THE LIMIT OF A LEVELWISE WEAK EQUIVALENCE OF TOWERS OF FIBRATIONS

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Abstract. We present an elementary proof of the well known theorem that the limit of a levelwise weak equivalence of towers of fibrations of topological spaces is a weak equivalence, without mentioning $\lim^1$.

1. The theorem

Theorem 1.1. *If*

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "f_{n+1}"] & X_n \arrow[r, "p_n"] \arrow[d, "f_n"] & X_{n-1} \arrow[r, "p_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \arrow[r, "p_1"] & X_0 \arrow[d, "f_0"] \\
\cdots \arrow[r] & Y_{n+1} \arrow[r, "q_{n+1}"'] & Y_n \arrow[r, "q_n"'] & Y_{n-1} \arrow[r, "q_{n-1}"'] & \cdots \arrow[r, "q_1"'] & Y_0
\end{tikzcd}$$

is a map of towers of fibrations of topological spaces such that f_n is a weak equivalence for every $n \geq 0$, then the induced map of limits $f\colon \lim_n X_n \to \lim_n Y_n$ is a weak equivalence.

The proof of Theorem 1.1 will make use of the following proposition, whose proof will be presented in Section 3.

Proposition 1.2. *For every choice of basepoint in $\lim_n X_n$ (whose image is then taken as the basepoint in each of $\lim_n Y_n$, X_n for $n \geq 0$, and Y_n for $n \geq 0$) and every $k \geq 0$ there is a map of short exact sequences*

$$\begin{tikzcd}
1 \arrow[r] & K_X \arrow[r] \arrow[d, "\phi"'] & \pi_k \lim\limits_n X_n \arrow[r, "P"] \arrow[d, "f_*"'] & \lim\limits_n \pi_k X_n \arrow[r] \arrow[d, "\psi"] & 1 \\
1 \arrow[r] & K_Y \arrow[r] & \pi_k \lim\limits_n Y_n \arrow[r, "Q"] & \lim\limits_n \pi_k Y_n \arrow[r] & 1
\end{tikzcd} \tag{1.3}$$

of groups if $k > 0$ and of pointed sets if $k = 0$, where K_X is the kernel of P, K_Y is the kernel of Q, and both ϕ and ψ are isomorphisms (of groups, if $k > 0$ and of pointed sets, if $k = 0$).

Proof of Theorem 1.1. For $k > 0$ the five lemma (non-abelian, if $k = 1$) applied to Diagram 1.3 shows that for every choice of basepoint in $\lim_n X_n$, the map $\pi_k \lim_n X_n \to \pi_k \lim_n Y_n$ is an isomorphism, and so it remains only to show that $\lim_n X_n$ is nonempty if and only if $\lim_n Y_n$ is nonempty and, in the case that they are nonempty, that the set of path components of $\lim_n X_n$ maps isomorphically onto the set of path components of $\lim_n Y_n$.

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If $\lim_n X_n$ is nonempty, then the existence of the natural map $\lim_n X_n \to \lim_n Y_n$ implies that $\lim_n Y_n$ is nonempty. Conversely, if $\lim_n Y_n$ is nonempty, then we can choose a point $(y_n)_{n\geq 0}$ in $\lim_n Y_n$, and for each $n \geq 0$ we can choose a point $x_n \in X_n$ such that $f_n(x_n)$ is in the same path component as y_n. We will now inductively define points $x'_n \in X_n$ for all $n \geq 0$ such that x'_n is in the same path component as x_n and $p_{n+1}(x'_{n+1}) = x'_n$. We begin the induction by letting $x'_0 = x_0$. If $n \geq 0$ and we've defined x'_n, then $f_n p_{n+1}(x_{n+1}) = q_{n+1} f_{n+1}(x_{n+1})$ is in the same path component as $y_n = q_{n+1}(y_{n+1})$, and so (since f_n is a weak equivalence) $p_{n+1}(x_{n+1})$ is in the same path component as x'_n. Thus, we can choose a path $\alpha\colon I \to X_n$ from $p_{n+1}(x_{n+1})$ to x'_n and then lift α to a path $\widetilde{\alpha}\colon I \to X_{n+1}$ such that $\widetilde{\alpha}(0) = x_{n+1}$; we let $x'_{n+1} = \widetilde{\alpha}(1)$. This completes the induction, and so $\lim_n X_n$ contains the point $(x'_n)_{n\geq 0}$ and is thus nonempty.

