fix some notation. For the most part we follow [CP].

- \mathfrak{g} is a complex simple Lie algebra with Cartan subalgebra \mathfrak{h} , and $A = (a_{ij})_{i,j \in I}$ is its Cartan matrix.

- $\langle \cdot, \cdot \rangle$ denotes the pairing between \mathfrak{h} and \mathfrak{h}^* and (\cdot, \cdot) denotes the usual symmetric bilinear form on either \mathfrak{h} or \mathfrak{h}^* . Fix the usual bases α_i for \mathfrak{h}^* and H_i for \mathfrak{h} , and recall that $\langle H_i, \alpha_j \rangle = a_{ij}$.

- $d_i = (\alpha_i, \alpha_i)/2$, so that $(H_i, H_j) = d_j^{-1} a_{ij}$. Let B denote the matrix $(d_j^{-1} a_{ij})$.

- $q_i = q^{d_i}$.

- ρ is the weight satisfying $(\alpha_i, \rho) = d_i$ for all i .

- H_ρ is the element of \mathfrak{h} such that $\langle \alpha_i, H_\rho \rangle = d_i = (\alpha_i, \rho)$ for all i .

- θ is the diagram automorphism such that $w_0(\alpha_i) = -\alpha_{\theta(i)}$, where w_0 is the longest element in the Weyl group.

- $U_q(\mathfrak{g})$ is the quantized universal enveloping algebra associated to \mathfrak{g} , generated over $\mathbb{C}(q)$ by E_i, F_i for all $i \in I$, and K_w for w in the co-weight lattice of \mathfrak{g} . As usual, let $K_i = K_{H_i}$. For convenience, we recall the exact formula for the coproduct:

$$(3) \quad \begin{cases} \Delta E_i &= E_i \otimes K_i + 1 \otimes E_i \\ \Delta F_i &= F_i \otimes 1 + K_i^{-1} \otimes F_i \\ \Delta K_i &= K_i \otimes K_i \end{cases}$$

- $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$, and $X^{(n)} = \frac{X^n}{[n][n-1] \cdots [2]}$.

- V_λ is an irreducible integrable representation of $U_q(\mathfrak{g})$ with highest weight λ .

- B_λ is a fixed global basis for V_λ , in the sense of Kashiwara (see [K]). b_λ and b_λ^{low} are the highest weight and lowest weight elements of B_λ respectively.

3. THE R-MATRIX

We briefly recall the definition of a universal R -matrix, and the related notion of a braiding.

Definition 3.1. *A braided monoidal category is a monoidal category \mathcal{C} , along with a natural system of isomorphisms $\sigma_{V,W}^{br} : V \otimes W \rightarrow W \otimes V$ for each pair $V, W \in \mathcal{C}$, such that for any $U, V, W \in \mathcal{C}$, the following two equalities hold:*

$$(4) \quad \sigma_{U,W}^{br} \otimes Id \circ Id \otimes \sigma_{V,W}^{br} = \sigma_{U \otimes V, W}^{br}$$

$$(5) \quad Id \otimes \sigma_{U,W}^{br} \circ \sigma_{U,V}^{br} \otimes Id = \sigma_{U,V \otimes W}^{br}.$$

The system $\sigma^{br} := \{\sigma_{V,W}^{br}\}$ is called a braiding on \mathcal{C} .

Let $U_q(\mathfrak{g}) \widetilde{\otimes} U_q(\mathfrak{g})$ be the completion of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ in the weak topology defined by all matrix elements of representations $V_\lambda \otimes V_\mu$, for all ordered pairs (λ, μ) .

Definition 3.2. A universal R -matrix is an element R of $U_q(\mathfrak{g}) \widetilde{\otimes} U_q(\mathfrak{g})$ such that $\sigma_{V,W}^{Br} := Flip \circ R$ is a braiding on the category of $U_q(\mathfrak{g})$ representations.

The following theorem is central to the theory of quantized universal enveloping algebra. See [CP] for a thorough discussion when \mathfrak{g} is of finite type, and [L] for the general case.

