# 18.02 LECTURE NOTES, SPRING 2013

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## 1. February 5

1.1. **Vectors.** A vector $\mathbf{v}$ in $\mathbb{R}^3$ is an ordered triple of real numbers. Example: $\langle 2, 3, 5\rangle$. (You can have vectors in $\mathbb{R}^2$ or $\mathbb{R}^{95786}$ too, if you want.) Geometrically, a vector is an arrow with a length and a direction; its position does not matter.

The standard basis vectors for $\mathbb{R}^3$:

$$\mathbf{i} := \langle 1, 0, 0\rangle$$
$$\mathbf{j} := \langle 0, 1, 0\rangle$$
$$\mathbf{k} := \langle 0, 0, 1\rangle$$

These are vectors in the directions of the three axes. Also, $\mathbf{0} := \langle 0, 0, 0\rangle$.

If $P$ is a point in space, then the position vector $\mathbf{P}$ is the vector pointing from $(0, 0, 0)$ to $P$.

The length (or magnitude) of $\mathbf{v} = \langle a, b, c\rangle$ is $|\mathbf{v}| := \sqrt{a^2 + b^2 + c^2}$. This formula can be explained by using the Pythagorean theorem twice.

A unit vector is a vector of length 1.

Addition: $\langle 3, 1\rangle + \langle 1, 4\rangle = \langle 4, 5\rangle$. Subtraction: $\langle 3, 1\rangle - \langle 1, 4\rangle = \langle 2, -3\rangle$. Geometrically: parallelogram law for $+$, triangle for $-$. Important: If $A$ and $B$ are two points, and $\mathbf{A}$ and $\mathbf{B}$ are their position vectors, then the vector from $A$ to $B$ is $\mathbf{B} - \mathbf{A}$, because this is what you have to add to $\mathbf{A}$ to get to $\mathbf{B}$.

Scalar multiplication: $-10\,\langle 3, 1\rangle = \langle -30, -10\rangle$. Scalar means number (one uses this word when one wants to emphasize that it is not a vector). Scalar multiplication is a scalar times a vector, and the result is a vector. Geometrically: $c\mathbf{v}$ has the same (or opposite) direction as $\mathbf{v}$, but possibly a different length. Two vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel if one of them is a scalar multiple of the other.

**Question 1.1.** How do you write a nonzero vector $\mathbf{v}$ such as $\langle 3, 4\rangle$ as a (positive) scalar times a unit vector?

Answer: The scalar is the length, and the unit vector is the original vector divided by its length:

$$\mathbf{v} = \underbrace{|\mathbf{v}|}_{\text{length}} \underbrace{\frac{\mathbf{v}}{|\mathbf{v}|}}_{\text{unit vector}} .$$

In our example,

$$\langle 3,4\rangle = \underbrace{5}_{\text{length}} \underbrace{\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle}_{\text{unit vector}}.$$

The unit vector is giving the direction of $\mathbf{v}$.

**Question 1.2.** Does the zero vector $\mathbf{0} := \langle 0,0,0\rangle$ have a direction? It is best to say that it has *every* direction, and hence to say that it is parallel to every other vector, and perpendicular to every other vector.

Another example involving addition and scalar multiplication:

$$\underbrace{2\,\mathbf{i} + 3\,\mathbf{j} + 5\,\mathbf{k}}_{\text{linear combination of } \mathbf{i},\mathbf{j},\mathbf{k}} = \langle 2,3,5\rangle.$$

In general, a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is any vector obtained by multiplying the vectors by (possibly different) scalars and adding the results, i.e., an expression of the form $c_1\mathbf{v}_1 + \cdots + c_2\mathbf{v}_n$ for some scalars $c_1, \ldots, c_n$.

**Question 1.3.** Suppose that $M$ is the midpoint of segment $AB$. In terms of the position vectors $\mathbf{A}$ and $\mathbf{B}$, what is the position vector $\mathbf{M}$?

To get to $M$, start at $A$ and go halfway from $A$ to $B$. The vector from $A$ to $B$ is $\mathbf{B} - \mathbf{A}$, so the vector from $A$ to $M$ is $\frac{1}{2}(\mathbf{B} - \mathbf{A})$. Thus

$$\mathbf{M} = \mathbf{A} + \frac{1}{2}(\mathbf{B} - \mathbf{A}) = \frac{\mathbf{A} + \mathbf{B}}{2}.$$

**Question 1.4.** Prove that the midpoints of the sides of a space quadrilateral form a parallelogram.

How should one approach a problem like this?