To see that the set of path components of $\lim_n X_n$ maps onto that of $\lim_n Y_n$ when those limits are nonempty, choose some basepoint for $\lim_n X_n$ and consider Diagram 1.3. Let $a \in \pi_0 \lim_n Y_n$; then we can choose $b \in \pi_0 \lim_n X_n$ such that $P(b) = \psi^{-1}Q(a)$, and we will have that $Q\big(f_*(b)\big) = Q(a)$. Now choose a new basepoint for $\lim_n X_n$ that is in the path component b of $\lim_n X_n$, and consider this version of Diagram 1.3. The path component $f_*(b)$ of $\lim_n Y_n$ is now the path component of the basepoint, and so $a \in K_Y$. Thus, there is an element $a' \in K_X$ such that $\phi(a') = a$, and a' is an element of $\pi_0 \lim_n X_n$ that goes to a under f_*.

To see that the set of path components is mapped injectively, let a and b be path components of $\lim_n X_n$ that go to the same path component of $\lim_n Y_n$. Choose a basepoint in the path component a, and consider Diagram 1.3. We have $f_*(a) = f_*(b)$, and so $P(a) = P(b)$, and since a is the path component of the basepoint both a and b are elements of K_X. If $a \neq b$, then $\phi(a) \neq \phi(b)$, and so $\phi(a)$ and $\phi(b)$ are distinct elements of $\pi_0 \lim_n Y_n$. Since those are the same element of $\pi_0 \lim_n Y_n$, it must be that $a = b$ in $\lim_n X_n$. □

2. Homotopy groups and homotopic homotopies

Notation 2.1. We let I denote the interval $[0,1]$, and we let I^0 denote the single point space $\{*\}$. If $k \geq 1$ we will often denote a point of I^k as (p,t), where $p \in I^{k-1}$ and $t \in I$.

Definition 2.2. If X is a space and $k \geq 0$, we will represent elements of $\pi_k X$ by maps $\alpha\colon I^k \to X$ that take the boundary of I^k to the basepoint, and we will denote the element of $\pi_k X$ represented by α as $[\alpha]$.

We will multiply and take inverses of elements of $\pi_k X$ using the last coordinate of I^k.

(1) If $k \geq 1$ and $\alpha\colon I^k \to X$ is a map, then the *inverse* $\alpha^{-1}\colon I^k \to X$ of α is the map defined by $\alpha^{-1}(p,t) = \alpha(p, 1-t)$. If $k = 1$ this is the usual definition of the inverse of a path.

(2) If $k \geq 1$ and $\alpha, \beta\colon I^k \to X$ are maps such that $\alpha(p,1) = \beta(p,0)$ for all $p \in I^{k-1}$, then the *composition* of α and β is the map $\alpha * \beta\colon I^k \to X$ defined by

$$(\alpha * \beta)(p,t) = \begin{cases} \alpha(p,2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(p,2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} .$$

If $k = 1$, this is the usual definition of the composition of paths.

Definition 2.3. If $k \geq 0$ and $\alpha, \beta \colon I^k \to X$ are maps such that $\alpha(p) = \beta(p)$ for all $p \in \partial(I^k)$, then

(1) a *homotopy from α to β* is a map $\gamma \colon I^{k+1} \to X$ such that

$$\gamma(p,t) = \begin{cases} \alpha(p) & \text{if } (p,t) \in (I^k \times \{0\}) \cup (\partial I^k \times I) \\ \beta(p) & \text{if } (p,t) \in I^k \times \{1\} \end{cases}$$

and

(2) $\alpha \simeq \beta$ means that α is homotopic to β relative to the boundary of I^k. If $k = 1$, this is the usual notion of path homotopy.

Note that we do not require that α and β take the boundary of I^k to the basepoint. Thus, if $\alpha_1, \alpha_2 \colon I^k \to X$ take the boundary of I^k to the basepoint and $\beta_1, \beta_2 \colon I^{k+1} \to X$ are both homotopies from α_1 to α_2, then $\beta_2 \simeq \beta_2$ means that the homotopies β_1 and β_2 are themselves homotopic through homotopies from α_1 to α_2.

