# GONALITY OF DYNATOMIC CURVES AND STRONG UNIFORM BOUNDEDNESS OF PREPERIODIC POINTS

JOHN R. DOYLE AND BJORN POONEN

Abstract. Fix $d \geq 2$ and a field $k$ such that $\operatorname{char} k \nmid d$. Assume that $k$ contains the $d$th roots of 1. Then the irreducible components of the curves over $k$ parameterizing preperiodic points of polynomials of the form $z^d + c$ are geometrically irreducible and have gonality tending to $\infty$. This implies the function field analogue of the strong uniform boundedness conjecture for preperiodic points of $z^d + c$. It also has consequences over number fields: it implies strong uniform boundedness for preperiodic points of bounded eventual period, which in turn reduces the full conjecture for preperiodic points to the conjecture for periodic points.

## 1. Introduction

1.1. **Dynatomic curves.** Fix an integer $d \geq 2$. Let $k$ be a field such that $\operatorname{char} k \nmid d$. View $f = f_c := z^d + c$ as a polynomial in $z$ with coefficients in $k[c]$. Let $f^n(z)$ be the $n$th iterate of $f$; in particular, $f^0(z) := z$. If $n$ and $m$ are nonnegative integers with $n > m$, then any irreducible factor of $f^n(z) - f^m(z) \in k[z, c]$ defines an affine curve over $k$. By a dynatomic curve over $k$, we mean any such curve, or its smooth projective model. Any $k$-point on such a curve yields $c_0 \in k$ equipped with a preperiodic point in $k$, that is, an element $z_0 \in k$ that under iteration of $x^d + c_0$ eventually enters a cycle; the length of the cycle is called the eventual period. We consider two dynatomic curves to be different if the corresponding closed subschemes of $\mathbb{A}^2_k$ are distinct. Section 2 describes all dynatomic curves in characteristic 0 explicitly.

1.2. **Gonality.** Let $\overline{k}$ be an algebraic closure of $k$. Let $\mu_d = \{x \in \overline{k} : x^d = 1\}$.

For a curve $X$ over $k$, let $X_{\overline{k}} = X \times_k \overline{k}$. If $X$ is irreducible, define the gonality $\gamma(X)$ of $X$ as the least possible degree of a dominant rational map $X \dashrightarrow \mathbb{P}^1_k$. If $X$ is geometrically irreducible, define its $\overline{k}$-gonality as $\gamma(X_{\overline{k}})$.

**Theorem 1.1.** *Fix $d \geq 2$ and $k$ such that $\operatorname{char} k \nmid d$. Suppose that $\mu_d \subset k$.*

(a) *Every dynatomic curve over $k$ is geometrically irreducible.*
(b) *If the dynatomic curves over $k$ are listed in any order, their gonalities tend to $\infty$.*

*Remark* 1.2. Part (a) of Theorem 1.1 can fail if $\mu_d \not\subset k$. See Remark 2.2.

*Remark* 1.3. In proving Theorem 1.1 in positive characteristic, we face the challenge that we do not know explicitly what the dynatomic curves are, since we are not sure whether

*Date*: November 11, 2017.

2010 *Mathematics Subject Classification.* Primary 37P35; Secondary 11G30, 14H51, 37P05, 37P15.

*Key words and phrases.* Gonality, dynatomic curve, preperiodic point.

The second author was supported in part by National Science Foundation grants DMS-1069236 and DMS-1601946 and Simons Foundation grants #402472 (to Bjorn Poonen) and #550033.

the known factors of the polynomials $f^n(z) - f^m(z)$ are irreducible. Some partial results regarding the irreducibility of dynatomic curves in positive characteristic may be found in [DKO$^+$17], but we will not use them.

Let us now introduce notation for our second main result. Let $\mu(n)$ denote the Möbius $\mu$-function. Then $f^n(z) - z = \prod_{e|n} \Phi_e(z, c)$, where

$$\Phi_n(z, c) := \prod_{e|n} (f^e(z) - z)^{\mu(n/e)} \in k[z, c]. \tag{1}$$

Let $Y_1^{\mathrm{dyn}}(n)$ be the curve defined by $\Phi_n(z, c) = 0$ in $\mathbb{A}^2_k$, and let $X_1^{\mathrm{dyn}}(n)$ be the normalization of its projective closure. To simplify notation, we omit the superscript dyn from now on. General points of $X_1(n)$ parametrize polynomials of the form $z^d + c$ equipped with a point of exact order $n$.

