THE LIMIT OF A LEVELWISE WEAK EQUIVALENCE
OF TOWERS OF FIBRATIONS

PHILIP S. HIRSCHHORN

Abstract. We present an elementary proof of the well known theorem that
the limit of a levelwise weak equivalence of towers of fibrations of topological
spaces is a weak equivalence, without mentioning lim^1.

1. The theorem

Theorem 1.1. If

\[
\cdots \rightarrow X_{n+1} \xrightarrow{p_{n+1}} X_n \xrightarrow{p_n} X_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} X_0 \\
\downarrow f_{n+1} \quad \downarrow f_n \quad \downarrow f_{n-1} \quad \downarrow f_0 \\
\cdots \rightarrow Y_{n+1} \xrightarrow{q_{n+1}} Y_n \xrightarrow{q_n} Y_{n-1} \xrightarrow{q_{n-1}} \cdots \xrightarrow{q_1} Y_0
\]

is a map of towers of fibrations of topological spaces such that \( f_n \) is a weak equivalence for every \( n \geq 0 \), then the induced map of limits \( f : \lim_n X_n \to \lim_n Y_n \) is a weak equivalence.

The proof of Theorem 1.1 will make use of the following proposition, whose proof
will be presented in Section 3.

Proposition 1.2. For every choice of basepoint in \( \lim_n X_n \) (whose image is then
taken as the basepoint in each of \( \lim_n Y_n, X_n \) for \( n \geq 0 \), and \( Y_n \) for \( n \geq 0 \)) and
every \( k \geq 0 \) there is a map of short exact sequences

\[
1 \xrightarrow{} K_X \xrightarrow{} \pi_k \lim_n X_n \xrightarrow{} \lim_n \pi_k X_n \xrightarrow{} 1 \\
\phi \downarrow \quad f_* \downarrow \quad \psi \\
1 \xrightarrow{} K_Y \xrightarrow{} \pi_k \lim_n Y_n \xrightarrow{} \lim_n \pi_k Y_n \xrightarrow{} 1
\]

(1.3)

of groups if \( k > 0 \) and of pointed sets if \( k = 0 \), where \( K_X \) is the kernel of \( P \), \( K_Y \)
is the kernel of \( Q \), and both \( \phi \) and \( \psi \) are isomorphisms (of groups, if \( k > 0 \) and of
pointed sets, if \( k = 0 \)).

Proof of Theorem 1.1. For \( k > 0 \) the five lemma (non-abelian, if \( k = 1 \)) applied
to Diagram 1.3 shows that for every choice of basepoint in \( \lim_n X_n \), the map
\( \pi_k \lim_n X_n \to \pi_k \lim_n Y_n \) is an isomorphism, and so it remains only to show that
\( \lim_n X_n \) is nonempty if and only if \( \lim_n Y_n \) is nonempty and, in the case that they
are nonempty, that the set of path components of \( \lim_n X_n \) maps isomorphically
onto the set of path components of \( \lim_n Y_n \).

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If \( \text{lim}_n X_n \) is nonempty, then the existence of the natural map \( \text{lim}_n X_n \to \text{lim}_n Y_n \) implies that \( \text{lim}_n Y_n \) is nonempty. Conversely, if \( \text{lim}_n Y_n \) is nonempty, then we can choose a point \((y_n)_{n\geq 0}\) in \( \text{lim}_n Y_n \), and for each \( n \geq 0 \) we can choose a point \( x_n \in X_n \) such that \( f_n(x_n) \) is in the same path component as \( y_n \). We will now inductively define points \( x'_n \in X_n \) for all \( n \geq 0 \) such that \( x'_n \) is in the same path component as \( x_n \) and \( p_{n+1}(x'_{n+1}) = x'_n \). We begin the induction by letting \( x'_0 = x_0 \). If \( n \geq 0 \) and we’ve defined \( x'_n \), then \( f_n p_{n+1}(x_{n+1}) = q_{n+1} f_{n+1}(x_{n+1}) \) is in the same path component as \( y_n = q_{n+1}(y_{n+1}) \), and so (since \( f_n \) is a weak equivalence) \( p_{n+1}(x_{n+1}) \) is in the same path component as \( x'_n \). Thus, we can choose a path \( \alpha: I \to X_n \) from \( p_{n+1}(x_{n+1}) \) to \( x'_n \) and then lift \( \alpha \) to a path \( \tilde{\alpha}: I \to X_{n+1} \) such that \( \tilde{\alpha}(0) = x_{n+1} \); we let \( x'_{n+1} = \tilde{\alpha}(1) \). This completes the induction, and so \( \text{lim}_n X_n \) contains the point \((x'_n)_{n\geq 0}\) and is thus nonempty.