Definition 2.4. If $k \geq 0$ and $\alpha \colon I^k \to X$ is a map taking the boundary of I^k to the basepoint of X, then by a *nullhomotopy* of α we will mean a homotopy from the constant map to the basepoint to α, i.e., a map $\beta \colon I^{k+1} \to X$ such that

$$\beta(p,t) = \begin{cases} * & \text{if } (p,t) \in (I^k \times \{0\}) \cup (\partial I^k \times I) \\ \alpha(p) & \text{if } (p,t) \in I^k \times \{1\} \end{cases} .$$

There exists such a nullhomotopy for α if and only if $[\alpha] = 0$ in $\pi_k X$.

Definition 2.5. If $k \geq 0$ and $\alpha \colon I^k \to X$ takes the boundary of I^k to the basepoint, then the *constant homotopy of* α is the map $\mathrm{C}_\alpha \colon I^{k+1} \to X$ defined by

$$\mathrm{C}_\alpha(t_1, t_2, \ldots, t_{k+1}) = \begin{cases} \alpha(t_1, t_2, \ldots, t_k) & \text{if } k > 0 \\ \alpha(*) & \text{if } k = 0. \end{cases}$$

Lemma 2.6. *Let $k \geq 0$ and let $\alpha_1, \alpha_2 \colon I^k \to X$ take the boundary of I^k to the basepoint. If β is a homotopy from α_1 to α_2, then*

(1) $\beta^{-1} * \beta \simeq \mathrm{C}_{\alpha_2}$,
(2) $\mathrm{C}_{\alpha_1} * \beta \simeq \beta$, *and*
(3) $\beta * \mathrm{C}_{\alpha_2} \simeq \beta$.

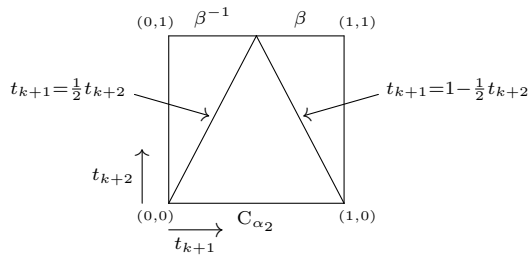
If $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \colon I^k \to X$ take the boundary of I^k to the basepoint, β_1 is a homotopy from α_1 to α_2, β_2 is a homotopy from α_2 to α_3, and β_3 is a homotopy from α_3 to α_4, then

(4) $(\beta_1 * \beta_2) * \beta_3 \simeq \beta_1 * (\beta_2 * \beta_3)$ . .

Proof. The proofs are the same as the ones for compositions of paths in a space (and they reduce to those assertions when $k = 0$). We will prove part 1.

Define $\gamma \colon I^{k+2} \to X$ by

$$\gamma(t_1, t_2, \ldots, t_{k+2}) = \begin{cases} \beta^{-1}(t_1, t_2, \ldots, t_k, 2t_{k+1}) & \text{if } t_{k+1} \leq \frac{1}{2} t_{k+2} \\ \beta^{-1}(t_1, t_2, \ldots, t_k, t_{k+2}) & \text{if } \frac{1}{2} t_{k+2} \leq t_{k+1} \leq 1 - \frac{1}{2} t_{k+2} \\ \beta(t_1, t_2, \ldots, t_k, 2t_{k+1} - 1) & \text{if } 1 - \frac{1}{2} t_{k+2} \leq t_{k+1} \end{cases} ;$$



then

$$\gamma(t_1, t_2, \ldots, t_{k+2}) = \begin{cases} \mathrm{C}_{\alpha_2}(t_1, t_2, \ldots, t_{k+1}) & \text{if } t_{k+2} = 0 \\ (\beta^{-1} * \beta)(t_1, t_2, \ldots, t_{k+1}) & \text{if } t_{k+2} = 1 \\ * & \text{if any of } t_1, t_2, \ldots, t_k \text{ equal 0 or 1} \end{cases}$$

i.e., γ is a homotopy relative to the boundary of I^{k+1} from C_{α_2} to $\beta^{-1} * \beta$. □

Lemma 2.7. *If $k \geq 0$, $\alpha_1, \alpha_2 \colon I^k \to X$ take the boundary of I^k to the basepoint, and β_1 and β_2 are homotopies from α_1 to α_2, then $\beta_1 * \beta_2^{-1} \simeq \mathrm{C}_{\alpha_1}$ if and only if $\beta_1 \simeq \beta_2$.*

