THE LOCAL-GLOBAL PRINCIPLE FOR INTEGRAL POINTS ON STACKY CURVES

MANJUL BHARGAVA AND BJORN POONEN

Abstract. We construct a stacky curve of genus $1/2$ (i.e., Euler characteristic 1) over $\mathbb{Z}$ that has an $\mathbb{R}$-point and a $\mathbb{Z}_p$-point for every prime $p$ but no $\mathbb{Z}$-point. This is best possible: we also prove that any stacky curve of genus less than $1/2$ over a ring of $S$-integers of a global field satisfies the local-global principle for integral points.

1. Introduction

Let $k$ be a global field, i.e., a finite extension of either $\mathbb{Q}$ or $\mathbb{F}_p(t)$. For each nontrivial place $v$ of $k$, let $k_v$ be the completion of $k$ at $v$. Let $X$ be a smooth projective geometrically integral curve of genus $g$ over $k$. If $X$ has a $k$-point, then of course $X$ has a $k_v$-point for every $v$. The converse holds if $g = 0$ (by the Hasse–Minkowski theorem), but there are well-known counterexamples of higher genus; in fact, counterexamples exist over every global field [Poo10]. This motivates the question: What is the smallest $g$ such that there exists a counterexample of genus $g$ over some global field? The answer is 1. Indeed, the first counterexample discovered was a genus 1 curve, the smooth projective model of $2y^2 = 1 - 17x^4$ over $\mathbb{Q}$ [Lin40,Rei42]. In fact, a positive proportion of genus 1 curves in the weighted projective space $\mathbb{P}(1,1,2)$ given by $z^2 = f(x,y)$, where $f(x,y)$ is an integral binary quartic form, violate the local-global principle over $\mathbb{Q}$ [Bha13].

Let us now generalize to allow $X$ to be a stacky curve over $k$. (See Sections 2 and 3 for our conventions.) Then the genus $g$ of $X$ — defined by the formula $\chi = 2 - 2g$, where $\chi$ is the topological Euler characteristic of $X$ — is no longer constrained to be a natural number; certain fractional values are also possible. Therefore we may now ask: What is the smallest $g$ such that there exists a stacky curve of genus $g$ over some global field $k$ violating the local-global principle? It turns out that if we formulate the local-global principle using rational points over $k$ and its completions, then the answer is not interesting, because rational points are almost the same as rational points on the coarse moduli space of $X$: see Section 4.

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Therefore we will answer our question in the context of a local-global principle for integral points on a stacky curve.

Our first theorem gives a proper stacky curve of genus 1/2 over \( \mathbb{Z} \) that violates the local-global principle.

**Theorem 1.** Let \( p, q, r \) be primes congruent to 7 (mod 8) such that \( p \) is a square (mod \( q \)) and (mod \( r \)), and \( q \) is a square (mod \( r \)). Let \( f(x, y) = ax^2 + bxy + cy^2 \) be a positive definite integral binary quadratic form of discriminant \(-pqr\) such that \( a \) is a nonzero square (mod \( q \)) but a nonsquare (mod \( p \)) and (mod \( r \)). Let \( \mathcal{Y} := \text{Proj} \mathbb{Z}[x, y, z]/(z^2 - f(x, y)) \). Define a \( \mu_2 \)-action on \( \mathcal{Y} \) by letting \( \lambda \in \mu_2 \) act as \((x : y : z) \mapsto (x : y : \lambda z)\). Let \( \mathcal{X} \) be the quotient stack \( [\mathcal{Y}/\mu_2] \). Then

(a) the genus of \( \mathcal{X} \) is 1/2 (i.e., \( \chi(\mathcal{X}) = 1 \));

(b) \( \mathcal{X}(\mathbb{Z}_\ell) \neq \emptyset \) for every rational prime \( \ell \) and \( \mathcal{X}(\mathbb{R}) \neq \emptyset \);

(c) \( \mathcal{X}(\mathbb{Z}) = \emptyset \), and even \( \mathcal{X}(\mathbb{Z}[1/(2pqr)]) = \emptyset \).

