THE ANALYTIC CLASS NUMBER FORMULA FOR ORDERS IN PRODUCTS OF NUMBER FIELDS

BRUCE W. JORDAN AND BJORN POONEN

Abstract. We derive an analytic class number formula for an arbitrary order in a product of number fields.

1. Introduction

Let $\mathcal{O}$ be an order in a product of m number fields for some nonnegative integer m. The 1-dimensional scheme $\operatorname{Spec}\mathcal{O}$ has m irreducible components; in particular, it is irreducible if and only if $m = 1$. The scheme $\operatorname{Spec}\mathcal{O}$ is regular if and only if $\mathcal{O}$ is the product of the full rings of integers of the m number fields. If $\operatorname{Spec}\mathcal{O}$ fails to be regular at some point, we call that point a singularity and say that $\operatorname{Spec}\mathcal{O}$ is singular. Let $\zeta_{\mathcal{O}}(s)$ be the zeta function of $\mathcal{O}$ as defined by Serre [Ser65, p. 83] (see Section 3).

In the case where $\mathcal{O}$ is the ring of integers of a number field, that is, the case in which $\operatorname{Spec}\mathcal{O}$ is regular and irreducible, Dedekind [Dir94, Supplement XI, §184, IV], generalizing work of Dirichlet, proved an analytic class number formula for the leading term of the Laurent series of $\zeta_{\mathcal{O}}(s)$ at $s = 1$ (see also Hilbert's *Zahlbericht* [Hil97, Theorem 56]). The generalization to the regular and reducible case is immediate. In this paper we generalize further by proving an analytic class number formula for an arbitrary order in a product of number fields, thereby extending Dedekind's result to orders $\mathcal{O}$ for which $\operatorname{Spec}\mathcal{O}$ is singular. We conclude by verifying our formula in an example with a singularity: the fiber product of rings $\mathbb{Z} \times_{\mathbb{F}_p} \mathbb{Z}$.

Remark 1. We expect a similar formula in the function field case, for a singular curve over a finite field, but we do not investigate this here. Various authors have defined other zeta functions attached to such a curve and have computed their leading terms at $s = 0$ or $s = 1$ [Gal73, Gre89, ZG97a, ZG97b, Stö98], but these zeta functions are different from the zeta function in [Ser65] in general.

2. Orders in products of number fields

If F is a number field with ring of integers $\mathcal{O}_F$, classical algebraic number theory defines the following invariants: the number of real embeddings $r_1(F)$, the number of pairs of complex embeddings $r_2(F)$, the discriminant $\operatorname{Disc}\mathcal{O}_F$, the regulator $R(\mathcal{O}_F)$, the class number $h(\mathcal{O}_F)$ defined as the order of the Picard group $\operatorname{Pic}\mathcal{O}_F$, and the number $w(\mathcal{O}_F)$ of roots of unity in $\mathcal{O}_F$. All of these invariants occur in the analytic class number formula. In this section

Date: August 21, 2017.

2010 *Mathematics Subject Classification.* Primary 11R54; Secondary 11R29.

The second author was supported in part by National Science Foundation grant DMS-1069236 and DMS-1601946 and grants from the Simons Foundation (#340694 and #402472 to Bjorn Poonen).

we extend each of these definitions to the case of an arbitrary order in a product of number fields.

2.1. **Orders.** Let K be a finite étale $\mathbb{Q}$-algebra; in other words, $K = K_1 \times \cdots \times K_m$ for some number fields K_i. Let $\mathcal{O} \subseteq K$ be an order, i.e., a subring of K finitely generated as a $\mathbb{Z}$-module such that $\mathbb{Q}\mathcal{O} = K$.

Equivalently, let $\mathcal{O}$ be a reduced ring that is free of finite rank as a $\mathbb{Z}$-module, let $K := \mathcal{O} \otimes \mathbb{Q}$, and let $m := \# \operatorname{Spec} K$.

2.2. **The invariants n, r_1, r_2, and r of K.** Let $n := [K : \mathbb{Q}]$, so $\mathcal{O}$ has rank n over $\mathbb{Z}$. Define $r_1, r_2 \in \mathbb{Z}_{\geq 0}$ by $K \otimes \mathbb{R} \simeq \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Thus r_1 is the number of ring homomorphisms $K \to \mathbb{R}$, and $2r_2$ is the number of ring homomorphisms $K \to \mathbb{C}$ whose image is not contained in $\mathbb{R}$. Let $r = r(K) := r_1 + r_2 - m$.

2.3. **Roots of unity, the Picard group $\operatorname{Pic} \mathcal{O}$, the class number $h(\mathcal{O})$, and the discriminant $\operatorname{Disc} \mathcal{O}$.** Let $\mu(\mathcal{O})$ be the torsion subgroup of $\mathcal{O}^\times$, so $\mu(\mathcal{O})$ is the group of roots of unity in $\mathcal{O}^\times$. Let $w(\mathcal{O}) := \#\mu(\mathcal{O})$.