Proof. If $\beta_1 * \beta_2^{-1} \simeq \mathrm{C}_{\alpha_1}$, then we have

$$\begin{aligned} \beta_1 &\simeq \beta_1 * \mathrm{C}_{\alpha_2} \\ &\simeq \beta_1 * (\beta_2^{-1} * \beta_2) \\ &\simeq (\beta_1 * \beta_2^{-1}) * \beta_2 \\ &\simeq \mathrm{C}_{\alpha_1} * \beta_2 \\ &\simeq \beta_2 \ . \end{aligned}$$

Conversely, if $\beta_1 \simeq \beta_2$, then $\beta_1 * \beta_2^{-1} \simeq \beta_2 \times \beta_2^{-1} \simeq \mathrm{C}_{\alpha_1}$. □

Proposition 2.8. *Let $k \geq 0$, let $\alpha \colon I^k \to X$ take the boundary of I^k to the basepoint, and let β_1 and β_2 be nullhomotopies of α, so that $\beta_1 * \beta_2^{-1} \colon I^{k+1} \to X$ is defined and takes the boundary of I^{k+1} to the basepoint. If $[\beta_1 * \beta_2^{-1}] = 1$ in $\pi_{k+1}X$, then $\beta_1 \simeq \beta_2$.*

Proof. This follows from Lemma 2.7. □

Proposition 2.9. *Let $k \geq 0$ and let $\alpha_1, \alpha_2 \colon I^k \to X$ take the boundary of I^k to the basepoint. If β_1 is a homotopy from α_1 to itself and β_2 is a homotopy from α_1 to α_2, then β_1 is homotopic to the constant homotopy of α_1 if and only if $\beta_2^{-1} * \beta_1 * \beta_2$ is homotopic to the constant homotopy of α_2.*

Proof. If $\beta_1 \simeq \mathrm{C}_{\alpha_1}$, then

$$\beta_2^{-1} * \beta_1 * \beta_2 \simeq \beta_2^{-1} * \mathrm{C}_{\alpha_1} * \beta_2 \simeq \beta_2^{-1} * \beta_2 \simeq \mathrm{C}_{\alpha_2} \ .$$

Conversely, if $\beta_2^{-1} * \beta_1 * \beta_2 \simeq \mathrm{C}_{\alpha_2}$, then

$$\begin{aligned} \mathrm{C}_{\alpha_1} \simeq \beta_2 * \beta_2^{-1} \simeq \beta_2 * \mathrm{C}_{\alpha_2} * \beta_2^{-1} &\simeq \beta_2 * (\beta_2^{-1} * \beta_1 * \beta_2) * \beta_2^{-1} \\ &\simeq (\beta_2 * \beta_2^{-1}) * \beta_1 * (\beta_2 * \beta_2^{-1}) \simeq \mathrm{C}_{\alpha_1} * \beta_1 * \mathrm{C}_{\alpha_1} \simeq \beta_1 \ . \end{aligned}$$ □

3. Proof of Proposition 1.2

Notation 3.1. For each $i \geq 0$ we let

$$P_i\colon \lim_n X_n \longrightarrow X_i \qquad \text{and} \qquad Q_i\colon \lim_n Y_n \longrightarrow Y_i$$

denote the natural projections from the limit, and we let $f\colon \lim_n X_n \to \lim_n Y_n$ be the limit of the f_n.

Proposition 3.2. *For every choice of basepoint in $\lim_n X_n$ and every $k \geq 0$, the natural map $P\colon \pi_k \lim_n X_n \to \lim_n \pi_k X_n$ is surjective.*

Proof. Every element of $\lim_n \pi_k X_n$ can be represented by $([\alpha_n])_{n\geq 0}$, where $\alpha_n\colon I^k \to X_n$ for $n \geq 0$ is a map taking the boundary of I^k to the basepoint of X_n and such that $p_{n+1} \circ \alpha_{n+1} \simeq \alpha_n$. We will inductively define maps $\bar{\alpha}_n\colon I^k \to X_n$ for $n \geq 0$ such that $\bar{\alpha}_n \simeq \alpha_n$ and $p_{n+1} \circ \bar{\alpha}_{n+1} = \bar{\alpha}_n$; the collection $(\bar{\alpha}_n)_{n\geq 0}$ will then define a map $\bar{\alpha}\colon I^k \to \lim_n X_n$ such that $P([\bar{\alpha}]) = ([\alpha_n])_{n\geq 0}$.