Theorem 3.3. Let \mathfrak{g} be a symmetrizable Kac-Moody algebra. Then $U_q(\mathfrak{g})$ has a universal R -matrix. \square

Comment 3.4. The universal R -matrix is not truly unique. However there is a well studied standard choice. We discuss how this R can be realized In Section 7.

4. CONSTRUCTING ISOMORPHISMS USING SYSTEMS OF ENDOMORPHISMS

Here and throughout this note a representation of $U_q(\mathfrak{g})$ will mean an integrable highest weight representation. When \mathfrak{g} is of finite type, this means a finite dimensional representation. In this section we review a method for constructing natural systems of isomorphisms $\sigma_{V,W} : V \otimes W \rightarrow W \otimes V$ for any representations V and W of $U_q(\mathfrak{g})$. This idea was used by Henriques and Kamnitzer in [HK], and was further developed in [KT]. The data needed to construct such a system is:

- (i) An algebra automorphism C_ξ of $U_q(\mathfrak{g})$ which is also a co-algebra anti-automorphism.
- (ii) A natural system of invertible (vector space) endomorphisms ξ_V of each representation V of $U_q(\mathfrak{g})$ which is compatible with C_ξ in the sense that the following diagram commutes for all V :

$$\begin{array}{ccc} V & \xrightarrow{\xi_V} & V \\ \curvearrowright & & \curvearrowright \\ U_q(\mathfrak{g}) & \xrightarrow{C_\xi} & U_q(\mathfrak{g}). \end{array}$$

It follows immediately from the definition of coalgebra anti-automorphism that

$$(6) \quad \sigma_{V,W}^\xi := Flip \circ (\xi_V^{-1} \otimes \xi_W^{-1}) \circ \xi_{V \otimes W}$$

is an isomorphism of $U_q(\mathfrak{g})$ representations from $V \otimes W$ to $W \otimes V$.

We will normally denote the system $\{\xi_V\}$ simply by ξ , and will denote the action of ξ on the tensor product of two representations by $\Delta(\xi)$. This is justified since, as explained in [KT], ξ in fact belongs to a completion of $U_q(\mathfrak{g})$, and the action of ξ on $V \otimes W$ is calculated using the coproduct. With this notation $\sigma^\xi := \{\sigma_{V,W}^\xi\}$ can be expressed as

$$(7) \quad \sigma^\xi = Flip \circ (\xi \otimes \xi) \circ \Delta(\xi).$$

In the current work we require a little more freedom: we will sometimes use automorphisms C_ξ of $U_q(\mathfrak{g})$ which are not linear over $\mathbb{C}(q)$, but instead are bar-linear (i.e. invert q). This causes some technical difficulties, which we deal with in Section 6. Once we make this precise, we will use all the same notation for a bar-linear C_ξ and compatible system of \mathbb{C} vector space automorphisms ξ as we do in the linear case, including using $\Delta(\xi)$ to denote ξ acting on a tensor product.

Comment 4.1. To describe the data (\mathcal{C}_ξ, ξ) , it is sufficient to describe C_ξ , and to give the action of ξ_{V_λ} on any one vector v in each irreducible representation V_λ . This is usually more convenient than describing ξ_{V_λ} explicitly. Of course, the choice of C_ξ imposes a restriction on the possibilities for $\xi_{V_\lambda}(v)$, so when we give a description of ξ in this way we are always claiming that the action on our chosen vector in each V_λ is compatible with C_ξ .

5. CRYSTAL BASES AND GLOBAL BASES

In order to extend the construction described in the Section 4 to include bar linear ξ , we will need to use some results concerning crystal bases and global bases, which we review here. We state only what is most relevant to us, and refer the reader to [K] for a more complete exposition.

Unfortunately, the conventions in [K] and [CP] do not quite agree. In particular, the theorems from [K] that we will need are stated in terms of a different coproduct, so we have modified them to match our conventions.