Let $X := \operatorname{Spec} \mathcal{O}$. Then $\operatorname{Pic} \mathcal{O} := \operatorname{Pic} X = H^1(X, \mathscr{O}_X^*)$ [Har77, Exercise III.4.5]. Let $h(\mathcal{O}) := \# \operatorname{Pic} \mathcal{O}$.

Let $\operatorname{Tr}_{K/\mathbb{Q}} \colon K \to \mathbb{Q}$ be the trace map. Let $e_1, \ldots, e_n$ be a $\mathbb{Z}$-basis of $\mathcal{O}$. As usual, the discriminant is defined by

$$\operatorname{Disc} \mathcal{O} := \det \left(\operatorname{Tr}_{K/\mathbb{Q}}(e_i e_j) \right)_{1 \leq i,j \leq n} \in \mathbb{Z}.$$

2.4. **The normalization.** Let $\widetilde{\mathcal{O}}_i$ be the ring of integers in K_i. Let $\widetilde{\mathcal{O}}$ be the normalization of $\mathcal{O}$. Since $\mathcal{O}$ is a finite $\mathbb{Z}$-module, the normalization of $\mathcal{O}$ equals the normalization of $\mathbb{Z}$ in K; thus $\widetilde{\mathcal{O}} = \prod_{i=1}^m \widetilde{\mathcal{O}}_i$ in K. Also, $\widetilde{\mathcal{O}}$ is finite as an $\mathcal{O}$-module, so $\widetilde{\mathcal{O}}$ is another order in K. Thus $\#(\widetilde{\mathcal{O}}/\mathcal{O}) < \infty$. The invariants n, r_1, r_2, and r depend only on K, so they are the same for $\mathcal{O}$ as for $\widetilde{\mathcal{O}}$.

Let $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}$. Localizing the $\mathcal{O}$-module $\widetilde{\mathcal{O}}$ at $\mathfrak{p}$ yields a semilocal ring $\widetilde{\mathcal{O}}_\mathfrak{p}$. The quotient $\widetilde{\mathcal{O}}_\mathfrak{p}/\mathcal{O}_\mathfrak{p} \simeq (\widetilde{\mathcal{O}}/\mathcal{O})_\mathfrak{p}$ is finite, and is trivial for all $\mathfrak{p}$ except the finitely many corresponding to singularities of $\operatorname{Spec} \mathcal{O}$. Each maximal ideal $\mathfrak{P}$ of $\widetilde{\mathcal{O}}_\mathfrak{p}$ lies above the maximal ideal $\mathfrak{p}\mathcal{O}_\mathfrak{p}$ of $\mathcal{O}_\mathfrak{p}$. Therefore, given $a \in \mathcal{O}_\mathfrak{p}$, saying that a lies outside every $\mathfrak{P}$ is the same as saying that a lies outside $\mathfrak{p}\mathcal{O}_\mathfrak{p}$; since $\widetilde{\mathcal{O}}_\mathfrak{p}$ is semilocal and $\mathcal{O}_\mathfrak{p}$ is local, this means that $a \in \widetilde{\mathcal{O}}_\mathfrak{p}^\times$ if and only if $a \in \mathcal{O}_\mathfrak{p}^\times$. In other words, the map of sets $\widetilde{\mathcal{O}}_\mathfrak{p}^\times/\mathcal{O}_\mathfrak{p}^\times \to \widetilde{\mathcal{O}}_\mathfrak{p}/\mathcal{O}_\mathfrak{p}$ is injective. Hence $\widetilde{\mathcal{O}}_\mathfrak{p}^\times/\mathcal{O}_\mathfrak{p}^\times$ is finite too, and trivial for all but finitely many $\mathfrak{p}$. Injectivity of

$$\widetilde{\mathcal{O}}/\mathcal{O} \longrightarrow \bigoplus_{\mathfrak{p}} \widetilde{\mathcal{O}}_\mathfrak{p}/\mathcal{O}_\mathfrak{p}$$

implies that if $a, a^{-1} \in \widetilde{\mathcal{O}}$ are such that their images in $\widetilde{\mathcal{O}}_\mathfrak{p}$ land in $\mathcal{O}_\mathfrak{p}$ for every $\mathfrak{p}$, then $a, a^{-1} \in \mathcal{O}$. Thus

$$\widetilde{\mathcal{O}}^\times/\mathcal{O}^\times \longrightarrow \bigoplus_{\mathfrak{p}} \widetilde{\mathcal{O}}_\mathfrak{p}^\times/\mathcal{O}_\mathfrak{p}^\times$$

is injective. Hence $\widetilde{\mathcal{O}}^\times/\mathcal{O}^\times$ is finite.

Remark 2. The finiteness of $\widetilde{\mathcal{O}}^\times/\mathcal{O}^\times$ follows from the more general fact, Theorem 3.8 in [BL17], that for any ring homomorphism $f\colon R \longrightarrow S$ having finite kernel and image of finite additive index, the induced map $R^\times \longrightarrow S^\times$ has finite kernel and image of finite index.