The morphism $(z, c) \mapsto (f(z), c)$ restricts to an order $n$ automorphism of $Y_1(n)$, so it induces an order $n$ automorphism $\sigma$ of $X_1(n)$. The quotient of $X_1(n)$ by the cyclic group generated by $\sigma$ is called $X_0(n)$. If $\operatorname{char} k = 0$, it is known that $X_1(n)$ is geometrically irreducible, so $X_0(n)$ is too.

**Theorem 1.4.** *Fix $d \geq 2$ and a field $k$ of characteristic $0$. Then*

$$\gamma(X_0(n)) > \left(\frac{1}{2} - \frac{1}{2d} - o(1)\right) n$$

*as $n \to \infty$. In particular, $\gamma(X_0(n)) \to \infty$.*

*Remark* 1.5. Our definition of dynatomic curve does not include quotient curves such as $X_0(n)$, so the conclusion $\gamma(X_0(n)) \to \infty$ of Theorem 1.4 does not follow from Theorem 1.1(b).

To prove Theorem 1.4, we use that $X_0(n)$ already has a morphism to $\mathbb{P}^1$ of degree lower than expected for its genus, namely $X_0(n) \xrightarrow{c} \mathbb{P}^1$. If it also had a morphism to $\mathbb{P}^1$ of bounded degree, then the Castelnuovo–Severi inequality would make the genus of $X_0(n)$ smaller than it actually is, a contradiction. See Section 3 for details.

To prove Theorem 1.1(b), we use different arguments in characteristic 0 and characteristic $> 0$.

In characteristic 0, we use that each dynatomic curve dominates $X_1(n)$ and hence also $X_0(n)$ for some $n$, so its gonality is large when $n$ is large. On the other hand, for a fixed small $n$, the dynatomic curves above $X_1(n)$ come in towers and we use the Castelnuovo–Severi inequality to work our way up each tower. See Section 3.

In characteristic $p$, we prove that the irreducible components of $f^n(z) - f^m(z)$ have large degree over the $c$-line, and we use that to prove that over the finite field $\mathbb{F}_q := \mathbb{F}_p(\mu_d)$ their smooth projective models have so many $\mathbb{F}_q$-points over $c = \infty$ that their $\mathbb{F}_q$-gonalities must be large. Finally, we use a result controlling how gonality of a curve changes when the base field is enlarged. See Section 4.

1.3. **Uniform boundedness of preperiodic points.** The growth of gonalities of classical modular curves implies the strong uniform boundedness theorem for torsion points on elliptic curves over function fields (the function field analogue of Merel's theorem [Mer96]); see [NS96, Theorem 0.3]. Similarly, from Theorem 1.1 we will deduce the following function field analogue of a case of the Morton–Silverman conjecture [MS94, p. 100]:

**Theorem 1.6** (Strong uniform boundedness theorem for preperiodic points over function fields)**.** *Fix $d \geq 2$ and a field $k$ such that* $\operatorname{char} k \nmid d$*. Let $K$ be the function field of an integral curve over $k$. Fix a positive integer $D$. Then there exists $B = B(d, K, D) > 0$ such that for every field extension $L \supseteq K$ of degree $\leq D$ and every $c \in L$ not algebraic over $k$, the number of preperiodic points of $z^d + c$ in $L$ is at most $B$. If $k$ is finite, the same holds with the words "not algebraic over $k$" deleted.*

See [DKO$^+$17, Section 4] for some related results and arguments.

Another application of Theorem 1.1 is the following, which over *number fields* provides a uniform bound on preperiodic points having a bounded eventual period:

**Theorem 1.7.** *Fix integers $d \geq 2$, $D \geq 1$, and $N \geq 1$. Then there exists $B = B(d, D, N) > 0$ such that for every number field $K$ satisfying $[K : \mathbb{Q}] \leq D$ and every $c \in K$, the number of preperiodic points of $z^d + c$ in $K$ with eventual period at most $N$ is at most $B$.*

Theorems 1.6 and 1.7 are proved in Section 5. Theorem 1.7 implies that the strong uniform boundedness conjecture for *periodic* points over number fields implies the strong uniform boundedness conjecture for *preperiodic* points over number fields, as we now explain:

**Corollary 1.8.** *Fix integers $d \geq 2$ and $D \geq 1$. Suppose that there exists a bound $N = N(d, D)$ such that for every number field $K$ satisfying $[K : \mathbb{Q}] \leq D$ and every $c \in K$, every periodic point of $z^d + c$ in $K$ has period at most $N$. Then there exists a bound $B' = B'(d, D)$ such that for every number field $K$ satisfying $[K : \mathbb{Q}] \leq D$ and every $c \in K$, the number of preperiodic points of $z^d + c$ in $K$ is at most $B'$.*

*Proof.* By assumption, if $[K : \mathbb{Q}] \leq D$ and $c \in K$, then the preperiodic points of $z^d + c$ in $K$ having eventual period at most $N$ are *all* the preperiodic points in $K$. Therefore the bound $B(d, D, N)$ of Theorem 1.7 is actually a bound on the *total* number of preperiodic points in $K$. Take $B' = B(d, D, N) = B(d, D, N(d, D))$. $\square$

## 2. Classification of dynatomic curves

For $m, n \geq 1$, let $Y_1(m, n)$ be the curve over $k$ whose general points parametrize polynomials $z^d + c$ equipped with a preperiodic point that after exactly $m$ steps enters an $n$-cycle. This curve is the zero locus in $\mathbb{A}^2_k$ of the polynomial

$$\Phi_{m,n}(z, c) := \frac{\Phi_n(f^m(z), c)}{\Phi_n(f^{m-1}(z), c)}.$$

For a general point $(z, c) \in Y_1(1, n)$, the elements $z$ and $f^n(z)$ are distinct preimages of $f(z)$, so $z = \zeta f^n(z)$ for some $\zeta \in \mu_d - \{1\}$. Suppose that $\mu_d \subseteq k$. For each $\zeta \in \mu_d - \{1\}$, let $Y_1(1, n)^\zeta$ be the subscheme of $Y_1(1, n)$ defined by the condition $z = \zeta f^n(z)$, so

$$Y_1(1, n) = \bigcup_{\zeta \in \mu_d - \{1\}} Y_1(1, n)^\zeta.$$

Both $(z, c) \mapsto (f(z), c)$ and $(z, c) \mapsto (\zeta^{-1} z, c)$ define isomorphisms $Y_1(1, n)^\zeta \to Y_1(n)$. In particular, $Y_1(1, n)^\zeta$ equals the curve $\Phi_n(\zeta^{-1} z, c) = 0$ in $\mathbb{A}^2_k$. For $m \geq 2$, let $Y_1(m, n)^\zeta$ be the inverse image of $Y_1(1, n)^\zeta$ under

$$\begin{aligned} Y_1(m, n) &\longrightarrow Y_1(1, n) \\ (z, c) &\longmapsto (f^{m-1}(z), c). \end{aligned}$$

Then for any $m, n \geq 1$,

$$Y_1(m,n) = \bigcup_{\zeta \in \mu_d - \{1\}} Y_1(m,n)^\zeta, \tag{2}$$

and $Y_1(m,n)^\zeta$ equals the curve $\Phi_n(\zeta^{-1} f^{m-1}(z), c) = 0$ in $\mathbb{A}^2_k$. The decomposition (2) corresponds to a factorization

$$\Phi_{m,n}(z,c) = \prod_{\zeta \in \mu_d - \{1\}} \Phi_n(\zeta^{-1} f^{m-1}(z), c). \tag{3}$$

The following is a collection of results from [Bou92, LS94, Mor96, Gao16].

**Theorem 2.1.** *Let $k$ be a field of characteristic $0$ such that $\mu_d \subset k$. Then the curves $Y_1(n)$ for $n \geq 1$ and the curves $Y_1(m,n)^\zeta$ for $m, n \geq 1$ and $\zeta \in \mu_d - \{1\}$ are irreducible, so they are all the dynatomic curves over $k$.*

Computer experiments (at least for $d = 2$) suggest that Theorem 2.1 holds for any field $k$ such that $\operatorname{char} k \nmid d$ and $\mu_d \subset k$, but in positive characteristic this remains unproved. See [DKO$^+$17], especially Theorems B and D, for some progress in this direction.