The same conclusions hold if instead we define \( \mathcal{X} \) as \([\mathcal{Y}/(\mathbb{Z}/2\mathbb{Z})]\), where \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \mathcal{Y} \) through the nontrivial homomorphism \( \mathbb{Z}/2\mathbb{Z} \to \mu_2 \); this \( \mathcal{X} \) is a Deligne–Mumford stack even over \( \mathbb{Z} \).

**Remark 2.** The hypotheses in Theorem 1 can be satisfied. For example, let \( p = 7 \), \( q = 47 \), \( r = 31 \), and \( f(x, y) = 3x^2 + xy + 850y^2 \).

**Remark 3.** The reason for considering \( \mathbb{Z}[1/(2pqr)] \) in [c] is that \( \mathcal{X} \) is smooth over that base.

**Remark 4.** Section 8 of [DG95] implicitly contains a similar counterexample, but of genus 2/3.

Let

\[
\mathcal{Y} := \text{Spec} \frac{\mathbb{Z}[x, y, z]}{(x^2 + 29y^2 - 3z^2)} - \{x = y = z = 0\},
\]

so \( \mathcal{Y}(\mathbb{Z}) \) consists of primitive integer solutions to \( x^2 + 29y^2 = 3z^2 \), those such that no prime divides all of \( x, y, z \). Let each \( \lambda \in \mathbb{G}_m \) act on \( \mathcal{Y} \) as \((x, y, z) \mapsto (\lambda^3 x, \lambda^3 y, \lambda^2 z)\). The quotient stack \( \mathcal{X} := [\mathcal{Y}/\mathbb{G}_m] \) is a proper stacky curve. Since every \( \mathbb{G}_m \)-torsor over \( \text{Spec} \mathbb{Z} \) is trivial, the map \( \mathcal{Y}(\mathbb{Z}) \to \mathcal{X}(\mathbb{Z}) \) is surjective, and likewise with \( \mathbb{Z} \) replaced by \( \mathbb{R} \) or \( \mathbb{Z}_p \) for any prime \( p \). Thus Section 8 of [DG95] says that \( \mathcal{X} \) is a counterexample to the local-global principle.

Our second theorem shows that any stacky curve of genus less than 1/2 over a ring of \( S \)-integers of a global field satisfies the local-global principle. Let \( k \) be a global field, and let \( k_v \) denote the completion of \( k \) at \( v \). Let \( S \) be a finite nonempty set of places of \( k \) containing all the archimedean places. Let \( \mathcal{O} \) be the ring of \( S \)-integers in \( k \); that is, \( \mathcal{O} := \{x \in k : v(x) \geq 0 \text{ for all } v \notin S\} \). For each \( v \notin S \), let \( \mathcal{O}_v \) be the completion of \( \mathcal{O} \) at \( v \). For each \( v \in S \), let \( \mathcal{O}_v = k_v \).

**Theorem 5.** Let \( \mathcal{X} \) be a stacky curve over \( \mathcal{O} \) of genus less than 1/2 (i.e., \( \chi(\mathcal{X}) > 1 \)). If \( \mathcal{X}(\mathcal{O}_v) \neq \emptyset \) for all places \( v \) of \( k \), then \( \mathcal{X}(\mathcal{O}) \neq \emptyset \).
2. Stacks

By a stack, we mean an algebraic (Artin) stack $\mathcal{X}$ over a scheme $S$ [SP, Tag 026O]. For any object $T \in (\text{Sch}/S)_{\text{fppf}}$, we write $\mathcal{X}(T)$ for the set of isomorphism classes of $S$-morphisms $T \to \mathcal{X}$, or equivalently (by the 2-Yoneda lemma [SP, Tag 04SS]), the set of isomorphism classes of the fiber category $\mathcal{X}_T$. If $T = \text{Spec} A$, we write $\mathcal{X}(A)$ for $\mathcal{X}(T)$.

3. Stacky curves

Let $k$ be an algebraically closed field. Let $X$ be a stacky curve over $k$, i.e., a smooth separated irreducible 1-dimensional Deligne–Mumford stack over $k$ containing a nonempty open substack isomorphic to a scheme. (This definition is slightly more general than [VZB19, Definition 5.2.1] in that we require only separatedness instead of properness, to allow punctures.)