Proposition 3 (Dirichlet unit theorem for orders)**.** *The unit group $\mathcal{O}^\times$ is a finitely generated abelian group of rank r.*

Proof. If $\mathcal{O}$ is the ring of integers in a number field, this is the Dirichlet unit theorem. In general, $\widetilde{\mathcal{O}}$ is a product of such rings of integers, so the result holds for $\widetilde{\mathcal{O}}$. Since $\mathcal{O}^\times$ is of finite index in $\widetilde{\mathcal{O}}^\times$, the result holds for $\mathcal{O}$ too. ☐

2.5. **The logarithmic embedding and the regulator $R(\mathcal{O})$.** For $x \in \mathbb{R}^\times$, let $\lambda_\mathbb{R}(x) = \ln|x|$. For $x \in \mathbb{C}^\times$, let $\lambda_\mathbb{C}(x) = 2\ln|x|$. Let

$$(K \otimes \mathbb{R})^\times = (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2} \xrightarrow{\lambda} \mathbb{R}^{r_1+r_2}$$

be the homomorphism that applies $\lambda_\mathbb{R}$ or $\lambda_\mathbb{C}$ coordinate-wise, as appropriate. Let ϕ be the composition

$$\mathcal{O}^\times \longrightarrow K^\times \longrightarrow (K \otimes \mathbb{R})^\times \xrightarrow{\lambda} \mathbb{R}^{r_1+r_2}.$$

Since $\ker\lambda$ is bounded in $K \otimes \mathbb{R}$, $\ker\phi$ is finite; on the other hand, the codomain of ϕ is torsion-free; thus $\ker\phi = \mu(\mathcal{O})$.

Suppose that K is a field. The proof of the classical Dirichlet unit theorem shows that the image $\phi(\widetilde{\mathcal{O}}^\times)$ is a full lattice in the hyperplane in $\mathbb{R}^{r_1+r_2}$ where the coordinates sum to 0. Under the projection to $\mathbb{R}^r = \mathbb{R}^{r_1+r_2-1}$ defined by forgetting one coordinate, the hyperplane maps isomorphically to $\mathbb{R}^r$, and $\phi(\widetilde{\mathcal{O}}^\times)$ maps to a full lattice in $\mathbb{R}^r$; the covolume of this lattice is called the regulator, $R(\widetilde{\mathcal{O}})$.

In the general case, $\phi(\widetilde{\mathcal{O}}^\times)$ is a direct product of lattices in $\prod_{i=1}^m \mathbb{R}^{r(K_i)} = \mathbb{R}^r$. As proved in Section 2.4, $\mathcal{O}^\times$ is of finite index in $\widetilde{\mathcal{O}}^\times$, so $\phi(\mathcal{O}^\times)$ is again a full lattice $L(\mathcal{O})$ in $\mathbb{R}^r$; its covolume is denoted $R(\mathcal{O})$.

3. The zeta function

Retain the notation of the previous section. In what follows, $\mathfrak{p}$ ranges over prime ideals of $\mathcal{O}$ with finite residue field. Since $\mathcal{O}$ is finitely generated as a $\mathbb{Z}$-algebra, these prime ideals are the same as the maximal ideals of $\mathcal{O}$, which correspond to the closed points of $\operatorname{Spec}\mathcal{O}$. Define $\mathrm{N}\mathfrak{p} := \#\mathcal{O}/\mathfrak{p}$. Since $\operatorname{Spec}\mathcal{O}$ is of finite type over $\mathbb{Z}$, it has a zeta function defined as an Euler product, as in [Ser65, p. 83]:

$$\zeta_\mathcal{O}(s) = \prod_\mathfrak{p} \left(1 - \mathrm{N}\mathfrak{p}^{-s}\right)^{-1}.$$

Work of Hecke implies that $\zeta_\mathcal{O}(s)$ has a meromorphic continuation to the entire complex plane, and that $\zeta_\mathcal{O}(s)$ has a pole at $s = 1$ of order m. The analytic class number formula proposed below gives the leading term of $\zeta_\mathcal{O}(s)$ at $s = 1$.

Theorem 4 (Analytic class number formula for orders)**.** *Let $\mathcal{O}$ be an order in a product of number fields $K = K_1 \times \cdots \times K_m$. Then*

$$\lim_{s\to1}(s-1)^m\zeta_\mathcal{O}(s) = \frac{2^{r_1}(2\pi)^{r_2}}{w(\mathcal{O})\sqrt{|\operatorname{Disc}\mathcal{O}|}}h(\mathcal{O})R(\mathcal{O}). \tag{1}$$

In the classical case when $\mathcal{O}$ is the ring of integers of a number field, $\zeta_{\mathcal{O}}$ is the Dedekind zeta function, and Theorem 4 was proved by Dedekind, as mentioned already in Section 1. Each factor in (1) is multiplicative if $\mathcal{O}$ is a product of rings, so Theorem 4 holds for any product of rings of integers, and in particular for the normalization $\widetilde{\mathcal{O}}$ of any $\mathcal{O}$. To prove Theorem 4 for a general order $\mathcal{O}$, we will relate the formulas for $\mathcal{O}$ and $\widetilde{\mathcal{O}}$.