*Remark* 2.2. If in Theorem 2.1 we drop the hypothesis that $\mu_d \subset k$, then the irreducible components of $Y_1(m,n)$ over $k$ are in bijection with the $\operatorname{Gal}(k(\mu_d)/k)$-orbits in $\mu_d - \{1\}$. An irreducible component corresponding to an orbit of size greater than 1 is not geometrically irreducible.

## 3. Gonality in characteristic 0

Given a geometrically irreducible curve $X$ over $k$, let $g(X)$ denote the genus of its smooth projective model. Let $D_0(n)$ be the degree of the morphism $X_0(n) \xrightarrow{c} \mathbb{P}^1$. Similarly, let $D_1(n) = \deg\left(X_1(n) \xrightarrow{c} \mathbb{P}^1\right)$.

**Proposition 3.1.** *Fix $d \geq 2$, and fix a field $k$ of characteristic $0$.*

(a) *We have $D_1(n) = (1 + o(1))d^n$ as $n \to \infty$.*
(b) *We have $D_0(n) = (1 + o(1))d^n/n$ as $n \to \infty$.*
(c) *We have $g(X_0(n)) > \left(\frac{1}{2} - \frac{1}{2d} - o(1)\right) d^n$ as $n \to \infty$.*
(d) *The Galois group of (the Galois closure of) the covering $X_0(n) \xrightarrow{c} \mathbb{P}^1$ is the full symmetric group $S_{D_0(n)}$.*

*Proof.* We may assume $k = \mathbb{C}$.

(a) First, $D_1(n) = \deg_z \Phi_n \leq \deg(f^n(z) - z) = d^n$. On the other hand, one can show (e.g., as in the proof of [Doy16, Lemma 2.2]) that $D_1(n) > d^n - d^{n/2+1}/(d-1)$.
(b) The first morphism in the tower $X_1(n) \to X_0(n) \xrightarrow{c} \mathbb{P}^1$ has degree $n$, so $D_0(n) = D_1(n)/n$. Substitute (a) into this.
(c) More precisely, $g(X_0(n)) > \left(\frac{1}{2} - \frac{1}{2d} - \frac{1}{n}\right) d^n + O(nd^{n/2})$ as $n \to \infty$, by [Mor96, Theorem 13(d)].
(d) This is a consequence of work of Bousch [Bou92] for $d = 2$, and Lau and Schleicher [LS94] for $d > 2$. See also [Mor98, Theorem B] and [Sch17]. $\square$

**Proposition 3.2** (Castelnuovo–Severi inequality)**.** *Let $F$, $F_1$, $F_2$ be function fields of curves over $k$, of genera $g$, $g_1$, $g_2$, respectively. Suppose that $F_i \subseteq F$ for $i = 1, 2$ and the compositum of $F_1$ and $F_2$ in $F$ equals $F$. Let $d_i = [F : F_i]$ for $i = 1, 2$. Then*

$$g \leq d_1 g_1 + d_2 g_2 + (d_1 - 1)(d_2 - 1).$$

*Proof.* See [Sti93, III.10.3]. □

*Proof of Theorem 1.4.* Let $X_0(n) \xrightarrow{h} \mathbb{P}^1$ be a dominant rational map of minimal degree.

*Case I: $h$ factors through $X_0(n) \xrightarrow{c} \mathbb{P}^1$.* Then

$$\deg h \geq D_0(n) = (1 + o(1)) \frac{d^n}{n}$$

by Proposition 3.1(b), so $\deg h$ is much larger than $n$ when $n$ is large.

*Case II: $h$ does not factor through $X_0(n) \xrightarrow{c} \mathbb{P}^1$.* Then the compositum of $k(c)$ and $k(h)$ in the function field $k(X_0(n))$ is strictly larger than $k(c)$. Because of the Galois group (Proposition 3.1(d)), the only nontrivial extension of $k(c)$ in $k(X_0(n))$ is the whole field $k(X_0(n))$. Thus $k(c)$ and $k(h)$ generate $k(X_0(n))$. By Proposition 3.2,

$$g(X_0(n)) \leq (D_0(n) - 1)(\deg h - 1).$$

Thus

$$\deg h \geq 1 + \frac{g(X_0(n))}{D_0(n) - 1} = \left( \frac{1}{2} - \frac{1}{2d} - o(1) \right) n$$

as $n \to \infty$, by Proposition 3.1(b,c). □

**Lemma 3.3.** *Let $k$ be a field of characteristic $0$ such that $\mu_d \subseteq k$. Let $m$ and $n$ be positive integers, and let $\zeta \in \mu_d - \{1\}$. Then the polynomial $\Phi_n(\zeta^{-1} f^{m-1}(0), c) \in k[c]$ has only simple roots, and their number is $d^{m-2} D_1(n)$ if $m \geq 2$.*