We begin the induction by letting $\bar{\alpha}_0 = \alpha_0$. If $n \geq 0$ and we've defined $\bar{\alpha}_n$, then $p_{n+1} \circ \alpha_{n+1} \simeq \alpha_n \simeq \bar{\alpha}_n$, and so there is a homotopy $H\colon I^{k+1} \to X_n$ such that

$$H(p,t) = \begin{cases} (p_{n+1} \circ \alpha_{n+1})(p) & \text{if } t = 0 \\ \bar{\alpha}_n(p) & \text{if } t = 1 \\ * & \text{if } p \in \partial I^k \end{cases} .$$

We define $H'\colon (I^k \times \{0\}) \cup (\partial I^k \times I) \to X_{n+1}$ by $H'(p,t) = \alpha_{n+1}(p)$, and H' is a lift of the restriction of H. Since there is a homeomorphism of I^{k+1} to itself that takes $(I^k \times \{0\}) \cup (\partial I^k \times I)$ onto $I^k \times \{0\}$, we can extend H' to a lift of H on all of I^{k+1}, and we define $\bar{\alpha}_{n+1}(p) = H'(p,1)$. □

Thus, for every choice of basepoint in $\lim_n X_n$ the rows of Diagram 1.3 are short exact sequences. Since each $f_n\colon X_n \to Y_n$ is a weak equivalence, the vertical map ψ on the right is an isomorphism, and so it remains only to show that the map $\phi\colon K_X \to K_Y$ is an isomorphism. We show that ϕ is a monomorphism in Proposition 3.3 and that it is an epimorphism in Proposition 3.4.

Proposition 3.3. *$\phi : K_X \to K_Y$ is injective.*

Proof. Let $\alpha\colon I^k \to \lim_n X_n$ represent an element $[\alpha]$ of K_X such that $\phi[\alpha] = [f \circ \alpha] = 1$ in $K_Y \subset \pi_k \lim_n Y_n$. If $\bar{\alpha} = f \circ \alpha\colon I^k \to \lim_n Y_n$, then $[\bar{\alpha}] = 1$, and so we can choose a nullhomotopy $\bar{\beta}\colon I^{k+1} \to \lim_n Y_n$ of $\bar{\alpha}$. We let $\bar{\alpha}_n = Q_n \circ \bar{\alpha}\colon I^k \to Y_n$ for all $n \geq 0$ and $\bar{\beta}_n = Q_n \circ \bar{\beta}\colon I^{k+1} \to Y_n$ for all $n \geq 0$. For every $n \geq 0$ the map $\bar{\beta}_n$ is then a nullhomotopy of $\bar{\alpha}_n$, and $q_{n+1} \circ \bar{\beta}_{n+1} = \bar{\beta}_n$.

Since $[\alpha] \in K_X$, for every $n \geq 0$ the map $\alpha_n = P_n \circ \alpha\colon I^k \to X_n$ is nullhomotopic, and so we can choose a nullhomotopy $\beta_n\colon I^{k+1} \to X_n$ of α_n. Since $\bar{\alpha}_n = Q_n \circ f \circ \alpha = f_n \circ P_n \circ \alpha$, the maps $f_n \circ \beta_n$ and $\bar{\beta}_n$ are both nullhomotopies of $\bar{\alpha}_n$, and so $\bar{\beta}_n * (f_n \circ \beta_n^{-1})$ is defined and takes the boundary of I^{k+1} to the basepoint, and thus defines an element $[\bar{\beta}_n * (f_n \circ \beta_n^{-1})]$ of $\pi_{k+1} Y_n$; since $f_n\colon X_n \to Y_n$ is a weak equivalence, there exists a map $\gamma_n\colon I^{k+1} \to X_n$ that takes the boundary of I^{k+1} to the basepoint and is such that $(f_n)_*[\gamma_n] = [\bar{\beta}_n * (f_n \circ \beta_n^{-1})]$. Let $\beta'_n = \gamma_n * \beta_n$; then

β'_n is a nullhomotopy of α_n, and

$$\begin{aligned}[\bar{\beta}_n * (f_n \circ (\beta'_n)^{-1})] &= [\bar{\beta}_n * (f_n \circ \beta_n^{-1}) * (f_n \circ \gamma_n^{-1})] \\ &= [\bar{\beta}_n * (f_n \circ \beta_n^{-1})][f_n \circ \gamma_n^{-1}] \\ &= [\bar{\beta}_n * (f_n \circ \beta_n^{-1})][\bar{\beta}_n * (f_n \circ \beta_n^{-1})]^{-1} \\ &= 1 \ .\end{aligned}$$