Definition 5.1. Fix a representation V of \mathfrak{g} , and $i \in I$. Define the Kashiwara operators $\tilde{F}_i, \tilde{E}_i : V \rightarrow V$ by linearly extending the following definition:

$$(8) \quad \begin{cases} \tilde{F}_i(F_i^{(n)}(v)) = F_i^{(n+1)}(v) \\ \tilde{E}_i(F_i^{(n)}(v)) = F_i^{(n-1)}(v). \end{cases}$$

for all $v \in V$ such that $E_i(v) = 0$.

Definition 5.2. Let $\mathcal{A}_\infty = \mathbb{C}[q^{-1}]_0$ be the algebra of rational functions in q^{-1} over \mathbb{C} whose denominators are not divisible by q^{-1} .

Definition 5.3. A crystal basis of a representation V (at $q = \infty$) is a pair (\mathcal{L}, B) , where \mathcal{L} is an \mathcal{A}_∞ -lattice of V and B is a basis for $\mathcal{L}/q^{-1}\mathcal{L}$, such that

- (i) \mathcal{L} and B are compatible with the weight decomposition of V .
- (ii) \mathcal{L} is invariant under the Kashiwara operators and $B \cup 0$ is invariant under their residues $e_i := \tilde{E}_i^{(\text{mod } q^{-1}\mathcal{L})}, f_i := \tilde{F}_i^{(\text{mod } q^{-1}\mathcal{L})} : \mathcal{L}/q^{-1}\mathcal{L} \rightarrow \mathcal{L}/q^{-1}\mathcal{L}$.
- (iii) For any $b, b' \in B$, we have $e_i b = b'$ if and only if $f_i b' = b$.

Definition 5.4. Let (\mathcal{L}, B) be a crystal basis for V . The highest weight elements of B are those $b \in B$ such that, for all i , $e_i(b) = 0$

Theorem 5.5 ([K], Theorem 1). Let V, W be representations with crystal bases (\mathcal{L}, A) and (\mathcal{M}, B) respectively. Then $(\mathcal{L} \otimes \mathcal{M}, A \otimes B)$ is a crystal basis of $V \otimes W$. Furthermore, the highest weight elements of $A \otimes B$ are all of the form $a^{\text{high}} \otimes b$, where a^{high} is a highest weight element of A . \square

Definition 5.6. Let $(\mathcal{L}_\lambda, B_\lambda)$ and (\mathcal{L}_μ, B_μ) be crystal bases for V_λ and V_μ . Set:

$$(9) \quad S_{\lambda, \mu}^\nu := \{b \in B_\mu : b_\lambda \otimes b \text{ is a highest weight element of } B_\lambda \otimes B_\mu \text{ of weight } \nu\}.$$

We will in fact need more than just the notion of a crystal basis. For any V_λ , and any choice of highest weight vector $b_\lambda \in V_\lambda$, there is a canonical choice of a basis B_λ for V_λ , which contains b_λ , and such that $(B_\lambda + q\mathcal{L}, \mathcal{L})$ is a crystal basis for V , where \mathcal{L} is the \mathcal{A}_∞ -span of B_λ . That is not to say there is a unique basis for V_λ satisfying these two conditions, only that one can find a canonical “good” choice. This is known as the global crystal basis for V_λ . A complete construction can be found in [K], although here we more closely follow the presentation from [CP, Chapter 14.1C]. In the present work we simply use the fact that the global basis exists, and state the properties of B_λ that we use. First we need the following

Definition 5.7. $C_{bar} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ is the \mathbb{C} -algebra involution defined by

$$(10) \quad \begin{cases} C_{bar}q = q^{-1} \\ C_{bar}K_i = K_i^{-1} \\ C_{bar}E_i = E_i \\ C_{bar}F_i = F_i. \end{cases}$$

Theorem 5.8. Kashiwara [K] Fix a highest weight vector $b_\lambda \in V_\lambda$. There is a canonical choice of a “global” basis B_λ of V_λ . This has the properties (although is not defined by these alone) that:

- (i) $b_\lambda \in B_\lambda$.
- (ii) B_λ is a weight basis for V_λ .
- (iii) Let \mathcal{L}_∞ be the \mathcal{A}_∞ span of B_λ . Then $(B_\lambda + q^{-1}\mathcal{L}_\infty, \mathcal{L}_\infty)$ is a crystal basis for V_λ at $q = \infty$.
- (iv) Define the involution bar_{V_λ} of V_λ by $bar_{V_\lambda}(f(q)b) = f(q^{-1})b$ for all $f(q) \in \mathbb{C}(q)$ and $b \in B_\lambda$. Then bar_{V_λ} is compatible with C_{bar} , in the sense discussed in Section 4. \square

Definition 5.9. If V is any (possibly reducible) representation of $U_q(\mathfrak{g})$, we say a basis B_V of V is a global basis for V if there is a decomposition of V into irreducible components such that B is a union of global bases for the irreducible pieces.

6. THE ENDOMORPHISM Θ AND THE ISOMORPHISM σ^Θ .

We now introduce a \mathbb{C} -algebra automorphism C_Θ of $U_q(\mathfrak{g})$. Notice that this inverts q , so it is not a $\mathbb{C}(q)$ algebra automorphism, but is instead bar linear:

$$(11) \quad \begin{cases} C_\Theta(E_i) = E_i K_i^{-1} \\ C_\Theta(F_i) = K_i F_i \\ C_\Theta(K_i) = K_i^{-1} \\ C_\Theta(q) = q^{-1}. \end{cases}$$

One can check that C_Θ is an algebra involution and a coalgebra anti-involution. We must now define a \mathbb{C} -vector space automorphism Θ_{V_λ} of each V_λ which is compatible with C_Θ . This is complicated by the fact that C_Θ does not preserve the $\mathbb{C}(q)$ algebra structure, but instead inverts q . In order to define Θ_{V_λ} , we must first modify the category of representations slightly. We will actually work in the category of representations with a chosen basis for the highest weight space of each isotypic component. By Theorem 5.8, this is the same as the category of representations with a chosen global basis. An element of this category will be denoted (V, B) , where B is the chosen global basis of V .

Definition 6.1. *The action of $\Theta_{(V_\lambda, B_\lambda)}$ on V_λ is defined by requiring that it be compatible with C_Θ , and that it send b_λ^{low} to $q^{-(\lambda, \lambda)/2 - (\lambda, \rho)} b_\lambda^{\text{low}}$. This is extended by naturality to define $\Theta_{(V, B)}$ for any (possibly reducible) V .*

Comment 6.2. One can show that $\Theta_{(V, B)}$ acts on any element $b \in B$ of weight μ as multiplication by $q^{-(\mu, \mu)/2 + (\mu, \rho)}$. Thus we could have normalized our definition of Θ by setting $\Theta_{(V_\lambda, B_\lambda)}(b_\lambda) = q^{-(\lambda, \lambda)/2 + (\lambda, \rho)} b_\lambda$. We have chosen to use b_λ^{low} instead because it simplifies certain computations.

Comment 6.3. We have not specified a chosen a global basis for $\Theta_{(V_\lambda, B_\lambda)} V_\lambda$, and in fact it is not clear to us what the natural choice should be. For this reason we consider $\Theta_{V, B}$ as a map to V considered in the category of representation without a chosen basis. We are careful to always apply Θ to representation where a global basis for the original representation has been chosen, or to apply Θ^{-1} to representations where a global basis for the image has been chosen.

The construction described in Section 4 uses the action of $\xi_{V \otimes W}$ on $V \otimes W$. Thus we will need to define how Θ acts on a tensor product. In particular, we need a well defined notion of tensor product in the category of representations with a chosen global basis. This will use the existence of global bases in a somewhat stronger way than the definition of $\Theta_{(V, B)}$. We first need a few preliminary definitions.