4. Relating the invariants for $\mathcal{O}$ and $\widetilde{\mathcal{O}}$

Let $X = \operatorname{Spec} \mathcal{O}$ and $\widetilde{X} = \operatorname{Spec} \widetilde{\mathcal{O}}$. The inclusion $\mathcal{O} \hookrightarrow \widetilde{\mathcal{O}}$ induces a morphism $\pi \colon \widetilde{X} \to X$ that is an isomorphism above the complement of a finite subset $Z \subseteq X$. For maximal ideals $\mathfrak{p} \subseteq \mathcal{O}$ and $\mathfrak{P} \subseteq \widetilde{\mathcal{O}}$, we write $\mathfrak{P}|\mathfrak{p}$ when $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}$, i.e., when π maps the closed point $\mathfrak{P}$ to the closed point $\mathfrak{p}$.

4.1. The zeta functions of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$.

Proposition 5. *We have*

$$\lim_{s \to 1} \frac{\zeta_{\widetilde{\mathcal{O}}}(s)}{\zeta_{\mathcal{O}}(s)} = \prod_{\mathfrak{p}} \frac{\prod_{\mathfrak{P}|\mathfrak{p}} (1 - \mathrm{N}\mathfrak{P}^{-1})^{-1}}{(1 - \mathrm{N}\mathfrak{p}^{-1})^{-1}}.$$

Proof. By definition,

$$\frac{\zeta_{\widetilde{\mathcal{O}}}(s)}{\zeta_{\mathcal{O}}(s)} = \prod_{\mathfrak{p}} \frac{\prod_{\mathfrak{P}|\mathfrak{p}} (1 - \mathrm{N}\mathfrak{P}^{-s})^{-1}}{(1 - \mathrm{N}\mathfrak{p}^{-s})^{-1}},$$

where, for all but finitely many $\mathfrak{p}$, the fraction on the right is 1; cf. [Jen69, Theorem]. □

4.2. The discriminants of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$.

Proposition 6. *We have*

$$\frac{\operatorname{Disc} \widetilde{\mathcal{O}}}{\operatorname{Disc} \mathcal{O}} = \left(\# \frac{\widetilde{\mathcal{O}}}{\mathcal{O}} \right)^{-2}.$$

Proof. This is standard: Let $A \in \mathrm{M}_2(\mathbb{Z})$ be the change-of-basis matrix expressing the $\mathbb{Z}$-basis of $\mathcal{O}$ in terms of the $\mathbb{Z}$-basis of $\widetilde{\mathcal{O}}$. Then $\#(\widetilde{\mathcal{O}}/\mathcal{O}) = \det A$. On the other hand, the matrix whose determinant is $\operatorname{Disc} \mathcal{O}$ is obtained from the matrix whose determinant is $\operatorname{Disc} \widetilde{\mathcal{O}}$ by multiplying by A on the right and A^T on the left, so $\operatorname{Disc} \mathcal{O} = (\det A)^2 \operatorname{Disc} \widetilde{\mathcal{O}}$. □

4.3. The regulators of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$.

Proposition 7.

$$\frac{R(\widetilde{\mathcal{O}})}{R(\mathcal{O})} \cdot \# \frac{\widetilde{\mathcal{O}}^\times}{\mathcal{O}^\times} = \frac{w(\widetilde{\mathcal{O}})}{w(\mathcal{O})}.$$

Proof. Let $L = L(\mathcal{O})$ be as in Section 2.5, and let $\widetilde{L} = L(\widetilde{\mathcal{O}})$; these are lattices in the same $\mathbb{R}^r$. Applying the snake lemma to

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu(\mathcal{O}) & \longrightarrow & \mathcal{O}^\times & \longrightarrow & L & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & \mu(\widetilde{\mathcal{O}}) & \longrightarrow & \widetilde{\mathcal{O}}^\times & \longrightarrow & \widetilde{L} & \longrightarrow & 0
\end{array}$$

yields an exact sequence

$$1 \to \frac{\mu(\widetilde{\mathcal{O}})}{\mu(\mathcal{O})} \to \frac{\widetilde{\mathcal{O}}^\times}{\mathcal{O}^\times} \to \frac{\widetilde{L}}{L} \to 0$$

of finite groups, the last of which has order $R(\mathcal{O})/R(\widetilde{\mathcal{O}})$. □