To see that the set of path components of \( \text{lim}_n X_n \) maps onto that of \( \text{lim}_n Y_n \) when those limits are nonempty, choose some basepoint for \( \text{lim}_n X_n \) and consider Diagram 1.3. Let \( a \in \pi_0 \text{lim}_n Y_n \); then we can choose \( b \in \pi_0 \text{lim}_n X_n \) such that \( P(b) = \psi^{-1} Q(a) \), and we will have that \( Q(f_*(b)) = Q(a) \). Now choose a new basepoint for \( \text{lim}_n X_n \) that is in the path component \( b \) of \( \text{lim}_n X_n \), and consider this version of Diagram 1.3. The path component \( f_*(b) \) of \( \text{lim}_n Y_n \) is now the path component of the basepoint, and so \( a \in K_Y \). Thus, there is an element \( a' \in K_X \) such that \( \phi(a') = a \), and \( a' \) is an element of \( \pi_0 \text{lim}_n X_n \) that goes to \( a \) under \( f_* \).

To see that the set of path components is mapped injectively, let \( a \) and \( b \) be path components of \( \text{lim}_n X_n \) that go to the same path component of \( \text{lim}_n Y_n \). Choose a basepoint in the path component \( a \), and consider Diagram 1.3. We have \( f_*(a) = f_*(b) \), and so \( P(a) = P(b) \), and since \( a \) is the path component of the basepoint both \( a \) and \( b \) are elements of \( K_X \). If \( a \neq b \), then \( \phi(a) \neq \phi(b) \), and so \( \phi(a) \) and \( \phi(b) \) are distinct elements of \( \pi_0 \text{lim}_n Y_n \). Since those are the same element of \( \pi_0 \text{lim}_n Y_n \), it must be that \( a = b \) in \( \text{lim}_n X_n \). \( \square \)

2. Homotopy groups and homotopic homotopies

**Notation 2.1.** We let \( I \) denote the interval \([0, 1]\), and we let \( I^0 \) denote the single point space \( \{\ast\} \). If \( k \geq 1 \) we will often denote a point of \( I^k \) as \((p,t)\), where \( p \in I^{k-1} \) and \( t \in I \).

**Definition 2.2.** If \( X \) is a space and \( k \geq 0 \), we will represent elements of \( \pi_k X \) by maps \( \alpha: I^k \to X \) that take the boundary of \( I^k \) to the basepoint, and we will denote the element of \( \pi_k X \) represented by \( \alpha \) as \([\alpha]\).

We will multiply and take inverses of elements of \( \pi_k X \) using the last coordinate of \( I^k \).

1. If \( k \geq 1 \) and \( \alpha: I^k \to X \) is a map, then the inverse \( \alpha^{-1}: I^k \to X \) of \( \alpha \) is the map defined by \( \alpha^{-1}(p,t) = \alpha(p,1-t) \). If \( k = 1 \) this is the usual definition of the inverse of a path.

2. If \( k \geq 1 \) and \( \alpha, \beta: I^k \to X \) are maps such that \( \alpha(p,1) = \beta(p,0) \) for all \( p \in I^{k-1} \), then the composition of \( \alpha \) and \( \beta \) is the map \( \alpha \circ \beta: I^k \to X \) defined by

\[
(\alpha \circ \beta)(p,t) = \begin{cases} 
\alpha(p,2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\
\beta(p,2t-1) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

If \( k = 1 \), this is the usual definition of the composition of paths.
Definition 2.3. If $k \geq 0$ and $\alpha, \beta: I^k \to X$ are maps such that $\alpha(p) = \beta(p)$ for all $p \in \partial(I^k)$, then

1. a homotopy from $\alpha$ to $\beta$ is a map $\gamma: I^{k+1} \to X$ such that
   \[
   \gamma(p, t) = \begin{cases}
   \alpha(p) & \text{if } (p, t) \in (I^k \times \{0\}) \cup (\partial I^k \times I) \\
   \beta(p) & \text{if } (p, t) \in I^k \times \{1\}
   \end{cases}
   \]

   and

2. $\alpha \simeq \beta$ means that $\alpha$ is homotopic to $\beta$ relative to the boundary of $I^k$. If $k = 1$, this is the usual notion of path homotopy.