By the Keel–Mori theorem [KM97] in the form given in [Con05] and [Ols16, Theorem 11.1.2], $X$ has a morphism to a coarse moduli space $X_{\text{coarse}}$ that is a smooth integral curve over $k$. By the Keel–Mori theorem in the form given in [Con05] and [Ols16, Theorem 11.3.1], each $P \in X_{\text{coarse}}(k)$ has an étale neighborhood $U$ above which $X \to X_{\text{coarse}}$ has the form $[V/G] \to U$ for some possibly ramified finite $G$-Galois cover $V \to U$ (by a scheme), where $G$ is the stabilizer of $X$ above $P$. The stacky curve $X$ is called tame above $P$ if $\text{char } k \nmid |G|$, and tame if it is tame above every $P$. Let $\mathcal{P} \subset X_{\text{coarse}}(k)$ be the (finite) set above which the stabilizer is nontrivial; then the morphism $X \to X_{\text{coarse}}$ is an isomorphism above $X_{\text{coarse}} - \mathcal{P}$.

Let $\tilde{g}_{\text{coarse}}$ be the genus of $\tilde{X}_{\text{coarse}}$; then the Euler characteristic $\chi(X_{\text{coarse}})$ is $(2 - 2\tilde{g}_{\text{coarse}}) - \#Z$. We now follow [Kob20] to define $\chi(X)$ and $g(X)$. For $P, U, V, G$ as above, let $G_i \leq G$ be the ramification subgroups for $V \to U$ above $P$, and define

$$\delta_P := \sum_{i \geq 0} \frac{|G_i| - 1}{|G|}$$

(which simplifies to only the first term $(|G| - 1)/|G|$ if $X$ is tame above $P$). Then define the Euler characteristic by

$$\chi(X) := \chi(X_{\text{coarse}}) - \sum_{P \in \mathcal{P}} \delta_P.$$

(This is motivated by the Riemann–Hurwitz formula. See [VZB19, Kob20] for other motivation.) Finally, define the genus $g = g(X)$ by $\chi(X) = 2 - 2g$.

**Lemma 6.** Let $X$ be a stacky curve over an algebraically closed field $k$ with $g < 1/2$. Then $X_{\text{coarse}} \simeq \mathbb{P}^1$ and $\#\mathcal{P} \leq 1$ and $X$ is tame.

**Proof.** Since $g < 1/2$, we have $\chi(X) > 1$. For each $P \in \mathcal{P}$, note that $\delta_P \geq (|G| - 1)/|G| \geq 1/2$. Now

$$\chi(X) = 2 - 2\tilde{g}_{\text{coarse}} - \#Z - \sum_{P \in \mathcal{P}} \delta_P,$$
which is \(\leq 1\) if \(\tilde{g}_{\text{coarse}} \geq 1\) or \(\#Z \geq 1\) or \(\#\mathcal{P} \geq 2\). Thus \(\tilde{g}_{\text{coarse}} = 0\), \(\#Z = 0\), and \(\#\mathcal{P} \leq 1\). Furthermore, if \(X\) is not tame, then there exists \(P \in \mathcal{P}\) with \(\delta_P \geq (|G| - 1)/|G| + 1/|G| \geq 1\), which again forces \(\chi(X) \leq 1\), a contradiction. \(\square\)

Now let \(k\) be any field. Let \(\overline{k}\) be an algebraic closure of \(k\), and let \(k_s\) be the separable closure of \(k\) in \(\overline{k}\). By a **stacky curve** over \(k\), we mean an algebraic stack \(X\) over \(k\) such that the base extension \(X_{\overline{k}}\) is a stacky curve over \(\overline{k}\). Define \(\chi(X) := \chi(X_{\overline{k}})\) and \(g(X) := g(X_{\overline{k}})\).

**Lemma 7.** If \(X\) is a tame stacky curve over \(k\), then the set \(\mathcal{P} \subset X_{\text{coarse}}(k)\) for \(X_{\overline{k}}\) consists of points whose residue fields are separable over \(k\).