Thus, $f_n \circ \beta'_n \simeq \bar{\beta}_n$ for all $n \geq 0$ (see Proposition 2.8), and so

$$\begin{aligned} f_n \circ p_{n+1} \circ \beta'_{n+1} &= q_{n+1} \circ f_{n+1} \circ \beta'_{n+1} \\ &\simeq q_{n+1} \circ \bar{\beta}_{n+1} \\ &= \bar{\beta}_n \\ &\simeq f_n \circ \beta'_n \ .\end{aligned}$$

Thus, $[(f_n \circ \beta'_n) * (f_n \circ p_{n+1} \circ (\beta'_{n+1})^{-1})] = (f_n)_*[\beta'_n * (p_{n+1} \circ (\beta'_{n+1})^{-1})] = 1$; since f_n is a weak equivalence, $[\beta'_n * (p_{n+1} \circ (\beta'_{n+1})^{-1})] = 1$, and so $\beta'_n \simeq p_{n+1} \circ \beta'_{n+1}$ (see Proposition 2.8).

We will now inductively define, for every $n \geq 0$, a nullhomotopy β''_n of α_n such that $\beta''_n \simeq \beta'_n$ and, if $n > 0$, $p_n \circ \beta''_n = \beta''_{n-1}$. We begin the induction by letting $\beta''_0 = \beta'_0$. If $n \geq 0$ and we've defined β''_n, then $p_{n+1} \circ \beta'_{n+1} \simeq \beta'_n \simeq \beta''_n$, and so there is a homotopy $H\colon I^{k+2} \to X_n$ from $p_{n+1} \circ \beta'_{n+1}$ to β''_n. If we define $H'\colon (I^{k+1} \times \{0\}) \cup (\partial I^{k+1} \times I) \to X_{n+1}$ by $H'(p,t) = \beta'_{n+1}(p)$, then H' is a lift of the restriction of H to $(I^{k+1} \times \{0\}) \cup (\partial I^{k+1} \times I)$. Since there is a homeomorphism of I^{k+2} to itself that takes $(I^{k+1} \times \{0\}) \cup (\partial I^{k+1} \times I)$ onto $I^{k+1} \times \{0\}$, we can extend H' to a lift of H on all of I^{k+2}, and we define $\beta''_{n+1}\colon I^{k+1} \to X_{n+1}$ by $\beta''_{n+1}(t_1, t_2, \ldots, t_{k+1}) = H'(t_1, t_2, \ldots, t_{k+1}, 1)$. That completes the induction, and the nullhomotopies $(\beta''_n)_{n \geq 0}$ combine to define a nullhomotopy $\beta''\colon I^{k+1} \to \lim_n X_n$ of α, and so $[\alpha] = 1$. □

Proposition 3.4. *$\phi\colon K_X \to K_Y$ is surjective.*

Proof. Let $\bar{\alpha}\colon I^k \to \lim_n Y_n$ represent an element of $K_Y \subset \pi_k \lim_n Y_n$. For each $n \geq 0$ we let $\bar{\alpha}_n = Q_n \circ \bar{\alpha}\colon I^k \to Y_n$, and we can choose a nullhomotopy $\bar{\beta}_n\colon I^{k+1} \to Y_n$ of $\bar{\alpha}_n$. Since $q_{n+1} \circ \bar{\alpha}_{n+1} = \bar{\alpha}_n$, the map $q_{n+1} \circ \bar{\beta}_{n+1}$ is a nullhomotopy of $\bar{\alpha}_n$, and so the map $\bar{\gamma}_n = \bar{\beta}_n * (q_{n+1} \circ \bar{\beta}_{n+1}^{-1})\colon I^{k+1} \to Y_n$ is defined and takes the boundary of I^{k+1} to the basepoint of Y_n and thus defines an element $[\bar{\gamma}_n] = [\bar{\beta}_n * (q_{n+1} \circ \bar{\beta}_{n+1}^{-1})]$ of $\pi_{k+1} Y_n$.