4.4. **Relating Pic $\mathcal{O}$ and Pic $\widetilde{\mathcal{O}}$ via the Leray spectral sequence.** View the abelian group $\widetilde{\mathcal{O}}_\mathfrak{p}^\times/\mathcal{O}_\mathfrak{p}^\times$ as a skyscraper sheaf on X supported at $\mathfrak{p}$; it is trivial for $\mathfrak{p} \notin Z$. We have an exact sequence of sheaves on X

$$0 \to \mathscr{O}_X^\times \to \pi_* \mathscr{O}_{\widetilde{X}}^\times \to \bigoplus_\mathfrak{p} \frac{\widetilde{\mathcal{O}}_\mathfrak{p}^\times}{\mathcal{O}_\mathfrak{p}^\times} \to 0.$$

The corresponding long exact sequence in cohomology is

$$0 \to \mathcal{O}^\times \to \widetilde{\mathcal{O}}^\times \to \bigoplus_\mathfrak{p} \frac{\widetilde{\mathcal{O}}_\mathfrak{p}^\times}{\mathcal{O}_\mathfrak{p}^\times} \to \operatorname{Pic} X \to H^1\left(X, \pi_* \mathscr{O}_{\widetilde{X}}^\times\right) \to 0. \tag{2}$$

Lemma 8. *We have* $H^1(X, \pi_* \mathscr{O}_{\widetilde{X}}^*) \simeq \operatorname{Pic} \widetilde{X}$.

Proof. From the Leray spectral sequence

$$H^p\left(X, R^q \pi_* \mathscr{F}\right) \implies H^{p+q}\left(\widetilde{X}, \mathscr{F}\right)$$

with $\mathscr{F} = \mathscr{O}_{\widetilde{X}}^\times$ we extract an exact sequence

$$0 \to H^1\left(X, \pi_* \mathscr{O}_{\widetilde{X}}^*\right) \to \operatorname{Pic} \widetilde{X} \to H^0\left(X, R^1 \pi_* \mathscr{O}_{\widetilde{X}}^\times\right).$$

Lemma 9 below completes the proof. □

Lemma 9. *The sheaf* $R^1 \pi_* \mathscr{O}_{\widetilde{X}}^\times$ *on* X *is* 0.

Proof. By [Har77, Proposition III.8.1], its stalk $(R^1 \pi_* \mathscr{O}_{\widetilde{X}}^\times)_x$ at a closed point x of X is $\varinjlim_U \operatorname{Pic} \pi^{-1} U$, where U ranges over open neighborhoods of x in X. Since $\pi^{-1}(x)$ is finite, every line bundle on $\pi^{-1} U$ becomes trivial on $\pi^{-1} U'$ for some smaller neighborhood U' of x in X. Thus $\varinjlim_U \operatorname{Pic} \pi^{-1} U = 0$. □

Substituting the isomorphism of Lemma 8 into (2) yields an exact sequence of finite groups

$$0 \to \frac{\widetilde{\mathcal{O}}^\times}{\mathcal{O}^\times} \to \bigoplus_\mathfrak{p} \frac{\widetilde{\mathcal{O}}_\mathfrak{p}^\times}{\mathcal{O}_\mathfrak{p}^\times} \to \operatorname{Pic} X \to \operatorname{Pic} \widetilde{X} \to 0. \tag{3}$$

Remark 10. For a more elementary derivation of (3), at least in the case where $\mathcal{O}$ is an integral domain; see [Neu99, Proposition I.12.9].

Next we compute the order of the second term in (3). Fix a nonzero ideal $\mathfrak{c}$ of $\widetilde{\mathcal{O}}$ such that $\mathfrak{c} \subseteq \mathcal{O}$; one possibility is $\mathfrak{c} = n\widetilde{\mathcal{O}}$, where $n := (\widetilde{\mathcal{O}} : \mathcal{O})$. (In fact, there is a largest $\mathfrak{c}$—the sum of all of them—called the conductor of $\mathcal{O}$.)

Lemma 11. *The natural map*

$$\frac{\widetilde{\mathcal{O}}_{\mathfrak{p}}^{\times}}{\mathcal{O}_{\mathfrak{p}}^{\times}} \to \frac{\left(\widetilde{\mathcal{O}}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}}\right)^{\times}}{\left(\mathcal{O}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}}\right)^{\times}}$$

is an isomorphism.

Proof. Case 1: $\mathfrak{c}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}$. Then $1 \in \mathfrak{c}_{\mathfrak{p}}$, so $\mathfrak{c}_{\mathfrak{p}} = \widetilde{\mathcal{O}}_{\mathfrak{p}}$ too; thus both sides are trivial.