**Proof.** Let \(\bar{P} \in \mathcal{P}\). Let \(P\) be the closed point of \(X_{\text{coarse}}\) associated to \(\bar{P}\). By working étale locally on \(X_{\text{coarse}}\), we may assume that \(X = [V/G]\) for a smooth curve \(V\) over \(k\) that is a \(G\)-Galois cover of \(X_{\text{coarse}}\) totally tamely ramified above \(P\). Analytically locally above \(P\), the tame cover is given by the equation \(y^n = \pi\) for some uniformizer \(\pi\) at \(P \in X_{\text{coarse}}\). After base change to \(\overline{k}\), however, \(\pi = u\bar{\pi}^i\), where \(u\) is a unit, \(\bar{\pi}\) is a uniformizer at \(\bar{P}\), and \(i\) is the inseparable degree of \(k(P)/k\). Thus \(V_{\overline{k}}\) is analytically locally given by \(y^n = u\overline{\pi}^i\). Since \(V_{\overline{k}}\) is smooth, \(i = 1\). Thus \(k(P)/k\) is separable. \(\square\)

Next, let \(\mathcal{O}\) be a ring of \(S\)-integers in a global field \(k\). By a **stacky curve** \(\mathcal{X}\) over \(\mathcal{O}\), we mean a separated finite-type algebraic stack over \(\text{Spec} \mathcal{O}\) such that \(\mathcal{X}_k\) is a stacky curve. (To be as general as possible, we do not impose Deligne–Mumford, tameness, smoothness, or properness conditions on the fibers above closed points of \(\text{Spec} \mathcal{O}\).) Define \(\chi(\mathcal{X}) := \chi(\mathcal{X}_{\overline{k}})\) and \(g(\mathcal{X}) := g(\mathcal{X}_{\overline{k}})\).

4. **Local-global principle for rational points**

We now explain why the local-global principle for rational points is not so interesting.

**Proposition 8.** Let \(k\) be a global field. Let \(X\) be a stacky curve over \(k\) with \(g < 1\). If \(X(k_v) \neq \emptyset\) for all nontrivial places \(v\) of \(k\), then \(X(k) \neq \emptyset\).

**Proof.** We have \(0 < \chi(X) \leq 2 - 2\tilde{g}_{\text{coarse}}\), so \(\tilde{g}_{\text{coarse}} = 0\). Thus \(X_{\text{coarse}}\) is a smooth geometrically integral curve of genus 0. Because of the morphism \(X \to X_{\text{coarse}}\), we have \(X_{\text{coarse}}(k_v) \neq \emptyset\) for every \(v\). By the Hasse–Minkowski theorem, \(X_{\text{coarse}}(k) \neq \emptyset\), so \(X_{\text{coarse}}\) is a dense open subscheme of \(\mathbb{P}^1_k\). In particular, \(X_{\text{coarse}}(k)\) is Zariski dense in \(X_{\text{coarse}}\), and all but finitely many of these \(k\)-points correspond to \(k\)-points on \(X\). \(\square\)

Because of Proposition 8, our main theorems are concerned with the local-global principle for integral points.
5. Proof of Theorem 1: Counterexample to the Local-Global Principle

(a) Since $(\mathcal{X}_Q)_{\text{coarse}}$ is dominated by the genus 0 curve $Y_Q$, we have $\tilde{g}_{\text{coarse}} = 0$. The action of $\mu_2$ on $\mathcal{Y}_Q$ fixes exactly two $\mathbb{F}$-points, namely those with $z = 0$; thus $P = 2$, and $\delta_P = 1/2$ for each $P \in \mathcal{P}$. Hence $\chi(\mathcal{X}) = (2 - 2 \cdot 0) - (1/2 + 1/2) = 1$. (Alternatively, $\chi(\mathcal{X}) = \chi(Y)/2 = 2/2 = 1$.)

(b) Let $R$ be a principal ideal domain. By definition of the quotient stack, a morphism $\text{Spec } R \to \mathcal{X}$ is given by a $\mu_2$-torsor $T$ equipped with a $\mu_2$-equivariant morphism $T \to \mathcal{Y}$. The torsors are classified by $H^1_{\text{fppf}}(R, \mu_2)$, which is isomorphic to $R^\times/R^\times^2$, since $H^1_{\text{fppf}}(R, \mathbb{G}_m) = \text{Pic } R = 0$. Explicitly, if $t \in R^\times$, the corresponding $\mu_2$-torsor is $T_t := \text{Spec } R[u]/(u^2 - t)$. Define the twisted cover