Definition 6.4. *Let $U_{\lambda, \mu}^{>\nu}$ be the submodule of $V_\lambda \otimes V_\mu$ consisting of the union of all isotypic components corresponding to highest weights $\rho > \nu$.*

Similarly, let $U_{\lambda, \mu}^{\geq \nu}$ be the submodule of $V_\lambda \otimes V_\mu$ consisting of the union of all isotypic components corresponding to highest weights $\rho \geq \nu$.

Definition 6.5. $Q_{\lambda, \mu}^\nu := U_{\lambda, \mu}^{\geq \nu} / U_{\lambda, \mu}^{>\nu}$.

Comment 6.6. $Q_{\lambda, \mu}^\nu$ is in fact isomorphic as a $U_q(\mathfrak{g})$ -module to the ν -isotypic component of $V_\lambda \otimes V_\mu$. However, it will be useful for us to think of it as a subquotient.

We can now define our tensor product:

Definition 6.7. *The tensor product $(V, B) \otimes (W, C)$ is defined to be $(V \otimes W, A)$, where A is the unique global basis of $V \otimes W$ such that the projections of the highest weight elements of A of weight ν in $Q_{\lambda, \mu}^\nu$ are equal to the projections of $b_\lambda \otimes b$ for those $b \in S_{\lambda, \mu}^\nu$.*

Comment 6.8. Lemma 5.5 implies that $\{b_\lambda \otimes b : b \in S_{\lambda, \mu}^\nu\}$ is a basis for the highest weight space of $Q_{\lambda, \mu}^\nu$, so Definition 6.7 makes sense.

Proposition 6.9. *Let (V, B) and (W, C) be two representation with chosen global bases, and let A for the global basis for $V \otimes W$ from Definition 6.7. Then*

$$(12) \quad \sigma_{(V, B), (W, C)}^\Theta := \text{Flip} \circ (\Theta_{(V, B)}^{-1} \otimes \Theta_{(W, C)}^{-1}) \Theta_{(V \otimes W, A)} : V \otimes W \rightarrow W \otimes V$$

is an isomorphism of representations, and furthermore is independent of the choice of global bases B and C .

Proof. That this is an isomorphism of representations follows from the fact that C_Θ is a coalgebra anti-automorphism (see section 4). To see that it independent of the global bases B and C , it is sufficient to consider that can when V and W are irreducible. Then the global bases are unique up to an overall rescaling. One simply checks that $\sigma_{V, W}^\Theta$ does not depend on the choice of scalars. \square

Definition 6.10. For any pair of representation V and W , set $\sigma_{V,W}^\Theta : V \otimes W \rightarrow W \otimes V$ to be $\text{Flip} \circ (\Theta_{(V,B)}^{-1} \otimes \Theta_{(W,C)}^{-1}) \Theta_{(V \otimes W, A)}$ for any choice of global bases B of V and C of W . This is well defined by Proposition 6.9.

Comment 6.11. As we did in Section 4, we will denote the action of Θ on a tensor product of two representations (with chosen global bases) by $\Delta(\Theta)$. Thus $\sigma^\Theta = \text{Flip} \circ (\Theta^{-1} \otimes \Theta^{-1}) \Delta(\Theta)$.

Comment 6.12. The choice of Θ in our construction is not canonical. For instance one can replace Θ with $K_w \circ \Theta$ for any weight w , and σ^Θ will remain unchanged.

7. PROOF THAT, WHEN \mathfrak{g} IS OF FINITE TYPE, σ^Θ IS THE STANDARD BRAIDING

For this section \mathfrak{g} is of finite type. The main result is that $(\Theta^{-1} \otimes \Theta^{-1}) \Delta(\Theta)$ is equal to the standard R -matrix, or equivalently that σ^Θ is the standard braiding. We prove this using a relationship between the R -matrix and the braid group element T_{w_0} first observed in [KR] and [LS]. For this reason we must restrict to finite type, although we hope the result will prove to be true in greater generality.