Note that we do not require that $\alpha$ and $\beta$ take the boundary of $I^k$ to the basepoint. Thus, if $\alpha_1, \alpha_2: I^k \to X$ take the boundary of $I^k$ to the basepoint and $\beta_1, \beta_2: I^{k+1} \to X$ are both homotopies from $\alpha_1$ to $\alpha_2$, then $\beta_2 \simeq \beta_2$ means that the homotopies $\beta_1$ and $\beta_2$ are themselves homotopic through homotopies from $\alpha_1$ to $\alpha_2$.

Definition 2.4. If $k \geq 0$ and $\alpha: I^k \to X$ is a map taking the boundary of $I^k$ to the basepoint of $X$, then by a nullhomotopy of $\alpha$ we will mean a homotopy from the constant map to the basepoint to $\alpha$, i.e., a map $\beta: I^{k+1} \to X$ such that

\[\beta(p, t) = \begin{cases}
\ast & \text{if } (p, t) \in (I^k \times \{0\}) \cup (\partial I^k \times I) \\
\alpha(p) & \text{if } (p, t) \in I^k \times \{1\}
\end{cases} .\]

There exists such a nullhomotopy for $\alpha$ if and only if $[\alpha] = 0$ in $\pi_k X$.

Definition 2.5. If $k \geq 0$ and $\alpha: I^k \to X$ takes the boundary of $I^k$ to the basepoint, then the constant homotopy of $\alpha$ is the map $C_\alpha: I^{k+1} \to X$ defined by

\[C_\alpha(t_1, t_2, \ldots, t_{k+1}) = \begin{cases}
\alpha(t_1, t_2, \ldots, t_k) & \text{if } k > 0 \\
\alpha(\ast) & \text{if } k = 0 .
\end{cases}\]

Lemma 2.6. Let $k \geq 0$ and let $\alpha_1, \alpha_2: I^k \to X$ take the boundary of $I^k$ to the basepoint. If $\beta$ is a homotopy from $\alpha_1$ to $\alpha_2$, then

1. $\beta^{-1} \ast \beta \simeq C_{\alpha_2}$,

2. $C_{\alpha_1} \ast \beta \simeq \beta$, and

3. $\beta \ast C_{\alpha_2} \simeq \beta$.

If $\alpha_1, \alpha_2, \alpha_3, \alpha_4: I^k \to X$ take the boundary of $I^k$ to the basepoint, $\beta_1$ is a homotopy from $\alpha_1$ to $\alpha_2$, $\beta_2$ is a homotopy from $\alpha_2$ to $\alpha_3$, and $\beta_3$ is a homotopy from $\alpha_3$ to $\alpha_4$, then

4. $(\beta_1 \ast \beta_2) \ast \beta_3 \simeq \beta_1 \ast (\beta_2 \ast \beta_3)$ . .

Proof. The proofs are the same as the ones for compositions of paths in a space (and they reduce to those assertions when $k = 0$). We will prove part 1.

Define $\gamma: I^{k+2} \to X$ by

\[\gamma(t_1, t_2, \ldots, t_{k+2}) = \begin{cases}
\beta^{-1}(t_1, t_2, \ldots, t_k, 2t_{k+1}) & \text{if } t_{k+1} \leq \frac{1}{2}t_{k+2} \\
\beta^{-1}(t_1, t_2, \ldots, t_k, t_{k+2}) & \text{if } \frac{1}{2}t_{k+2} \leq t_{k+1} \leq 1 - \frac{1}{2}t_{k+2} \\
\beta(t_1, t_2, \ldots, t_k, 2t_{k+1} - 1) & \text{if } 1 - \frac{1}{2}t_{k+2} \leq t_{k+1} ;
\end{cases}\]
then

\[ \gamma(t_1, t_2, \ldots, t_{k+2}) = \begin{cases} 
C_{\alpha_2}(t_1, t_2, \ldots, t_{k+1}) & \text{if } t_{k+2} = 0 \\
(\beta^{-1} \ast \beta)(t_1, t_2, \ldots, t_{k+1}) & \text{if } t_{k+2} = 1 \\
\ast & \text{if any of } t_1, t_2, \ldots, t_k \text{ equal } 0 \text{ or } 1 
\end{cases} \]

i.e., \( \gamma \) is a homotopy relative to the boundary of \( I^{k+1} \) from \( C_{\alpha_2} \) to \( \beta^{-1} \ast \beta \). \( \square \)