Case 2: $\mathfrak{c}_{\mathfrak{p}} \neq \mathcal{O}_{\mathfrak{p}}$. Then $\mathfrak{c}_{\mathfrak{p}} \subseteq \mathfrak{p}\mathcal{O}_{\mathfrak{p}} \subset \mathfrak{P}$ for every maximal ideal $\mathfrak{P}$ of $\widetilde{\mathcal{O}}_{\mathfrak{p}}$. If an element $\bar{a} \in (\widetilde{\mathcal{O}}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}})^{\times}$ is lifted to an element $a \in \widetilde{\mathcal{O}}_{\mathfrak{p}}$, then a lies outside each $\mathfrak{P}$, so $a \in \widetilde{\mathcal{O}}_{\mathfrak{p}}^{\times}$. Thus $\widetilde{\mathcal{O}}_{\mathfrak{p}}^{\times} \to (\widetilde{\mathcal{O}}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}})^{\times}$ is surjective. Similarly, $\mathcal{O}_{\mathfrak{p}}^{\times} \to (\mathcal{O}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}})^{\times}$ is surjective. Both surjections have the same kernel $1 + \mathfrak{c}_{\mathfrak{p}}$, so the result follows. □

Lemma 12. *If* $\mathfrak{c}_{\mathfrak{p}} \neq \mathcal{O}_{\mathfrak{p}}$*, then*

$$\#\left(\widetilde{\mathcal{O}}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}}\right)^{\times} = \#\left(\widetilde{\mathcal{O}}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}}\right)\prod_{\mathfrak{P}|\mathfrak{p}}\left(1 - \mathrm{N}\mathfrak{P}^{-1}\right),$$
$$\#\left(\mathcal{O}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}}\right)^{\times} = \#\left(\mathcal{O}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}}\right)\left(1 - \mathrm{N}\mathfrak{p}^{-1}\right).$$

Proof. The maximal ideals of $\widetilde{\mathcal{O}}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}}$ are the ideals $\mathfrak{P}\widetilde{\mathcal{O}}_{\mathfrak{p}}$ for $\mathfrak{P}|\mathfrak{p}$. An element of $\widetilde{\mathcal{O}}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}}$ is a unit if and only if it lies outside each maximal ideal. The probability that a random element of the finite group $\widetilde{\mathcal{O}}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}}$ lies outside $\mathfrak{P}\widetilde{\mathcal{O}}_{\mathfrak{p}}$ is $1 - \mathrm{N}\mathfrak{P}^{-1}$, and these events for different $\mathfrak{P}$ are independent by the Chinese remainder theorem, so the first equation follows. The second equation is similar (but easier). □

Lemma 13. *We have*

$$\prod_{\mathfrak{p}} \#\frac{\widetilde{\mathcal{O}}_{\mathfrak{p}}^{\times}}{\mathcal{O}_{\mathfrak{p}}^{\times}} = \#\frac{\widetilde{\mathcal{O}}}{\mathcal{O}} \cdot \prod_{\mathfrak{p}} \frac{\prod_{\mathfrak{P}|\mathfrak{p}}(1 - \mathrm{N}\mathfrak{P}^{-1})}{1 - \mathrm{N}\mathfrak{p}^{-1}}.$$

Proof. By Lemmas 11 and 12,

$$\#\frac{\widetilde{\mathcal{O}}_{\mathfrak{p}}^{\times}}{\mathcal{O}_{\mathfrak{p}}^{\times}} = \#\frac{\widetilde{\mathcal{O}}_{\mathfrak{p}}}{\mathcal{O}_{\mathfrak{p}}} \cdot \frac{\prod_{\mathfrak{P}|\mathfrak{p}}(1 - \mathrm{N}\mathfrak{P}^{-1})}{1 - \mathrm{N}\mathfrak{p}^{-1}};$$

this holds even if $\mathfrak{c}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}$ since both sides are 1 in that case. Now take the product of both sides and use the isomorphism of finite groups

$$\frac{\widetilde{\mathcal{O}}}{\mathcal{O}} \simeq \prod_{\mathfrak{p}} \frac{\widetilde{\mathcal{O}}_{\mathfrak{p}}}{\mathcal{O}_{\mathfrak{p}}}.$$ □

Proposition 14.

$$\#\frac{\widetilde{\mathcal{O}}^{\times}}{\mathcal{O}^{\times}} = \frac{h(\widetilde{\mathcal{O}})}{h(\mathcal{O})} \cdot \#\frac{\widetilde{\mathcal{O}}}{\mathcal{O}} \cdot \prod_{\mathfrak{p}} \frac{\prod_{\mathfrak{P}|\mathfrak{p}}(1 - \mathrm{N}\mathfrak{P}^{-1})}{1 - \mathrm{N}\mathfrak{p}^{-1}}.$$

Proof. Take the alternating product of the orders of the groups in (3) and use Lemma 13. □

4.5. **Conclusion of the proof.** To complete the proof of Theorem 4, we compare (1) for $\widetilde{\mathcal{O}}$ to (1) for $\mathcal{O}$. The ratio of the left side of (1) for $\widetilde{\mathcal{O}}$ to the left side of (1) for $\mathcal{O}$ is

$$\lim_{s\to 1} \frac{\zeta_{\widetilde{\mathcal{O}}}(s)}{\zeta_{\mathcal{O}}(s)}.$$