We will need to introduce a few more automorphisms of $U_q(\mathfrak{g})$, and of its representations:

Definition 7.1. C_Γ is the \mathbb{C} -hopf algebra automorphism of $U_q(\mathfrak{g})$ defined by

$$(13) \quad \begin{cases} C_\Gamma(E_i) = -K_{\theta(i)} F_{\theta(i)} \\ C_\Gamma(F_i) = -E_{\theta(i)} K_{\theta(i)}^{-1} \\ C_\Gamma(K_i) = K_{\theta(i)} \\ C_\Gamma(q) = q^{-1}. \end{cases}$$

Define the action of $\Gamma_{(V_\lambda, B_\lambda)}$ on V_λ as the unique \mathbb{C} -linear endomorphism of each V_λ which is compatible with C_Γ , and which is normalized so that $\Gamma(b_\lambda) = b_\lambda^{\text{low}}$. Extend this by naturallity to get the action of $\Gamma_{(V,B)}$ on any (possible reducible) representation V .

Comment 7.2. It is a simple exercise to check that C_Γ is in fact a hopf algebra automorphism, and is compatible with an \mathbb{C} vector space automorphism of V_λ which takes b_λ to b_λ^{low} . As with Θ , we do not specify a chosen global basis for $\Gamma_{(V,B)} V$.

Recall that the braid group acts on $U_q(\mathfrak{g})$ representations, and that there is a compatible action on $U_q(\mathfrak{g})$ by algebra automorphisms. Let T_{w_0} be the braid group element corresponding to the longest element in the Weyl group. We will use $C_{T_{w_0}}$ to denote the action of T_{w_0} on $U_q(\mathfrak{g})$, and reserve T_{w_0} to mean the action on a representation V . As discussed in [KT, Lemmas 5.3 and 5.6] we have:

$$(14) \quad \begin{cases} C_{T_{w_0}}(E_i) = -F_{\theta(i)} K_{\theta(i)} \\ C_{T_{w_0}}(F_i) = -K_{\theta(i)}^{-1} E_{\theta(i)} \\ C_{T_{w_0}}(K_H) = K_{w_0(H)}, \text{ so that } C_{T_{w_0}}(K_i) = K_{\theta(i)}^{-1}. \end{cases}$$

Furthermore, For all λ , we have $T_{w_0} b_\lambda^{\text{low}} = b_\lambda$.

Lemma 7.3. $\Gamma = \text{bar} \circ T_{w_0}^{-1}$, where bar is the involution defined in Theorem 5.8, part (iv).

Proof. A simple calculation shows that $C_\Gamma = C_{\text{bar}} C_{T_{w_0}}^{-1}$. Also, one can easily check that both maps take b_λ to b_λ^{low} for all λ . The lemma follows. \square

Definition 7.4. Let C_J be the algebra automorphism of $U_q(\mathfrak{g})$ defined by:

$$(15) \quad \begin{cases} C_J(E_i) = K_i E_i \\ C_J(F_i) = F_i K_i^{-1} \\ C_J(K_H) = K_H \end{cases}$$

As in [KT], this is compatible with the system of linear automorphisms J of each representation V defined by letting J act on a vector of weight μ as multiplication by $q^{(\mu, \mu)/2 + (\mu, \rho)}$.

Comment 7.5. In some cases J acts on a weight vector as multiplication by a fractional power of q . To be completely precise we should adjoin a fixed k^{th} root of unity to the base field $\mathbb{C}(q)$, where k is twice the dual Coxeter number of \mathfrak{g} . This causes no difficulties.

Comment 7.6. We will denote the action of Γ (Definition 7.1), “bar” (Theorem 5.8 part (iv)) and J (Definition 7.4) on a tensor product by $\Delta(\Gamma)$, $\Delta(\text{bar})$ and $\Delta(J)$ respectively. When a chosen canonical basis is needed for the tensor product, this is always chosen using Definition 6.7.

Lemma 7.7. $\Gamma^{-1}\Theta = JT_{w_0}$.