**Lemma 2.7.** If \( k \geq 0 \), \( \alpha_1, \alpha_2 : I^k \to X \) take the boundary of \( I^k \) to the basepoint, and \( \beta_1 \) and \( \beta_2 \) are homotopies from \( \alpha_1 \) to \( \alpha_2 \), then \( \beta_1 \ast \beta_2^{-1} \simeq C_{\alpha_1} \) if and only if \( \beta_1 \simeq \beta_2 \).

**Proof.** If \( \beta_1 \ast \beta_2^{-1} \simeq C_{\alpha_1} \), then we have

\[
\beta_1 \simeq \beta_1 \ast C_{\alpha_2} \\
\simeq \beta_1 \ast (\beta_2^{-1} \ast \beta_2) \\
\simeq (\beta_1 \ast \beta_2^{-1}) \ast \beta_2 \\
\simeq C_{\alpha_1} \ast \beta_2 \\
\simeq \beta_2 .
\]

Conversely, if \( \beta_1 \simeq \beta_2 \), then \( \beta_1 \ast \beta_2^{-1} \simeq \beta_2 \ast \beta_2^{-1} \simeq C_{\alpha_1} \). \( \square \)

**Proposition 2.8.** Let \( k \geq 0 \), let \( \alpha : I^k \to X \) take the boundary of \( I^k \) to the basepoint, and let \( \beta_1 \) and \( \beta_2 \) be nullhomotopies of \( \alpha \), so that \( \beta_1 \ast \beta_2^{-1} : I^{k+1} \to X \) is defined and takes the boundary of \( I^{k+1} \) to the basepoint. If \([\beta_1 \ast \beta_2^{-1}] = 1\) in \( \pi_{k+1}X \), then \( \beta_1 \simeq \beta_2 \).

**Proof.** This follows from Lemma 2.7. \( \square \)

**Proposition 2.9.** Let \( k \geq 0 \) and let \( \alpha_1, \alpha_2 : I^k \to X \) take the boundary of \( I^k \) to the basepoint. If \( \beta_1 \) is a homotopy from \( \alpha_1 \) to itself and \( \beta_2 \) is a homotopy from \( \alpha_1 \) to \( \alpha_2 \), then \( \beta_1 \) is homotopic to the constant homotopy of \( \alpha_1 \) if and only if \( \beta_2^{-1} \ast \beta_1 \ast \beta_2 \) is homotopic to the constant homotopy of \( \alpha_2 \).

**Proof.** If \( \beta_1 \simeq C_{\alpha_1} \), then

\[
\beta_2^{-1} \ast \beta_1 \ast \beta_2 \simeq \beta_2^{-1} \ast C_{\alpha_1} \ast C_{\alpha_2} \simeq \beta_2^{-1} \ast C_{\alpha_2} .
\]

Conversely, if \( \beta_2^{-1} \ast \beta_1 \ast \beta_2 \simeq C_{\alpha_2} \), then

\[
\begin{align*}
C_{\alpha_1} & \simeq \beta_2 \ast \beta_2^{-1} \simeq \beta_2 \ast C_{\alpha_2} \ast \beta_2^{-1} \simeq \beta_2 \ast (\beta_2^{-1} \ast \beta_1 \ast \beta_2) \ast \beta_2^{-1} \\
& \simeq (\beta_2 \ast \beta_2^{-1}) \ast \beta_1 \ast (\beta_2 \ast \beta_2^{-1}) \simeq C_{\alpha_1} \ast \beta_1 \ast C_{\alpha_2} \simeq \beta_1 .
\end{align*}
\]

\( \square \)
3. Proof of Proposition 1.2

Notation 3.1. For each $i \geq 0$ we let

\[ P_i : \lim_n X_n \rightarrow X_i \quad \text{and} \quad Q_i : \lim_n Y_n \rightarrow Y_i \]

denote the natural projections from the limit, and we let $f : \lim_n X_n \rightarrow \lim_n Y_n$ be the limit of the $f_n$.