Since each map $f_n\colon X_n \to Y_n$ is a weak equivalence, we can choose elements $[\gamma_n]$ of $\pi_{k+1} X_n$ such that $f_*([\gamma_n]) = [\bar{\gamma}_n]$. We will inductively define $\alpha_n\colon I^k \to X_n$ taking the boundary of I^k to the basepoint and a nullhomotopy $\beta_n\colon I^{k+1} \to X_n$ of α_n for all $n \geq 0$. We will arrange it so that

- $p_{n+1} \circ \alpha_{n+1} = \alpha_n$ for all $n \geq 0$, so that the $(\alpha_n)_{n \geq 0}$ will define a map $\alpha\colon I^k \to \lim_n X_n$ taking the boundary of I^k to the basepoint and thus defining an element $[\alpha]$ of K_X, and
- $[\beta_n * (p_{n+1} \circ \beta_{n+1}^{-1})] = [\gamma_n]$ in $\pi_{k+1} X_n$ for all $n \geq 0$.

We begin by letting $\alpha_0\colon I^k \to X_0$ and $\beta_0\colon I^{k+1} \to X_0$ both be constant maps to the basepoint.

For the inductive step, let $n \geq 0$ and assume that we've defined α_n and β_n. The map $\gamma_n^{-1} * \beta_n\colon I^{k+1} \to X_n$ is a nullhomotopy of α_n, and since there is a

homeomorphism of I^{k+1} to itself that takes $(I^k \times \{0\}) \cup (\partial I^k \times I)$ onto $I^k \times \{0\}$, we can lift the homotopy $\gamma_n^{-1} * \beta_n$ in X_n to a homotopy $\beta_{n+1} \colon I^{k+1} \to X_{n+1}$ such that $\beta_{n+1}(p,t)$ equals the basepoint for $(p,t) \in (I^k \times \{0\}) \cup (\partial I^k \times I)$. We define $\alpha_{n+1}(p) = \beta_{n+1}(p,1)$, and we have $p_{n+1} \circ \alpha_{n+1} = \alpha_n$ and

$$\begin{aligned}
[\beta_n * (p_{n+1} \circ \beta_{n+1}^{-1})] &= [\beta_n * (\gamma_n^{-1} * \beta_n)^{-1}] \\
&= [\beta_n * (\beta_n^{-1} * \gamma_n)] \\
&= [\beta_n * \beta_n^{-1}][\gamma_n] \\
&= [\gamma_n] \ .
\end{aligned}$$

Since $p_{n+1} \circ \alpha_{n+1} = \alpha_n$ for all $n \geq 0$, the maps $(\alpha_n)_{n \geq 0}$ define $\alpha \colon I^k \to \lim_n X_n$, which represents $[\alpha]$ in K_X. We will show that $f_*[\alpha] = [\bar{\alpha}]$.

For each $n \geq 0$, $(f_n \circ \beta_n^{-1}) * \bar{\beta}_n$ is a homotopy from $f_n \circ \alpha_n$ to $\bar{\alpha}_n$. We will now show that $q_{n+1} \circ \big((f_{n+1} \circ \beta_{n+1}^{-1}) * \bar{\beta}_{n+1}\big)$ is homotopic to $(f_n \circ \beta_n^{-1}) * \bar{\beta}_n$ for all $n \geq 0$. After that, we will inductively replace these homotopies with ones that commute with the $(q_n)_{n>0}$ on the nose, and thus define a homotopy from $f \circ \alpha$ to $\bar{\alpha}$.

The map $(f_n \circ \beta_n^{-1}) * \bar{\beta}_n$ is homotopic to $q_{n+1} \circ \big((f_{n+1} \circ \beta_{n+1}^{-1}) * \bar{\beta}_{n+1}\big)$ if and only if $\big(q_{n+1} \circ \big((f_{n+1} \circ \beta_{n+1}^{-1}) * \bar{\beta}_{n+1}\big)\big) * \big((f_n \circ \beta_n^{-1}) * \bar{\beta}_n\big)^{-1}$ is homotopic to the constant homotopy of $f_n \circ \alpha_n$ (see Lemma 2.7), and we have

$$\begin{aligned}
&\big(q_{n+1} \circ \big((f_{n+1} \circ \beta_{n+1}^{-1}) * \bar{\beta}_{n+1}\big)\big) * \big((f_n \circ \beta_n^{-1}) * \bar{\beta}_n\big)^{-1} \\
&= \big((q_{n+1} \circ f_{n+1} \circ \beta_{n+1}^{-1}) * (q_{n+1} \circ \bar{\beta}_{n+1})\big) * \big((f_n \circ \beta_n^{-1}) * \bar{\beta}_n\big)^{-1} \\
&= \big((f_n \circ p_{n+1} \circ \beta_{n+1}^{-1}) * (q_{n+1} \circ \bar{\beta}_{n+1})\big) * \big((f_n \circ \beta_n^{-1}) * \bar{\beta}_n\big)^{-1} \\
&= \big((f_n \circ p_{n+1} \circ \beta_{n+1}^{-1}) * (q_{n+1} \circ \bar{\beta}_{n+1})\big) * \big(\bar{\beta}_n^{-1} * (f_n \circ \beta_n)\big) \ .
\end{aligned}$$