Proof. It follows immediately from definitions that $C_{\Gamma^{-1}\Theta} = C_{JT_{w_0}}$. Furthermore, both $\Gamma^{-1}\Theta$ and JT_{w_0} take b_λ^{low} to $q^{(\lambda, \lambda)/2 + (\lambda, \rho)} b_\lambda$. \square

We will need to use two different expressions for the R -matrix, which we now recall. These are presented as elements in the h -adic completion of $U_h(\mathfrak{g})$, whereas here we are working in $U_q(\mathfrak{g})$. However, it is straightforward to check that these expressions for R give a well defined endomorphism of $V \otimes W$ for any finite dimensional $U_q(\mathfrak{g})$ -representations V and W .

Lemma 7.8. [CP, Theorem 8.3.9] *The standard universal R matrix is given by the expression*

$$(16) \quad R_h = \exp \left(h \sum_{i,j} (B^{-1})_{ij} H_i \otimes H_j \right) \prod_{\beta} \exp_{q_\beta} \left[(1 - q_\beta^{-2}) E_\beta \otimes F_\beta \right],$$

where the product is over all the positive roots of \mathfrak{g} , and the order of the terms is such that β_r appears to the left of β_s if $r > s$. \square

We will not explain all this terminology here, since the only thing we use is the fact that E_β always acts as 0 on any highest weight vector. We also need the following construction of the R matrix due to Kirillov-Reshetikhin and Levendorskii-Soibelman. Due to a different choice of conventions, our T_{w_0} is $K_\rho^{-1} T_{w_0}^{-1}$ in those papers, so we have modified the statement accordingly. We have also simplified the statement using the fact that K_ρ is group-like.

Theorem 7.9. [KR, Theorem 3], [LS, Theorem 1] *The standard universal R -matrix for $U_h(\mathfrak{g})$ can be realized as*

$$(17) \quad R = \exp \left(h \sum_{i,j \in I} (B^{-1})_{ij} H_i \otimes H_j \right) (T_{w_0}^{-1} \otimes T_{w_0}^{-1}) \Delta(T_{w_0}). \quad \square$$

Corollary 7.10.

$$(18) \quad (T_{w_0}^{-1} \otimes T_{w_0}^{-1})\Delta(T_{w_0}) = \prod_{\beta} \exp_{q_{\beta}} \left[(1 - q_{\beta}^{-2})E_{\beta} \otimes F_{\beta} \right],$$

where the product is over all the positive roots of \mathfrak{g} , and the order of the terms is such that β_r appears to the left of β_s if $r > s$.

Proof. Follows immediately from Theorems 7.8 and 7.9. \square

As discussed in [KT], the following is equivalent to Theorem 7.9:

Corollary 7.11. *Let $X = JT_{w_0}$. Then $R = (X^{-1} \otimes X^{-1})\Delta(X)$.* \square

Before proving our main result, we need the following technical lemma:

Lemma 7.12. *The operator $(\Gamma \otimes \Gamma)\Delta(\Gamma)^{-1}$ acts on $V_{\lambda} \otimes V_{\mu}$ as the identity.*

Proof. C_{Γ} is a hopf algebra automorphism of $U_q(\mathfrak{g})$, so, as in Section 4, it follows that

$$(19) \quad (\Gamma \otimes \Gamma)\Delta(\Gamma)^{-1} : V_{\lambda} \otimes V_{\mu} \rightarrow V_{\lambda} \otimes V_{\mu}$$

is an automorphism of $U_q(\mathfrak{g})$ representations. In particular, $(\Gamma \otimes \Gamma)\Delta(\Gamma)^{-1}$ preserves isotypic components of $V_{\lambda} \otimes V_{\mu}$ and hence acts on each sub-quotient $Q_{\lambda, \mu}^{\nu}$ (see Definition 6.5). The action of $(\Gamma \otimes \Gamma)\Delta(\Gamma)^{-1}$ on $Q_{\lambda, \mu}^{\nu}$ will agree with the action on the ν isotypic component of $V_{\lambda} \otimes V_{\mu}$, so it is sufficient to show that the action on $Q_{\lambda, \mu}^{\nu}$ is the identity, and in fact it is sufficient to consider the action on the highest weight space of $Q_{\lambda, \mu}^{\nu}$. This has a basis consisting of $\{\overline{b_{\lambda} \otimes b} : b \in S_{\lambda, \mu}^{\nu}\}$, where we use the notation $\overline{a \otimes b}$ to denote the image of $a \otimes b$ in $Q_{\lambda, \mu}^{\nu}$.