The ratio of the right sides is

$$\left|\frac{\operatorname{Disc}\widetilde{\mathcal{O}}}{\operatorname{Disc}\mathcal{O}}\right|^{-1/2} \left(\frac{w(\widetilde{\mathcal{O}})}{w(\mathcal{O})}\right)^{-1} \cdot \frac{h(\widetilde{\mathcal{O}})}{h(\mathcal{O})} \cdot \frac{R(\widetilde{\mathcal{O}})}{R(\mathcal{O})}.$$

By Propositions 5, 6, 7, and 14, both ratios equal

$$\prod_{\mathfrak{p}} \frac{\prod_{\mathfrak{P}|\mathfrak{p}}(1-\mathrm{N}\mathfrak{P}^{-1})^{-1}}{(1-\mathrm{N}\mathfrak{p}^{-1})^{-1}}.$$

5. An example with $\operatorname{Spec}\mathcal{O}$ singular: a fiber product

Consider the ring

$$\mathcal{O} := \mathbb{Z}\times_{\mathbb{F}_p}\mathbb{Z} = \{\,(a,b)\in\mathbb{Z}\times\mathbb{Z} : a\equiv b \pmod p\,\}.$$

The normalization $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ is the ring $\mathbb{Z}\times\mathbb{Z}$; inverting all non-zerodivisors of $\mathcal{O}$ gives its ring of fractions $K=\mathbb{Q}\times\mathbb{Q}$. The scheme $X:=\operatorname{Spec}\mathcal{O}$ consists of two copies of the "curve" $\operatorname{Spec}\mathbb{Z}$ crossing at the point $(p)\in\operatorname{Spec}\mathbb{Z}$. The scheme $\widetilde{X}:=\operatorname{Spec}\widetilde{\mathcal{O}}$ is a disjoint union of two copies of $\operatorname{Spec}\mathbb{Z}$. The conductor of $\mathcal{O}$ is the $\widetilde{\mathcal{O}}$-ideal

$$\mathfrak{p} := \{\,(a,b)\in\mathbb{Z}\times\mathbb{Z} : a\equiv b\equiv 0 \pmod p\,\}.$$

Above the prime $\mathfrak{p}$ of $\mathcal{O}$ there are two primes of $\widetilde{\mathcal{O}}$, the two copies of (p).

Proposition 15. *We have*

$$\lim_{s\to 1}(s-1)^2\zeta_{\mathcal{O}}(s) = 1-p^{-1}.$$

Proof. The Riemann zeta function $\zeta_{\mathbb{Z}}(s)$ has a pole of order 1 at $s=1$ with residue 1. Since the one ideal $\mathfrak{p}$ of norm p in $\mathcal{O}$ is replaced by two ideals of norm p in $\widetilde{\mathcal{O}}$,

$$\begin{aligned}
\zeta_{\mathcal{O}}(s) &= \left(1-p^{-s}\right)\zeta_{\widetilde{\mathcal{O}}}(s)\\
\lim_{s\to 1}(s-1)^2\zeta_{\mathcal{O}}(s) &= \left(1-p^{-1}\right)\lim_{s\to 1}(s-1)^2\zeta_{\widetilde{\mathcal{O}}}(s)\\
&= \left(1-p^{-1}\right)\left(\lim_{s\to 1}(s-1)\zeta_{\mathbb{Z}}(s)\right)^2\\
&= 1-p^{-1}.
\end{aligned}$$

□

Proposition 16. *We have*

$$\frac{2^{r_1}(2\pi)^{r_2}}{w(\mathcal{O})\sqrt{|\operatorname{Disc}\mathcal{O}|}}h(\mathcal{O})R(\mathcal{O}) = 1-p^{-1}.$$

Proof. First, $r_1 = 2$, $r_2 = 0$, and $r = 2 + 0 - 2 = 0$, so $R(\mathcal{O}) = 1$.

The trace map on $\widetilde{\mathcal{O}}$ or $\mathcal{O}$ sends (a, b) to $a + b$. The elements $(1, 1)$ and $(p, 0)$ form a basis of $\mathcal{O}$, so

$$\operatorname{Disc} \mathcal{O} = \det \begin{pmatrix} 2 & p \\ p & p^2 \end{pmatrix} = p^2.$$

Inside $\widetilde{\mathcal{O}}^\times = \mathbb{Z}^\times \times \mathbb{Z}^\times = \pm 1 \times \pm 1$ we have

$$\mathcal{O}^\times = \begin{cases} \pm(1, 1) & \text{if } p \text{ is odd,} \\ \pm 1 \times \pm 1 & \text{if } p = 2, \end{cases}$$

so

$$w(\mathcal{O}) = \begin{cases} 2 & \text{if } p \text{ is odd,} \\ 4 & \text{if } p = 2. \end{cases}$$

By Lemma 11, the exact sequence (3) is

$$1 \to \frac{\widetilde{\mathcal{O}}^\times}{\mathcal{O}^\times} \to \frac{\mathbb{F}_p^\times \times \mathbb{F}_p^\times}{\mathbb{F}_p^\times} \to \operatorname{Pic} \mathcal{O} \to \operatorname{Pic} \widetilde{\mathcal{O}}. \tag{4}$$

