THESIS SYNOPSIS

PETER TINGLEY

1. BACKGROUND

Quantum groups and in particular quantized universal enveloping algebras have provided a rich field of study. They have been used in many applications ranging from representation theory to topology to statistical mechanics. In the present work we study the representation theory of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ associated to a symmetrizable Kac-Moody algebra $\mathfrak{g}$.

We focus on two aspects of this theory. The first is the fact that representations of $U_q(\mathfrak{g})$ have the structure of a braided monoidal category. In particular, there is a natural system of isomorphisms $\sigma^{\text{br}}_{V,W} : V \otimes W \to W \otimes V$, for all pairs of representations V and W, which satisfies the braid relations:

$$\sigma^{\text{br}}_{1,2} \circ \sigma^{\text{br}}_{2,3} \circ \sigma^{\text{br}}_{1,2} = \sigma^{\text{br}}_{2,3} \circ \sigma^{\text{br}}_{1,2} \circ \sigma^{\text{br}}_{2,3}$$

as isomorphisms from $U \otimes V \otimes W$ to $W \otimes V \otimes U$, where $\sigma^{\text{br}}_{i,i+1}$ means apply the braiding to the i and $i+1$st factor of the tensor product. This implies that the action of the braiding on an N-fold tensor product factors through the N-strand braid group, and is one of the main ingredients used in constructing the well known quantum group knot invariants.

The other is the existence of crystal bases for representations of $U_q(\mathfrak{g})$. These are remarkable bases discovered by Kashiwara (see [19]). By allowing q to approach 0 (or ∞) in an appropriate way, one can obtain a combinatorial object called a crystal associated to each integrable highest weight representation of $U_q(\mathfrak{g})$. This crystal consists of an underlying set along with certain operators e_i and f_i corresponding to the generators E_i and F_i of $U_q(\mathfrak{g})$. A crystal can be depicted using a colored directed graph, where the underlying set of the crystal is represented as the vertices of the graph, and the operators are encoded with the edges. One can use these crystals to answer various questions about the original representations. For instance, there is a simple tensor product rule for crystals which allows one to calculate tensor product multiplicities for the original representations.

There are two parts to this work. The first (Chapter 3) is concerned with the braiding on the category of representations of $U_q(\mathfrak{g})$, the related notion of a coboundary structure, and how these interact with crystal bases. The second (Chapter 4) is concerned with crystals for the affine algebra $\widehat{\mathrm{sl}}_n$. We now give a very brief overview of the results in each. Please see the introduction of the thesis for more detail.

2. Results from Chapter 3

One might hope that the braiding on the category of $U_q(\mathfrak{g})$ representations would descend to a braiding on the category on $U_q(\mathfrak{g})$ crystals, but this is not the case. Instead, one should use the related notion of a coboundary monoidal category, introduced by Drinfeld in [6]. A coboundary structure is a system of isomorphisms $\sigma_{V,W} : V \otimes V \to W \otimes V$ for all V and W, but where the role played by the braid group above is replaced by the "cactus group."

Drinfeld showed that the standard braiding on $U_q(\mathfrak{g})$ representations can be modified to obtain a coboundary structure. When $\mathfrak{g}$ is of finite type, Henriques and Kamnitzer [9] define a coboundary structure on the category of $U_q(\mathfrak{g})$ crystals. In this chapter we study the relationship between these two coboundary structures. We extend Henriques and Kamnitzer's definition to include all symmetrizable Kac-Moody algebras, although we cannot show that the result is a coboundary structure in non-finite type cases. Our main results are:

• Drinfeld's coboundary structure respects crystal bases (up to signs), and thus gives rise to a coboundary structure on $\mathfrak{g}$-crystals.

• The coboundary structure on $\mathfrak{g}$ crystals arising from Drinfeld's construction agrees with the one constructed by Henriques and Kamnitzer in [9].

• The crystal commutor admits an alternative definition using Kashiwara's involution on the infinity crystal B_∞. This is well-defined for all symmetrizable Kac-Moody algebras, although we only show it is a coboundary structure in finite type cases (where it agrees with Henriques and Kamnitzer's definition). It has recently been proven by Savage [30, Theorem 6.4] that this is a coboundary structure in all cases.

• One of the main tools we use is a formula for the standard braiding due to Kirillov-Reshetikhin [23] and Levendorskii-Soibelman [24]:

$$\sigma^{\mathrm{br}} = \mathrm{Flip} \circ (X^{-1} \otimes X^{-1})\Delta(X),$$

where X is in a certain completion of $U_q(\mathfrak{g})$. The element X is only defined in finite type, which is one major obstacle in extending our work to other types. We show that there is an analogous formula which is valid for all symmetrizable Kac-Moody algebras $\mathfrak{g}$, but where the action of X on a representation is replaced with a bar-linear endomorphism. We feel this is the first step towards establishing a relationship between Drinfeld's coboundary structure and the coboundary structure on crystals in non-finite type cases.

3. Results from Chapter 4

The results in this chapter were motivated by the Hayashi realization for crystals of level one $\widehat{\mathrm{sl}}_n$ representations, originally developed by Misra and Miwa [28] using work of Hayashi [8]. In that realization, the underlying set of the crystal consists of partitions, and the operators f_i act by adding a box to the associated Young diagram. We wondered if there was a similar realization for representations of arbitrary level ℓ, where the operators f_i would act by adding ℓ boxes at once, arranged in an ℓ-ribbon. It turns out that there is. In studying this model, we are led to consider two other models as well. Our main results are:

• We explicitly construct combinatorial model for $\widehat{\mathrm{sl}}_n$ crystals whose vertices are parameterized by (i) partitions, (ii) configurations of beads on an abacus and (iii) cylindric plane partitions.

• We find the crystal corresponding to every irreducible integrable highest weight representation of $\widehat{\mathrm{sl}}_n$ as an explicitly defined sub-crystal of the abacus model.

• We show that the set of cylindric plane partitions with a given boundary is a union of infinitely many copies of the same $\widehat{\mathrm{sl}}_n$ crystal. These appear with a certain grading, which corresponds to the decomposition of a highest weight $\widehat{\mathrm{gl}}_n$ representation into $\widehat{\mathrm{sl}}_n$ representations. This allows us to show that the generating function for cylindric plane partitions, counted according to a natural weight, is given by the q-character of a certain $\widehat{\mathrm{gl}}_n$ representation.

• The generating function of cylindric plane partitions can be given either as the q-character of a certain level ℓ representation of $\widehat{\mathrm{sl}}_n$ or a certain level n representation of $\widehat{\mathrm{sl}}_\ell$, where ℓ and n are determined by the boundary. This allows us to recover a form of rank-level duality originally observed by I. Frenkel [7].