*Proof.* First suppose that $n$ does not divide $m - 1$. In this case, the roots of $\Phi_{m,n}(0, c)$ are distinct by [HT15, Theorem 1.1], and therefore the roots of $\Phi_n(\zeta^{-1} f^{m-1}(0), c)$ are distinct by (3).

Now suppose that $n$ divides $m - 1$. By [Eps12, Proposition A.1], the roots of $\Phi_n(0, c)$ are simple. If $c$ is such a root, then the polynomial $f = f_c$ satisfies $f^n(0) = 0$, so $f^{m-1}(0) = 0$ and $\Phi_n(\zeta^{-1} f^{m-1}(0), c) = 0$. Thus $\Phi_n(0, c)$ divides $\Phi_n(\zeta^{-1} f^{m-1}(0), c)$. The factorization (3) yields

$$\frac{\Phi_{m,n}(0, c)}{\Phi_n(0, c)^{d-1}} = \prod_{\zeta \in \mu_d - \{1\}} \frac{\Phi_n(\zeta^{-1} f^{m-1}(0), c)}{\Phi_n(0, c)}. \tag{4}$$

By [HT15, Theorem 1.1], $\Phi_{m,n}(0, c)/\Phi_n(0, c)^{d-1}$ has only simple roots, none of which are also roots of $\Phi_n(0, c)$. Combining this with (4) shows that $\Phi_n(\zeta^{-1} f^{m-1}(0), c)$ has only simple roots.

It remains to prove $\deg \Phi_n(\zeta^{-1} f^{m-1}(0), c) = d^{m-2} D_1(n)$. In fact, this is [Gao16, Lemma 4.8]. In our notation, the argument is as follows. By induction on $m$, the degree of the polynomial $f^{m-1}(0) \in k[c]$ is $d^{m-2}$ if $m \geq 2$. Hence, by induction on $e$, we have $\deg f^e(\zeta^{-1} f^{m-1}(0)) = d^e d^{m-2}$ for each $e \geq 0$. Thus the $c$-degree of $f^e(\zeta^{-1} f^{m-1}(0)) - \zeta^{-1} f^{m-1}(0)$ is $d^{m-2}$ times the $z$-degree of $f^e(z) - z$ for each $e \geq 1$. Substituting $\zeta^{-1} f^{m-1}(0)$ for $z$ in (1) shows that

$$\deg_c \Phi_n(\zeta^{-1} f^{m-1}(0), c) = d^{m-2} \deg_z \Phi_n(z, c) = d^{m-2} D_1(n).$$ □

*Proof of Theorem 1.1 in characteristic* 0.

(a) By Theorem 2.1, the dynatomic curves over $k$ are the curves $Y_1(n)$ and $Y_1(m,n)^\zeta$, and they are geometrically irreducible.

(b) The curves $Y_1(n)$ and $Y_1(m,n)^\zeta$ dominate $X_0(n)$, so their gonalities are at least the gonality of $X_0(n)$, by [Poo07, Proposition A.1(vii)]. In light of Theorem 1.4, it remains to prove that in each tower

$$\cdots \longrightarrow Y_1(m,n)^\zeta \longrightarrow Y_1(m-1,n)^\zeta \longrightarrow \cdots \longrightarrow Y_1(1,n)^\zeta$$

for fixed $n$ and $\zeta$, the gonality tends to $\infty$ as $m \to \infty$.

Let $g_m = g(Y_1(m,n)^\zeta)$ and $\gamma_m = \gamma(Y_1(m,n)^\zeta)$. Let $Y_1(m,n)^\zeta \xrightarrow{h} \mathbb{P}^1$ be a dominant rational map of degree $\gamma_m$.