$$\mathcal{Y}_t := \text{Proj } R[x, y, z]/(tz^2 - f(x, y))$$

with its morphism $\pi_t : \mathcal{Y}_t \to \mathcal{X}$. To give a $\mu_2$-equivariant morphism $T_t \to \mathcal{Y}$ is the same as giving a morphism $\text{Spec } R \to \mathcal{Y}_t$. Thus we obtain

$$\mathcal{X}(R) = \coprod_{t \in R^\times} \pi_t(\mathcal{Y}_t(R)).$$

For any $\ell \notin \{p, q, r\}$, the rank 3 form $z^2 - f(x, y)$ has good reduction at $\ell$, so $\mathcal{Y}(\mathbb{F}_\ell) \neq \emptyset$, and Hensel’s lemma yields $\mathcal{Y}(\mathbb{Z}_\ell) \neq \emptyset$. Since the discriminant of $f(x, y)$ is divisible only by $p$ and not $p^2$, the form is not identically 0 modulo $p$, so there exist $\bar{a}, \bar{b} \in \mathbb{F}_p$ with $f(\bar{a}, \bar{b}) \in \mathbb{F}_p^\times$. Lift $\bar{a}, \bar{b}$ to $a, b \in \mathbb{Z}_p$, so $f(a, b) \in \mathbb{Z}_p^\times$. Then $\mathcal{Y}_j(a, b)(\mathbb{Z}_p) \neq \emptyset$. The same argument applies at $q$ and $r$. Since $f$ is positive definite, $\mathcal{Y}(\mathbb{R}) \neq \emptyset$. Thus $\mathcal{X}(\mathbb{Z}_\ell) \neq \emptyset$ for all primes $\ell$, and $\mathcal{X}(\mathbb{R}) \neq \emptyset$.

(c) We now show that $\mathcal{X}(\mathbb{Z}[1/(2pqr)]) = \emptyset$, i.e., that $\mathcal{Y}_t(\mathbb{Z}[1/(2pqr)]) = \emptyset$ for all $t \in \mathbb{Z}[1/(2pqr)]^\times$, or equivalently, that the quadratic form $f(x, y)$ does not represent any element of $\mathbb{Z}[1/(2pqr)]^\times$ times a square in $\mathbb{Z}[1/(2pqr)]$.

Completing the square shows that $f$ is equivalent over $\mathbb{Q}$ to the diagonal form $[a, apqr]$. If we use $u = u_v$ to denote a unit nonresidue in $\mathbb{Z}_v$, then

- over $\mathbb{Q}_p$, the form $f$ is equivalent to $[u, up]$ and represents the squareclasses $u, up$;
- over $\mathbb{Q}_q$, the form $f$ is equivalent to $[1, uq]$ and represents the squareclasses $1, uq$;
- over $\mathbb{Q}_r$, the form $f$ is equivalent to $[u, ur]$ and represents the squareclasses $u, ur$.

Therefore,

- $f$ takes square values in $\mathbb{R}$ and $\mathbb{Q}_q$, but not in $\mathbb{Q}_p$ and $\mathbb{Q}_r$;
- $-f$ takes square values in $\mathbb{Q}_p$ and $\mathbb{Q}_r$, but not in $\mathbb{R}$ and $\mathbb{Q}_q$.

It follows that $f$ and $-f$ together represent squares locally at all places, but do not globally represent squares.
We now further check that $sf$, for every factor $s$ of $pq$, fails to globally represent a square (by quadratic reciprocity, $r$ is not a square \( \text{mod } p \) and \( \text{mod } q \), and $q$ is not a square \( \text{mod } p \)):

- $pf$ takes square values in $\mathbb{R}$ and $\mathbb{Q}_q$, but not in $\mathbb{Q}_p$ and $\mathbb{Q}_r$.
- $qf$ takes square values in $\mathbb{R}$ and $\mathbb{Q}_p$, but not in $\mathbb{Q}_q$ and $\mathbb{Q}_r$.
- $rf$ takes square values in $\mathbb{R}$ and $\mathbb{Q}_p$, but not in $\mathbb{Q}_q$ and $\mathbb{Q}_r$.
- $pqf$ takes square values in $\mathbb{R}$ and $\mathbb{Q}_p$, but not in $\mathbb{Q}_q$ and $\mathbb{Q}_r$.
- $prf$ takes square values in $\mathbb{R}$ and $\mathbb{Q}_p$, but not in $\mathbb{Q}_q$ and $\mathbb{Q}_r$.
- $qrf$ takes square values in $\mathbb{R}$ and $\mathbb{Q}_q$, but not in $\mathbb{Q}_p$ and $\mathbb{Q}_r$.
- $pqr$ takes square values in $\mathbb{R}$ and $\mathbb{Q}_q$, but not in $\mathbb{Q}_p$ and $\mathbb{Q}_r$.