Proposition 3.2. For every choice of basepoint in $\lim_n X_n$ and every $k \geq 0$, the natural map $P : \pi_k \lim_n X_n \rightarrow \lim_n \pi_k X_n$ is surjective.

Proof. Every element of $\lim_n \pi_k X_n$ can be represented by $([\alpha_n])_{n \geq 0}$, where $\alpha_n : I^k \rightarrow X_n$ for $n \geq 0$ is a map taking the boundary of $I^k$ to the basepoint of $X_n$ and such that $p_{n+1} \circ \alpha_{n+1} \simeq \alpha_n$. We will inductively define maps $\bar{\alpha}_n : I^k \rightarrow X_n$ for $n \geq 0$ such that $\bar{\alpha}_n \simeq \alpha_n$ and $p_{n+1} \circ \bar{\alpha}_{n+1} = \bar{\alpha}_n$; the collection $(\bar{\alpha}_n)_{n \geq 0}$ will then define a map $\bar{\alpha} : I^k \rightarrow \lim_n X_n$ such that $P(\bar{\alpha}) = ([\alpha_n])_{n \geq 0}$.

We begin the induction by letting $\bar{\alpha}_0 = \alpha_0$. If $n \geq 0$ and we’ve defined $\bar{\alpha}_n$, then $\bar{\alpha}_{n+1} \circ \bar{\alpha}_{n+1} \simeq \alpha_n \simeq \bar{\alpha}_n$, and so there is a homotopy $H : I^{k+1} \rightarrow X_n$ such that

\[
H(p, t) = \begin{cases} 
(p_{n+1} \circ \alpha_{n+1})(p) & \text{if } t = 0 \\
\bar{\alpha}_n(p) & \text{if } t = 1 \\
* & \text{if } p \in \partial I^k.
\end{cases}
\]

We define $H' : (I^k \times \{0\}) \cup (\partial I^k \times I) \rightarrow X_{n+1}$ by $H'(p, t) = \alpha_{n+1}(p)$, and $H'$ is a lift of the restriction of $H$. Since there is a homeomorphism of $I^{k+1}$ to itself that takes $(I^k \times \{0\}) \cup (\partial I^k \times I)$ onto $I^k \times \{0\}$, we can extend $H'$ to a lift of $H$ on all of $I^{k+1}$, and we define $\bar{\alpha}_{n+1}(p) = H'(p, 1)$. \hfill \Box

Thus, for every choice of basepoint in $\lim_n X_n$ the rows of Diagram 1.3 are short exact sequences. Since each $f_n : X_n \rightarrow Y_n$ is a weak equivalence, the vertical map $\psi$ on the right is an isomorphism, and so it remains only to show that the map $\phi : K_X \rightarrow K_Y$ is an isomorphism. We show that $\phi$ is a monomorphism in Proposition 3.3 and that it is an epimorphism in Proposition 3.4.

Proposition 3.3. $\phi : K_X \rightarrow K_Y$ is injective.

Proof. Let $\alpha : I^k \rightarrow \lim_n X_n$ represent an element $[\alpha]$ of $K_X$ such that $\phi[\alpha] = [\bar{f} \circ \alpha] = 1$ in $K_Y \subset \pi_k \lim_n Y_n$. If $\bar{\alpha} = f \circ \alpha : I^k \rightarrow \lim_n Y_n$, then $[\bar{\alpha}] = 1$, and so we can choose a nullhomotopy $\bar{\beta} : I^{k+1} \rightarrow \lim_n Y_n$ of $\bar{\alpha}$. We let $\bar{\alpha}_n = Q_n \circ \bar{\alpha} : I^k \rightarrow Y_n$ for all $n \geq 0$ and $\bar{\beta}_n = Q_n \circ \bar{\beta} : I^{k+1} \rightarrow Y_n$ for all $n \geq 0$. For every $n \geq 0$ the map $\bar{\beta}_n$ is then a nullhomotopy of $\bar{\alpha}_n$, and $q_{n+1} \circ \bar{\beta}_{n+1} = \bar{\beta}_n$.