*Case I: $h$ factors through a curve $Z$ with $k(Y_1(m-1,n)^\zeta) \subseteq k(Z) \subsetneq k(Y_1(m,n)^\zeta)$.* Then $\gamma_m \geq 2\gamma(Z) \geq 2\gamma_{m-1}$ by [Poo07, Proposition A.1(vii)], so $\gamma_m$ grows by induction on $m$.

*Case II: $h$ does not factor through any such curve $Z$.* Denote by $\pi_m$ the degree $d$ map

$$\begin{aligned} \pi_m\colon Y_1(m,n)^\zeta &\longrightarrow Y_1(m-1,n)^\zeta \\ (z,c) &\longmapsto (f(z),c), \end{aligned}$$

and let $\widetilde{\pi}_m\colon X_1(m,n)^\zeta \to X_1(m-1,n)^\zeta$ be its extension to the smooth projective models. Let $R_m$ be the ramification divisor of $\widetilde{\pi}_m$.

Applying Proposition 3.2 to $\pi_m$ and $h$ yields

$$g_m \leq d g_{m-1} + (d-1)(\gamma_m - 1),$$

so it suffices to show that $g_m - d g_{m-1} \to \infty$ as $m \to \infty$. By Riemann–Hurwitz, this is equivalent to showing that $\deg R_m \to \infty$ as $m \to \infty$. In the fiber product diagram

$$\begin{tikzcd} Y_1(m,n)^\zeta \arrow[r, "\pi_m"] \arrow[d, "z"'] & Y_1(m-1,n)^\zeta \arrow[d, "z"] \\ \mathbb{A}^1 \arrow[r, "f"] & \mathbb{A}^1, \end{tikzcd}$$

both vertical morphisms are étale above 0 by Lemma 3.3, while $f$ has ramification index $d$ at 0, so $\widetilde{\pi}^m$ has ramification index $d$ at each point of $Y_1(m,n)^\zeta$ where $z=0$. For $m \geq 2$, the number of such points is $d^{m-2}D_1(n)$ by Lemma 3.3. Thus $\deg R_m \geq (d-1)d^{m-2}D_1(n)$, which tends to $\infty$ as $m \to \infty$. $\square$

## 4. Gonality in characteristic $p$

### 4.1. **Reduction to the case of a finite field.** Let $\mathbb{F}_q = \mathbb{F}_p(\mu_d)$.

**Lemma 4.1.** *Theorem 1.1 for $\mathbb{F}_q$ implies Theorem 1.1 for any characteristic $p$ field $k$ containing $\mu_d$.*

*Proof.* Theorem 1.1(a) for $\mathbb{F}_q$ implies that the dynatomic curves over $k$ are just the base extensions of the dynatomic curves $X$ over $\mathbb{F}_q$, and that they are geometrically irreducible too. We will prove in Section 4.4 that the smooth projective model of each $X$ has an $\mathbb{F}_q$-point. Then Theorem 2.5(iii) and Proposition A.1(ii) of [Poo07] imply $\gamma(X_k) \geq \sqrt{\gamma(X)}$, which implies Theorem 1.1(b) for $k$. $\square$

4.2. **Symbolic dynamics.** View $f(z)$ as a polynomial in $z$ over the local field $\mathbb{F}_q((c^{-1}))$. Normalize the valuation $v$ on $\mathbb{F}_q((c^{-1}))$ so that $v(c^{-1}) = 1$, and extend $v$ to an algebraic closure. Let $t = c^{-1/d}$, so $\mathbb{F}_q((t))$ is a degree $d$ totally tamely ramified extension of $\mathbb{F}_q((c^{-1}))$.

**Lemma 4.2.**

(a) *For any nonnegative integers $n > m$, the polynomial $f^n(z) - f^m(z)$ over $\mathbb{F}_q(c)$ is separable and splits completely over $\mathbb{F}_q((t))$.*

(b) *Each zero of $f^n(z) - f^m(z)$ has valuation $-1/d$ and generates $\mathbb{F}_q((t))$ over $\mathbb{F}_q((c^{-1}))$.*

*Proof.* The ideas in the following argument are well known; cf. [Mor96, Lemma 1].