We now argue that $(\Gamma \otimes \Gamma)\Delta(\Gamma)^{-1}(\overline{b_{\lambda} \otimes b}) = \overline{b_{\lambda} \otimes b}$ for all $b \in S_{\lambda, \mu}^{\nu}$. By Lemma 7.3 and Corollary 7.10,

$$(20) \quad (\Gamma \otimes \Gamma)\Delta(\Gamma)^{-1} = (\text{bar} \otimes \text{bar})(T_{w_0}^{-1} \otimes T_{w_0}^{-1})\Delta(T_{w_0})\Delta \text{bar}$$

$$(21) \quad = (\text{bar} \otimes \text{bar}) \prod_{\beta} \exp_{q_{\beta}} \left[(1 - q_{\beta}^{-2})E_{\beta} \otimes F_{\beta} \right] \Delta \text{bar},$$

For convenience, set

$$(22) \quad \Psi := (\text{bar} \otimes \text{bar}) \prod_{\beta} \exp_{q_{\beta}} \left[(1 - q_{\beta}^{-2})E_{\beta} \otimes F_{\beta} \right].$$

Note that both $(\Gamma \otimes \Gamma)\Delta(\Gamma)^{-1}$ and $\Delta(\text{bar})$ preserve isotypic components of $V_{\lambda} \otimes V_{\mu}$, which implies that Ψ does as well. Hence it is sufficient to first apply $\Delta(\text{bar})$ to $\overline{b_{\lambda} \otimes b}$, then apply Ψ , and at each stage take the image in $Q_{\lambda, \mu}^{\nu}$.

The global basis of the tensor product was chosen so that $\Delta(\text{bar})(\overline{b_{\lambda} \otimes b}) = \overline{b_{\lambda} \otimes b}$ (see Definition 6.7). One can easily see that $\Psi(b_{\lambda} \otimes b) = b_{\lambda} \otimes b$, since all E_{β} kill b_{λ} , and $\text{bar} \otimes \text{bar}$ preserves $b_{\lambda} \otimes b$ by definition. Hence $\psi(\overline{b_{\lambda} \otimes b}) = \overline{b_{\lambda} \otimes b}$. The lemma follows. \square

Theorem 7.13. *$(\Theta^{-1} \otimes \Theta^{-1})\Delta(\Theta)$ acts on $V_{\lambda} \otimes V_{\mu}$ as the standard R-matrix.*

Proof. By Lemma 7.7, $\Gamma^{-1}\Theta = JT_{w_0} = X$. By Corollary 7.11, $(X^{-1} \otimes X^{-1})\Delta(X)$ is the standard R-matrix. Thus $R = (\Theta^{-1} \otimes \Theta^{-1})(\Gamma \otimes \Gamma)\Delta(\Gamma)^{-1}\Delta(\Theta)$. By Lemma 7.12, the $(\Gamma \otimes \Gamma)\Delta(\Gamma)^{-1}$ that appears acts as the identity. \square

8. FUTURE DIRECTIONS

For any symmetrizable Kac-Moody algebra \mathfrak{g} , $\sigma^\Theta : V \otimes W \rightarrow W \otimes V$ is a natural system of automorphisms for the category of integrable highest weight $U_q(\mathfrak{g})$ representations. The current work shows that, when \mathfrak{g} is of finite type, this is the standard braiding. We would like to understand σ^Θ in the other cases. In particular, we would like to answer the following

Question 1. Is $(\Theta^{-1} \otimes \Theta^{-1})\Delta(\Theta)$ a universal R -matrix for $U_q(\mathfrak{g})$ if \mathfrak{g} is a general symmetrizable Kac-Moody algebra \mathfrak{g} ? If yes, is it the standard R -matrix?

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