Since $[\alpha] \in K_X$, for every $n \geq 0$ the map $\alpha_n = P_n \circ \alpha : I^k \rightarrow X_n$ is nullhomotopic, and so we can choose a nullhomotopy $\beta_n : I^{k+1} \rightarrow X_n$ of $\alpha_n$. Since $\bar{\alpha}_n = Q_n \circ f \circ \alpha = f_n \circ P_n \circ \alpha$, the maps $f_n \circ \beta_n$ and $\bar{\beta}_n$ are both nullhomotopics of $\bar{\alpha}_n$, and so $\bar{\beta}_n * (f_n \circ \beta_n^{-1})$ is defined and takes the boundary of $I^{k+1}$ to the basepoint, and thus defines an element $[\bar{\beta}_n * (f_n \circ \beta_n^{-1})]$ of $\pi_{k+1} Y_n$; since $f_n : X_n \rightarrow Y_n$ is a weak equivalence, there exists a map $\gamma_n : I^{k+1} \rightarrow X_n$ that takes the boundary of $I^{k+1}$ to the basepoint and is such that $(f_n)_*[\gamma_n] = [\bar{\beta}_n * (f_n \circ \beta_n^{-1})]$. Let $\bar{\beta}_n = \gamma_n * \beta_n$; then
\(\beta'_n\) is a nullhomotopy of \(\alpha_n\), and

\[
[\bar{\beta}_n \ast (f_n \circ (\beta'_n)^{-1})] = [\bar{\beta}_n \ast (f_n \circ \beta_n^{-1}) \ast (f_n \circ \gamma_n^{-1})]
= [\beta_n \ast (f_n \circ \beta_n^{-1})] [f_n \circ \gamma_n^{-1}]
= [\beta_n \ast (f_n \circ \beta_n^{-1})] [\bar{\beta}_n \ast (f_n \circ \beta_n^{-1})]^{-1}
= 1 .
\]

Thus, \(f_n \circ \beta'_n \simeq \beta_n\) for all \(n \geq 0\) (see Proposition 2.8), and so

\[
f_n \circ p_{n+1} \circ \beta'_n = q_{n+1} \circ f_{n+1} \circ \beta_{n+1}'
\simeq q_{n+1} \circ \beta_{n+1}
= \bar{\beta}_n
\simeq f_n \circ \beta'_n.
\]

Thus, \([(f_n \circ \beta'_n) \ast (f_n \circ p_{n+1} \circ (\beta'_n)^{-1})] = (f_n) \ast [\beta'_n \ast (p_{n+1} \circ (\beta'_n)^{-1})] = 1\); since \(f_n\) is a weak equivalence, \([\beta'_n \ast (p_{n+1} \circ (\beta'_n)^{-1})] = 1\), and so \(\beta'_n \simeq p_{n+1} \circ \beta_{n+1}'\) (see Proposition 2.8).

We will now inductively define, for every \(n \geq 0\), a nullhomotopy \(\beta''_n\) of \(\alpha_n\) such that \(\beta''_n \simeq \beta'_n\) and, if \(n > 0\), \(p_n \circ \beta''_n = \beta''_{n-1}\). We begin the induction by letting \(\beta''_0 = \beta'_0\). If \(n \geq 0\) and we’ve defined \(\beta''_n\), then \(p_{n+1} \circ \beta''_{n+1} \simeq \beta''_n \simeq \beta'_n\), and so there is a homotopy \(H : I^{k+2} \to X_n\) from \(p_{n+1} \circ \beta''_{n+1}\) to \(\beta''_n\). If we define \(H' : (I^{k+1} \times \{0\}) \cup (\partial I^{k+1} \times I) \to X_{n+1}\) by \(H'(p, t) = \beta''_{n+1}(p)\), then \(H'\) is a lift of the restriction of \(H\) to \((I^{k+1} \times \{0\}) \cup (\partial I^{k+1} \times I)\). Since there is a homeomorphism of \(I^{k+2}\) to itself that takes \((I^{k+1} \times \{0\}) \cup (\partial I^{k+1} \times I)\) onto \(I^{k+1} \times \{0\}\), we can extend \(H'\) to a lift of \(H\) on all of \(I^{k+2}\), and we define \(\beta''_{n+1} : I^{k+1} \to X_{n+1}\) by \(\beta''_{n+1}(t_1, t_2, \ldots, t_{k+1}) = H'(t_1, t_2, \ldots, t_{k+1}, 0)\). That completes the induction, and the nullhomotopies \(\beta''_n\) combine to define a nullhomotopy \(\beta'' : I^{k+1} \to \lim_n X_n\) of \(\alpha\), and so \([\alpha] = 1\).