This is homotopic to a constant homotopy if and only if the result of conjugating it by $(f_n \circ p_{n+1} \circ \beta_{n+1}^{-1})$ is homotopic to a constant homotopy (see Proposition 2.9), and that conjugate is homotopic to

$$\begin{aligned}
&(f_n \circ p_{n+1} \circ \beta_{n+1}^{-1})^{-1} \\
&\qquad * \big((f_n \circ p_{n+1} \circ \beta_{n+1}^{-1}) * (q_{n+1} \circ \bar{\beta}_{n+1})\big) * \big(\bar{\beta}_n^{-1} * (f_n \circ \beta_n)\big) \\
&\qquad * (f_n \circ p_{n+1} \circ \beta_{n+1}^{-1}) \\
&\simeq \big((q_{n+1} \circ \bar{\beta}_{n+1}) * \bar{\beta}_n^{-1}\big) * \big((f_n \circ \beta_n) * (f_n \circ p_{n+1} \circ \beta_{n+1}^{-1})\big) \\
&= \big((q_{n+1} \circ \bar{\beta}_{n+1}) * \bar{\beta}_n^{-1}\big) * \big(f_n \circ \big(\beta_n * (p_{n+1} \circ \beta_{n+1}^{-1})\big)\big) \\
&\simeq (\bar{\gamma}_n)^{-1} * (f_n \circ \gamma_n) \\
&\simeq (\bar{\gamma}_n)^{-1} * \bar{\gamma}_n \\
&\simeq \mathrm{C}_*
\end{aligned}$$

(see Lemma 2.6), where C_* is the constant homotopy of the constant map to the basepoint. Thus, $(f_n \circ \beta_n^{-1}) * \bar{\beta}_n$ is homotopic to $q_{n+1} \circ \big((f_{n+1} \circ \beta_{n+1}^{-1}) * \bar{\beta}_{n+1}\big)$.

We will now inductively define homotopies $\bar{\beta}'_n$ from $f_n \circ \alpha_n$ to $\bar{\alpha}_n$ such that

- $\bar{\beta}'_n \simeq (f_n \circ \beta_n^{-1}) * \bar{\beta}_n$, and
- $q_{n+1} \circ \bar{\beta}'_{n+1} = \bar{\beta}'_n$.

We begin the induction by letting $\bar{\beta}'_0 = (f_n \circ \beta_n^{-1}) * \bar{\beta}_n$. For the inductive step, if $n \geq 0$ and we've defined $\bar{\beta}'_n$, then $q_{n+1} \circ \big((f_{n+1} \circ \beta_{n+1}^{-1}) * \bar{\beta}_{n+1}\big) \simeq (f_n \circ \beta_n^{-1}) * \bar{\beta}_n \simeq \bar{\beta}'_n$, and so we can choose a homotopy $H \colon I^{k+2} \to X_n$ from $q_{n+1} \circ \big((f_{n+1} \circ \beta_{n+1}^{-1}) * \bar{\beta}_{n+1}\big)$ to

$\bar{\beta}'_n$. There is a homeomorphism of I^{k+2} to itself that takes $(I^{k+1} \times \{0\}) \cup (\partial I^{k+1} \times I)$ to $I^{k+1} \times \{0\}$, and so we can lift H to a homotopy $H' \colon I^{k+2} \to X_{n+1}$ such that $H'(p,t) = \big((f_{n+1} \circ \beta_{n+1}^{-1}) * \bar{\beta}_{n+1}\big)(p)$ for $(p,t) \in (I^{k+1} \times \{0\}) \cup (\partial I^{k+1} \times I)$, and we let $\bar{\beta}'_{n+2}(p) = H'(p,1)$.

We now have $q_{n+1} \circ \bar{\beta}'_{n+1} = \bar{\beta}'_n$ for $n \geq 0$, and so the $\bar{\beta}'_n$ combine to define a homotopy $\bar{\beta}'$ from $f \circ \alpha$ to $\bar{\alpha}$. $\square$

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