Since 2 is a square in $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{Q}_q$, and $\mathbb{Q}_r$, multiplying each of the $sf$’s in the above statements by 2 would not change the truth of any these statements. Meanwhile, since $-1$ and $-2$ are nonsquares in $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{Q}_q$, and $\mathbb{Q}_r$, multiplying the $sf$’s in the statements above by $-1$ or $-2$ would simply reverse all the conditions (in particular, all would fail to represent squares in $\mathbb{R}$).

We conclude that $\mathcal{Y}_t(\mathbb{Z}[1/(2pqr)]) = \emptyset$ for all $t \in \mathbb{Z}[1/(2pqr)]^\times$, i.e., $\mathcal{X}(\mathbb{Z}[1/(2pqr)]) = \emptyset$, as claimed.

The same arguments apply to $\mathcal{X}' := [\mathcal{Y}/(\mathbb{Z}/2\mathbb{Z})]$; in particular,

$$
\mathcal{X}'(\mathbb{Z}[1/(2pqr)]) = \mathcal{X}(\mathbb{Z}[1/(2pqr)]) = \emptyset,
$$

because the homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mu_2$ is an isomorphism over $\mathbb{Z}[1/2]$ and hence over $\mathbb{Z}[1/(2pqr)]$.

6. Stacks over local rings

This section contains some results to be used in the proof of Theorem 5.

**Proposition 9.** Let $A$ be a noetherian local ring. Let $X$ be an algebraic stack of finite type over $A$. Let $x \in X(A)$. Then there exists a finite-type algebraic space $U$ over $A$, a smooth surjective morphism $f : U \to X$, and an element $u \in U(A)$ such that $f(u) = x$.

**Proof.** By definition, there exists a finite-type $A$-scheme $V$ and a smooth surjective morphism $V \to X$. Taking the 2-fiber product with $\text{Spec } A \to X$ yields an algebraic space $V_x \to \text{Spec } A$. Then $V_x \to \text{Spec } A$ is smooth, so it admits étale local sections. Thus we can find a Galois étale extension $A'$ of $A$, say with group $G$, such that $x$ lifts to a morphism $\text{Spec } A' \to V$ equipped with a compatible system of isomorphisms between the conjugates of $v$.

Let $n = \#G$. Let $V_X^n$ be the 2-fiber product over $X$ of $n$ copies of $V$, indexed by $G$. The left translation action of $G$ on $G$ induces a right $G$-action on $V_X^n$ respecting the morphism $V_X^n \to X$, and there is also a right $G$-action on $\text{Spec } A'$. Therefore we may twist $V_X^n$ to obtain a new algebraic space $U$ lying over $X$ (a quotient of $V_X^n \times_A A'$ by a twisted action of $G$) such
that the element of \( V_\alpha^r(A') \) given by the conjugates of \( v \) and the isomorphisms between them descends to an element of \( U(A) \).

\[ \square \]

**Remark 10.** Atticus Christensen, combining a variant of our proof with other arguments, has extended Proposition 9 to other rings \( A \), such as arbitrary products of complete noetherian local rings, and \( \text{adèle} \) rings of global fields [Chr20, Theorem 7.0.7 and Propositions 12.0.5 and 12.0.8].

For any valued field \( K \), let \( \widehat{K} \) denote its completion.

**Proposition 11.** Let \( A \) be an excellent henselian discrete valuation ring. Let \( K = \text{Frac} \, A \). Let \( U \) be a separated finite-type algebraic space over \( K \).

(a) The set \( U(K) \) has a topology inherited from the topology on \( K \).
(b) If \( U \) is smooth and irreducible, then any nonempty open subset of \( U(K) \) is Zariski dense in \( U \).