**Proposition 3.4.** \(\phi : K_X \to K_Y\) is surjective.

**Proof.** Let \(\bar{\alpha} : I^k \to \lim_n Y_n\) represent an element of \(K_Y \subset \pi_k \lim_n Y_n\). For each \(n \geq 0\) we let \(\bar{\alpha}_n = Q_n \circ \bar{\alpha} : I^k \to Y_n\), and we can choose a nullhomotopy \(\bar{\beta}_n : I^{k+1} \to Y_n\) of \(\bar{\alpha}_n\). Since \(q_{n+1} \circ \bar{\alpha}_n = \bar{\alpha}_{n+1}\), the map \(q_{n+1} \circ \bar{\beta}_n\) is a nullhomotopy of \(\bar{\alpha}_{n+1}\), and so the map \(\bar{\gamma}_n = \bar{\beta}_n \ast (q_{n+1} \circ \bar{\beta}_n^{-1}) : I^{k+1} \to Y_n\) is defined and takes the boundary of \(I^{k+1}\) to the basepoint of \(Y_n\) and thus defines an element \([\bar{\gamma}_n] = [\bar{\beta}_n \ast (q_{n+1} \circ \bar{\beta}_n^{-1})]\) of \(\pi_{k+1} Y_n\).

Since each map \(f_n : X_n \to Y_n\) is a weak equivalence, we can choose elements \([\gamma_n]\) of \(\pi_{k+1} X_n\) such that \(f_n([\gamma_n]) = [\bar{\gamma}_n]\). We will inductively define \(\alpha_n : I^k \to X_n\) taking the boundary of \(I^k\) to the basepoint and a nullhomotopy \(\beta_n : I^{k+1} \to X_n\) of \(\alpha_n\) for all \(n \geq 0\). We will arrange it so that

- \(p_{n+1} \circ \alpha_{n+1} = \alpha_n\) for all \(n \geq 0\), so that the \((\alpha_n)_{n \geq 0}\) will define a map \(\alpha : I^k \to \lim_n X_n\) taking the boundary of \(I^k\) to the basepoint and thus defining an element \([\alpha]\) of \(K_X\), and
- \([\beta_n \ast (p_{n+1} \circ \beta_{n+1}^{-1})] = [\gamma_n]\) in \(\pi_{k+1} X_n\) for all \(n \geq 0\).

We begin by letting \(\alpha_0 : I^k \to X_0\) and \(\beta_0 : I^{k+1} \to X_0\) both be constant maps to the basepoint.

For the inductive step, let \(n \geq 0\) and assume that we’ve defined \(\alpha_n\) and \(\beta_n\). The map \(\gamma_n^{-1} \ast \beta_n : I^{k+1} \to X_n\) is a nullhomotopy of \(\alpha_n\), and since there is a
homeomorphism of $I^{k+1}$ to itself that takes $(I^k \times \{0\}) \cup (\partial I^k \times I)$ onto $I^k \times \{0\}$, we can lift the homotopy $\gamma^{-1}_n \ast \beta_n$ in $X_n$ to a homotopy $\beta_{n+1}: I^{k+1} \to X_{n+1}$ such that $\beta_{n+1}(p, t)$ equals the basepoint for $(p, t) \in (I^k \times \{0\}) \cup (\partial I^k \times I)$. We define $\alpha_{n+1}(p) = \beta_{n+1}(p, 1)$, and we have $p_{n+1} \circ \alpha_{n+1} = \alpha_n$ and

$$
\beta_n \ast (p_{n+1} \circ \beta_{n+1}^{-1}) = [\beta_n \ast (\gamma^{-1}_n \ast \beta_n)^{-1}]
= [\beta_n \ast (\beta_n^{-1} \ast \gamma_n)]
= [\beta_n \ast \beta_n^{-1}][\gamma_n]
= [\gamma_n].
$$

Since $p_{n+1} \circ \alpha_{n+1} = \alpha_n$ for all $n \geq 0$, the maps $(\alpha_n)_{n \geq 0}$ define $\alpha: I^k \to \lim_n X_n$, which represents $[\alpha]$ in $K_X$. We will show that $f_*[\alpha] = [\bar{\alpha}]$.