(a) For each $d$th root of $-c$, interpreting $(-c+z)^{1/d}$ as $(-c)^{1/d}(1 - c^{-1}z)^{1/d}$ and expanding $(1 - c^{-1}z)^{1/d}$ in a binomial series defines a branch of the inverse of $z^d + c$ on the open disk $D := \{z \in \mathbb{F}_q((t)) : v(z) > v(c)\}$. Taking the derivative of $(-c+z)^{1/d}$ shows that each branch is a contracting map $D \to D$. These branches have disjoint images, each a smaller open disk around a different $d$th root of $-c$. Let $S$ be the set of these $d$ functions. Each finite sequence of elements of $S$ defines a composition of functions, and for each $m$, the images of the different $m$-fold compositions are disjoint open disks. For each infinite sequence $s_1, s_2, \cdots$ of elements of $S$, the images of $s_1 \cdots s_m$ for $m \geq 1$ are nested open disks whose radii tend to 0, so they have a unique point in their intersection; denote it $[s_1 s_2 \cdots]$. Any two distinct infinite sequences yield two points in disjoint disks, so these points are distinct. Since $f \circ s_1$ is the identity, $f$ maps $[s_1 s_2 \cdots]$ to $[s_2 s_3 \cdots]$. For fixed nonnegative integers $n > m$, any $n$-long sequence $s_1, \ldots, s_n$ in $S$ extends uniquely to an infinite sequence $(s_i)$ satisfying $s_{i+n} = s_{i+m}$ for all $i \geq 1$, and then $[s_1 s_2 \cdots]$ is a zero of $f^n(z) - f^m(z)$. There are $d^n$ of these, so they are *all* the zeros. In particular, these zeros are distinct elements of $\mathbb{F}_q((t))$. This implies that $f^n(z) - f^m(z)$ is separable.

(b) The image of each $s \in S$ consists of elements of valuation exactly $-1/d$. Thus each element $[s_1 s_2 \cdots]$ has valuation $-1/d$. In particular, each zero of $f^n(z) - f^m(z)$ generates an extension field of $\mathbb{F}_q((c^{-1}))$ of ramification index divisible by $d$; this extension field can only be the whole field $\mathbb{F}_q((t))$. □

4.3. **Dynatomic curves of low degree.**

**Lemma 4.3.** *For each $e \geq 1$, the set of dynatomic curves $X$ over $\mathbb{F}_q$ such that* $\deg(X \xrightarrow{c} \mathbb{P}^1) =$ *$e$ is finite.*

*Proof.* Suppose that $Q(z) = \sum_{r=0}^{e} q_r z^{e-r}$ is a monic degree $e$ factor of $f^n(z) - f^m(z)$ over $\mathbb{F}_q(c)$ for some $n$ and $m$. For each $r$, the coefficient $q_r$ is the $r$th elementary symmetric polynomial evaluated at the negatives of the zeros of $Q$; those zeros have valuation $-1/d$ by Lemma 4.2(b), so $v(q_r) \geq -r/d$. On the other hand, by Gauss's lemma, $q_r \in \mathbb{F}_q[c]$, so $\deg q_r \leq r/d$. Thus there are only finitely many possibilities for each $q_r$, and hence finitely many possibilities for $Q$, each of which yields one dynatomic curve. □

4.4. **Gonality of dynatomic curves.**

*Proof of Theorem 1.1.* By Lemma 4.1, we may assume that $k = \mathbb{F}_q$.

(a) Let $X$ be the smooth projective model of a dynatomic curve, corresponding to a factor of $f^n(z) - f^m(z)$ for some $n$ and $m$. By Lemma 4.2(b), $f^n(z) - f^m(z)$ splits completely over $\mathbb{F}_q((t))$, and the preimage of $\infty$ under $X \xrightarrow{c} \mathbb{P}^1$ consists of $\mathbb{F}_q$-points, each of ramification index $d$. Every irreducible component $Z$ of $X_{\overline{\mathbb{F}}_q}$ dominates $\mathbb{P}^1$ and hence must contain one of those $\mathbb{F}_q$-points, say $x$. Then each $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-conjugate of $Z$ contains $x$. On the other hand, since $X_{\overline{\mathbb{F}}_q}$ is smooth, its irreducible components are disjoint. Thus $Z$ equals each of its conjugates, so $Z$ descends to an irreducible component of $X$, which must be $X$ itself. (This proves also that $X$ has an $\mathbb{F}_q$-point, as promised in the proof of Lemma 4.1.)