**Proof.**

(a) In fact, much more is true: if \( K = \widehat{K} \), then the analytification of \( U \) exists as a rigid analytic space [CT09, Theorem 1.2.1]. If \( K \neq \widehat{K} \), equip \( U(K) \) with the subspace topology inherited from \( U(\widehat{K}) \).

(b) If \( K = \widehat{K} \), this follows from the fact that a nonzero power series in \( n \) variables over \( K \) cannot vanish on a nonempty open subset of \( K^n \). If \( K \neq \widehat{K} \), use Artin approximation: any point of \( U(\widehat{K}) \) can be approximated by a point of \( U(K) \). \[ \square \]

**Proposition 12.** Let \( A \) be an excellent henselian discrete valuation ring. Let \( K = \text{Frac} \, A \). Let \( U \) be a separated finite-type algebraic space over \( A \). Then \( U(A) \) is an open subset of \( U(K) \).

**Proof.** Since \( U \) is separated over \( A \), the map \( U(A) \to U(K) \) is injective. Let \( u \in U(A) \). Choose a separated \( A \)-scheme \( V \) with an étale surjective morphism \( f: V \to U \). Then \( u \) lifts to some \( v \in V(A') \) for some finite étale \( A \)-algebra \( A' \). Let \( K' = \text{Frac} \, A' \). Since \( V \) is a separated \( A \)-scheme, \( V(A') \) is an open subset of \( V(K') \). If \( A \) is complete, then the étale morphism \( V \to U \) induces an étale morphism of analytifications [CT09, Theorem 2.3.1], so \( V(K') \to U(K') \) is a local homeomorphism; in particular, it defines a homeomorphism from a neighborhood \( N_V \) of \( v \) in \( V(K') \) to a neighborhood \( N_U \) of \( u \) in \( U(K') \), and we may assume that \( N_V \subseteq V(A') \). In the general case, a given point of \( V(\widehat{K}') \) maps to some point of \( U(K') \) if and only if it is in \( V(K') \), so the homeomorphism for \( \widehat{K}' \)-points restricts to a homeomorphism for \( K' \)-points, which we again denote \( N_V \to N_U \). If \( u_1 \in N_U \cap U(K) \), then \( u_1 \) lies in the image of \( N_V \subseteq V(A') \), so \( u_1 \in U(A') \); now \( u_1 \in U(A') \cap U(K) \), which is \( U(A) \) since \( U \) is a sheaf on \( \text{Spec} \, A \)_{fppf}. Hence \( U(A) \) is open in \( U(K) \). \[ \square \]
7. Proof of Theorem 5

By Lemma 6, we have \((\mathcal{X}_T)_{\text{coarse}} \simeq \mathbb{P}^1_k\), and hence \((\mathcal{X}_k)_{\text{coarse}}\) is a smooth proper curve of genus 0. Since \(\mathcal{X}\) has an \(O_v\)-point for every \(v\), the stack \(\mathcal{X}_k\) has a \(k_v\)-point for every \(v\), so \((\mathcal{X}_k)_{\text{coarse}}\) has a \(k_v\)-point for every \(v\). Thus \((\mathcal{X}_k)_{\text{coarse}} \simeq \mathbb{P}^1_k\).

If \(\mathcal{X}_k \to (\mathcal{X}_k)_{\text{coarse}}\) is not an isomorphism, then by Lemma 6, there is a unique \(k\)-point above which it fails to be an isomorphism, and by Lemma 7, it is a \(k_v\)-point, and that point must be \(\text{Gal}(k_v/k)\)-stable, hence a \(k\)-point of \(\mathbb{P}^1\), which we may assume is \(\infty\). Thus \(\mathcal{X}_k\) contains an open substack isomorphic to \(\mathbb{A}^1_k\).

Since all the stacks are of finite presentation, the isomorphism just constructed extends above some affine open neighborhood of the generic point in \(\text{Spec} \, O\). That is, there exists a finite set of places \(S' \supseteq S\) such that if \(O'\) is the ring of \(S'\)-integers in \(k\), then the stack \(\mathcal{X}_{O'}\) contains an open substack isomorphic to \(\mathbb{A}^1_{O'}\).