For each $n \geq 0$, $(f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n$ is a homotopy from $f_n \circ \alpha_n$ to $\bar{\alpha}_n$. We will now show that $q_{n+1} \circ ((f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n)$ is homotopic to $(f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n$ for all $n \geq 0$. After that, we will inductively replace these homotopies with ones that commute with the $(q_n)_{n \geq 0}$ on the nose, and thus define a homotopy from $f \circ \alpha$ to $\bar{\alpha}$.

The map $(f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n$ is homotopic to $q_{n+1} \circ ((f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n)$ if and only if $(q_n \circ ((f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n)) \ast ((f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n) = 1$ is homotopic to the constant homotopy of $f_n \circ \alpha_n$ (see Lemma 2.7), and we have

$$
(q_{n+1} \circ ((f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n)) \ast ((f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n)
= ((q_n \circ ((f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n)) \ast ((f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n))
= ((f_n \circ p_{n+1} \circ \beta_n^{-1}) \ast (q_n \circ \beta_n)) \ast ((f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n)
= ((f_n \circ p_{n+1} \circ \beta_n^{-1}) \ast (q_n \circ \beta_n)) \ast ((f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n).
$$

This is homotopic to a constant homotopy if and only if the result of conjugating it by $(f_n \circ p_{n+1} \circ \beta_n^{-1})$ is homotopic to a constant homotopy (see Proposition 2.9), and that conjugate is homotopic to

$$
(f_n \circ p_{n+1} \circ \beta_n^{-1})^{-1}
\ast ((f_n \circ p_{n+1} \circ \beta_n^{-1}) \ast (q_n \circ \beta_n)) \ast ((f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n)
\simeq ((q_n \circ \beta_n) \ast (f_n \circ p_{n+1} \circ \beta_n^{-1})) \ast (f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n
\simeq (\gamma_n)^{-1} \ast (f_n \circ \gamma_n)
\simeq (\gamma_n)^{-1} \ast \gamma_n
\simeq C_*
$$

(see Lemma 2.6), where $C_*$ is the constant homotopy of the constant map to the basepoint. Thus, $(f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n$ is homotopic to $q_n \circ ((f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n)$.

We will now inductively define homotopies $\bar{\beta}'_n$ from $f_n \circ \alpha_n$ to $\bar{\alpha}_n$ such that

- $\bar{\beta}'_n \simeq (f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n$, and
- $q_n \circ \bar{\beta}'_n = \bar{\beta}'_n$.

We begin the induction by letting $\bar{\beta}'_0 = (f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n$. For the inductive step, if $n \geq 0$ and we’ve defined $\bar{\beta}'_n$, then $q_{n+1} \circ ((f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n) \simeq (f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n \simeq \bar{\beta}'_n$, and so we can choose a homotopy $H: I^{k+2} \to X_n$ from $q_{n+1} \circ ((f_n \circ \beta_n^{-1}) \ast \bar{\beta}_n)$ to
There is a homeomorphism of $I^{k+2}$ to itself that takes $(I^{k+1} \times \{0\}) \cup (\partial I^{k+1} \times I)$ to $I^{k+1} \times \{0\}$, and so we can lift $H$ to a homotopy $H': I^{k+2} \to X_{n+1}$ such that $H'(p, t) = ((f_{n+1} \circ \beta^{-1}_{n+1} )* \beta_{n+1})(p)$ for $(p, t) \in (I^{k+1} \times \{0\}) \cup (\partial I^{k+1} \times I)$, and we let $\beta'_n(p) = H'(p, 1)$.

We now have $q_{n+1} \circ \beta'_n = \beta'_n$ for $n \geq 0$, and so the $\beta'_n$ combine to define a homotopy $\beta'$ from $f \circ \alpha$ to $\bar{\alpha}$.

Department of Mathematics, Wellesley College, Wellesley, Massachusetts 02481
E-mail address: psh@math.mit.edu
URL: http://www-math.mit.edu/~psh