(b) To bound gonality from below, we use Ogg's method of counting points over a finite field; cf. [Ogg74], [Poo07, Section 3], and [DKO+17, Section 4]. Let $X$ be the smooth projective model of a dynatomic curve. Let $e = \deg(X \xrightarrow{c} \mathbb{P}^1)$. Each preimage of $\infty$ in $X$ is an $\mathbb{F}_q$-point of ramification index $d$, so there are $e/d$ such points. On the other hand, $\mathbb{P}^1$ has only $q+1$ points over $\mathbb{F}_q$, so any nonconstant morphism $X \to \mathbb{P}^1$ has degree at least $e/(d(q+1))$. As $X$ varies, $e \to \infty$ by Lemma 4.3. $\square$

## 5. Strong uniform boundedness of preperiodic points

*Proof of Theorem 1.6.* Without loss of generality, $K = k(u)$ for some indeterminate $u$. Given $L$, let $Y$ be the smooth projective integral curve over $k$ with function field $L$. The condition $[L : K] \le d$ implies that $Y$ has gonality at most $D$, so each irreducible component of $Y_{\overline{k}}$ has gonality at most $D$. For $c \in L$ not algebraic over $k$, if $z \in L$ and $n > m$ satisfy $f^n(z) - f^m(z) = 0$, then $(z, c)$ is a nonconstant and hence smooth $L$-point of the curve $f^n(z) - f^m(z) = 0$ in $\mathbb{A}^2$, so it yields an $L$-point on a dynatomic curve $X$ corresponding to a factor of $f^n(z) - f^m(z)$. This $L$-point defines a nonconstant $k$-morphism $Y \to X$, so $\gamma(X) \le \gamma(Y) \le D$. By Theorem 1.1(b), this places a uniform bound on $n$. For each $n$, the number of preperiodic points of $z^d + c$ corresponding to that value of $n$ is uniformly bounded by $\sum_{m=0}^{n-1} \deg(f^n(z) - f^m(z)) = nd^n$, so bounding $n$ bounds the number of preperiodic points too.

If $k$ is finite and $c$ lies in the maximal algebraic extension $\ell$ of $k$ in $L$, then all the preperiodic points of $x^d + c$ in $L$ are in $\ell$, but $[\ell : k] \le D$, so the number of preperiodic points is uniformly bounded by $(\#k)^D$. $\square$

To prove Theorem 1.7, we need the following result of Frey [Fre94, Proposition 2].

**Lemma 5.1.** *Let $C$ be a curve defined over a number field $K$. Let $D \ge 1$. If there are infinitely many points $P \in C(\overline{K})$ of degree $\le D$ over $K$, then $\gamma(C) \le 2D$.*

*Proof of Theorem 1.7.* If the conclusion fails, then there exists $n \le N$ such that there are infinitely many $m \ge 1$ such that $Y_1(m, n)$ has a point of degree $\le D$ over $\mathbb{Q}$. On the other hand, by Northcott's theorem, for any given $c \in \overline{\mathbb{Q}}$, the set of preperiodic points of $x^d + c$ is finite. Thus for any $m_0$, the points on $Y_1(m, n)$ in the first sentence above for all $m \ge m_0$ together must include infinitely many distinct $c$-coordinates. In particular, their images in $Y_1(m_0, n)$ include infinitely many distinct points. Thus some $\mathbb{Q}(\mu_d)$-irreducible component $Y_1(m_0, n)^\zeta$ contains infinitely many points of degree $\le D$ over $\mathbb{Q}(\mu_d)$. By Lemma 5.1, $\gamma(Y_1(m_0, n)^\zeta) \le 2D$. This argument applies for every $m_0$, contradicting Theorem 1.1(b). $\square$

## Acknowledgements

We thank Joseph Gunther, Holly Krieger, Andrew Obus, Padmavathi Srinivasan, and Isabel Vogt for comments and discussions.

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Department of Mathematics & Statistics, Louisiana Tech University, Ruston, LA 71272, USA

*E-mail address*: `jdoyle@latech.edu`

*URL*: `http://www2.latech.edu/~jdoyle/`

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA

*E-mail address*: `poonen@math.mit.edu`

*URL*: `http://math.mit.edu/~poonen/`