Let \(v \in S' - S\). Let \(O_{(v)}\) be the localization of \(O\) at \(v\), and let \(O_{v, h}\) be its henselization in \(O_v\), so \(O_{v, h}\) is the set of elements of \(O_v\) that are algebraic over \(k\). Let \(k_{v, h} = \text{Frac} \, O_{v, h}\). We are given \(x \in \mathcal{X}(O_v)\). Let \(U, f, u\) be as in Proposition 9 with \(A = O_v\). By Proposition 12, \(U(O_v)\) is open in \(U(k_v)\). Let \(U_0\) be the connected component of \(U_{k_v}\) containing \(u\), so \(U_0(k_v)\) is open in \(U(k_v)\). The morphisms \(U_0 \to U_{k_v} \to \mathcal{X}_{k_v} \to \text{Spec} \, k_v\) are smooth, so \(U_0\) is smooth and irreducible. Therefore, by Proposition 11, \(U(O_v) \cap U_0(k_v)\) is Zariski dense in \(U_0\). On the other hand, \(U_0\) dominates \(\mathcal{X}_{k_v}\), since \(U_0 \to \mathcal{X}_{k_v}\) is smooth and \(\mathcal{X}_{k_v}\) is irreducible.

By the previous two sentences, there exists \(u_0 \in U(O_v) \cap U_0(k_v)\) mapping into the subset \(\mathbb{A}^1(k_v)\) of \(\mathcal{X}(k_v)\). By Artin approximation, we may replace \(u_0\) by a nearby point to assume also that \(u_0 \in U(O_{v, h})\).

Let \(U_1\) be the inverse image of \(\mathbb{A}^1\) under \(U_{k_v, h} \to \mathcal{X}_{k_v, h}\). By Proposition 12, \(U(O_{v, h})\) is open in \(U(k_{v, h})\), so \(U(O_{v, h}) \cap U_1(k_{v, h})\) is an open neighborhood of \(u_0\) in \(U_1(k_{v, h})\). Since \(U_1 \to \mathbb{A}^1\) is smooth, the image of this neighborhood is a nonempty open subset \(B_v\) of \(\mathbb{A}^1(k_{v, h})\). By construction, \(B_v\) is contained in the image of \(U(O_{v, h}) \to \mathcal{X}(O_{v, h}) \subseteq \mathcal{X}(k_{v, h})\), so \(B_v \subseteq \mathcal{X}(O_{v, h})\).

By strong approximation, there exists \(x \in \mathbb{A}^1(O')\) such that \(x \in B_v\) for all \(v \in S' - S\). For each \(v \in S' - S\), since \(B_v \subseteq \mathcal{X}(O_{v, h})\), there exists \(x_v \in \mathcal{X}(O_{v, h})\) such that \(x\) and \(x_v\) become equal in \(\mathcal{X}(k_{v, h})\). Finally, the following lemma shows that \(x\) comes from an element of \(\mathcal{X}(O)\).

**Lemma 13.** If \(x \in \mathcal{X}(O')\) and \(x_v \in \mathcal{X}(O_{v, h})\) for each \(v \in S' - S\) are such that the images of \(x\) and \(x_v\) in \(\mathcal{X}(k_{v, h})\) are equal for every \(v \in S' - S\), then there exists an element of \(\mathcal{X}(O)\) mapping to \(x\) in \(\mathcal{X}(O')\) and to \(x_v\) in \(\mathcal{X}(O_{v, h})\) for each \(v \in S' - S\).

**Proof.** Since \(\mathcal{X}\) is of finite presentation over \(O\), the element \(x_v\) comes from an element \(\tilde{x}_v\) of some finitely generated \(O\)-subalgebra \(A_v\) of \(O_{v, h}\). The schemes \(\text{Spec} \, A_v\) together with \(\text{Spec} \, O'\) form an fpf covering of \(\text{Spec} \, O\), so the stack property of \(\mathcal{X}\) shows that \(x\) and \(\tilde{x}_v\) come from an element of \(\mathcal{X}(O)\). \(\square\)

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Remark 14. Inspired by an earlier draft of our article, Christensen has found a natural way to define a topology on the set of adelic points of a finite-type algebraic stack, and has proved a strong approximation theorem for a stacky curve with $\chi > 1$ [Chr20, Theorem 13.0.6]. His argument can substitute for the three paragraphs before Lemma 13 and hence give a partially independent proof of Theorem 5.

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