The image of $\dfrac{\widetilde{\mathcal{O}}^\times}{\mathcal{O}^\times}$ in $\dfrac{\mathbb{F}_p^\times \times \mathbb{F}_p^\times}{\mathbb{F}_p^\times} \simeq \mathbb{F}_p^\times$ is ± 1, even when $p = 2$ in which case these groups are trivial. On the other hand, $\operatorname{Pic} \widetilde{\mathcal{O}} = \operatorname{Pic} \mathbb{Z} \times \operatorname{Pic} \mathbb{Z} = 0$. Thus (4) yields $\operatorname{Pic} \mathcal{O} \simeq \mathbb{F}_p^\times / \pm 1$, and

$$h(\mathcal{O}) = \begin{cases} (p-1)/2 & \text{if } p \text{ is odd,} \\ 1 & \text{if } p = 2. \end{cases}$$

Combining the above calculations yields

$$\begin{aligned} \frac{2^{r_1} (2\pi)^{r_2}}{w(\mathcal{O}) \sqrt{|\operatorname{Disc} \mathcal{O}|}} h(\mathcal{O}) R(\mathcal{O}) &= \begin{cases} \dfrac{2^2}{2\sqrt{p^2}} \cdot \dfrac{p-1}{2} \cdot 1 & \text{if } p \text{ is odd} \\ \dfrac{2^2}{4\sqrt{2^2}} \cdot 1 \cdot 1 & \text{if } p = 2 \end{cases} \\ &= 1 - p^{-1}. \end{aligned}$$

□

Propositions 15 and 16 verify Theorem 4 for $\mathcal{O}$.

Remark 17. Fiber products such as $\mathcal{O}$ arise as integral Hecke algebras of elliptic modular forms. For example, the integral Hecke algebra $\mathbf{T}^*$ (cf. [Maz77, p. 37]) for modular forms of weight 2 for the congruence subgroup $\Gamma_0(11)$ is $\mathbb{Z} \times_{\mathbb{F}_5} \mathbb{Z}$.

Acknowledgments

It is a pleasure to thank Tony Scholl for helpful discussions. We thank Carlos J. Moreno for bringing the article [Stö98] to our attention.

References

[BL17] Alex Bartel and Hendrik W. Lenstra Jr., *Commensurability of automorphism groups*, Compositio Math. **153** (2017), no. 2, 323–346, DOI 10.1112/S0010437X1600823X.

[Dir94] P. G. Lejeune Dirichlet, *Vorlesungen über Zahlentheorie*, 4th ed., Braunschweig, 1894. Edited by and with supplements by R. Dedekind.

[Gal73] V. M. Galkin, *Zeta-functions of certain one-dimensional rings*, Izv. Akad. Nauk SSSR Ser. Mat. **37** (1973), 3–19 (Russian). MR0332729

[Gre89] Barry Green, *Functional equations for zeta functions of non-Gorenstein orders in global fields*, Manuscripta Math. **64** (1989), no. 4, 485–502, DOI 10.1007/BF01170941. MR1005249

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)

[Hil97] David Hilbert, *Die Theorie der algebraische Zahlkörper*, Jahresbericht der Deutschen Mathematiker-Vereinigung **4** (1897), 175–546; English transl., David Hilbert, *The theory of algebraic number fields* (1998), xxxvi+350. Translated from the German and with a preface by Iain T. Adamson; With an introduction by Franz Lemmermeyer and Norbert Schappacher. MR1646901 (99j:01027).

[Jen69] W. E. Jenner, *On zeta functions of number fields*, Duke Math. J. **36** (1969), 669–671. MR0249394

[Maz77] B. Mazur, *Modular curves and the Eisenstein ideal*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 33–186 (1978). MR488287 (80c:14015)

[Neu99] Jürgen Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher; With a foreword by G. Harder. MR1697859 (2000m:11104)

[Ser65] Jean-Pierre Serre, *Zeta and L functions*, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), Harper & Row, New York, 1965, pp. 82–92. MR0194396 (33 #2606)

[Stö98] Karl-Otto Stöhr, *Local and global zeta-functions of singular algebraic curves*, J. Number Theory **71** (1998), no. 2, 172–202, DOI 10.1006/jnth.1998.2240. MR1633801

[ZG97a] W. A. Zúñiga Galindo, *Zeta functions and Cartier divisors on singular curves over finite fields*, Manuscripta Math. **94** (1997), no. 1, 75–88, DOI 10.1007/BF02677839. MR1468935

[ZG97b] W. A. Zúñiga-Galindo, *Zeta functions of singular curves over finite fields*, Rev. Colombiana Mat. **31** (1997), no. 2, 115–124. MR1667594

Department of Mathematics, Baruch College, The City University of New York, One Bernard Baruch Way, New York, NY 10010-5526, USA

E-mail address: `bruce.jordan@baruch.cuny.edu`

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA

E-mail address: `poonen@math.mit.edu`

URL: `http://math.mit.edu/~poonen/`