Give variable names to the objects given in the problem. *Let $A, B, C, D$ be the vertices of the quadrilateral. Let $A'$ be the midpoint of $AB$, let $B'$ be the midpoint of $BC$, let $C'$ be the midpoint of $CD$, and let $D'$ be the midpoint of $DA$. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{A}', \mathbf{B}', \mathbf{C}', \mathbf{D}'$ be the corresponding position vectors.*

Write down known equations relating the variables. *We know that*

$$\mathbf{A}' = \frac{\mathbf{A} + \mathbf{B}}{2}, \quad \mathbf{B}' = \frac{\mathbf{B} + \mathbf{C}}{2}, \quad \mathbf{C}' = \frac{\mathbf{C} + \mathbf{D}}{2}, \quad \mathbf{D}' = \frac{\mathbf{D} + \mathbf{A}}{2}.$$

See if these equations imply the desired conclusion. *We compute*

$$\mathbf{B}' - \mathbf{A}' = \frac{\mathbf{C} - \mathbf{A}}{2}$$
$$\mathbf{C}' - \mathbf{D}' = \frac{\mathbf{C} - \mathbf{A}}{2}.$$

*Thus segments $B'A'$ and $C'D'$ are parallel and have the same length. This means that $A'B'C'D'$ is a parallelogram.*

1.2. **Some advice for success in 18.02.**

- Read the Stellar website now, especially Information.
- Do the reading assignments *before* lecture!
- Come to office hours! (Office hours generally consist of a small group of students discussing additional examples not covered in lecture or recitation, asking questions, getting started on difficult homework problems, etc. The recitation leaders for this class are some of the best math faculty and grad students worldwide, and office hours are your best chance to learn from them.)
- Homework:
  - It's long and has difficult problems, so start now! When I created the homework assignments this January, I labelled each problem with the date when you should be able to start it.
  - Work together in groups! It's OK if other people tell you how to solve a problem, but don't be looking at their solution as you write your own.
  - Do what it takes (come to office hours, discuss problems with others) so that when you hand in an assignment you are pretty sure that it is complete and correct.

1.3. **Dot product.** Do we multiply vectors coordinate-wise? No! Why not? This does not give a notion with useful geometric meaning. Instead:

Dot product (also called scalar product or inner product):

$$\underbrace{\langle 2,3,5\rangle}_{\text{vector}} \cdot \underbrace{\langle 7,8,9\rangle}_{\text{vector}} = (2)(7) + (3)(8) + (5)(9)$$
$$= 14 + 24 + 45$$
$$= \underbrace{83}_{\text{scalar}}.$$

Important special case: $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.

**Theorem 1.5** (Geometric interpretation of the dot product)**.** *If $\theta$ is the angle between nonzero vectors* **a** *and* **b***, then*

$$\boxed{\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta.}$$

*(If* $\mathbf{a}$ *or* $\mathbf{b}$ *is* $\mathbf{0}$*, then* $\theta$ *can be taken to be any real number, and the formula still holds, with both sides being* 0.*)*

Why?

*Proof (=explanation).* Let $\mathbf{c} = \mathbf{a} - \mathbf{b}$. Let $a = |\mathbf{a}|$, $b = |\mathbf{b}|$, $c = |\mathbf{c}|$. Then

$$\begin{aligned} c^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= a^2 + b^2 - 2\mathbf{a} \cdot \mathbf{b}. \end{aligned}$$

On the other hand, the law of cosines says

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Comparing shows that $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$. □

**Corollary 1.6.** *Vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are perpendicular* $\iff$ $\mathbf{a} \cdot \mathbf{b} = 0$.

**Definition 1.7** (Scalar component of a vector)**.** If $\mathbf{a}$ is a vector, and $\mathbf{b}$ is a nonzero vector, then the scalar component of $\mathbf{a}$ in the direction of $\mathbf{b}$ measures "the amount of $\mathbf{a}$ that is in the direction of $\mathbf{b}$", i.e., what the $x$-coordinate of $\mathbf{a}$ *would be* in a new coordinate system in which the $x$-axis were in the direction of $\mathbf{b}$. It is defined by the following formulas:

$$\boxed{\operatorname{comp}_{\mathbf{b}}\mathbf{a} := \mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|} = |\mathbf{a}|\cos\theta.}$$

The equality of the last two expressions is explained as follows:

$$\mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{1}{|\mathbf{b}|}(\mathbf{a} \cdot \mathbf{b}) = \frac{1}{|\mathbf{b}|}(|\mathbf{a}||\mathbf{b}|\cos\theta) = |\mathbf{a}|\cos\theta.$$

Geometric interpretation: If $\mathbf{a}$ and $\mathbf{b}$ are position vectors, then $\operatorname{comp}_{\mathbf{b}}\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto the line containing $\mathbf{b}$, except that if the endpoint of the projection is on the other side of $\mathbf{b}$, then $\operatorname{comp}_{\mathbf{b}}\mathbf{a}$ is negative.

**Example 1.8.** If $\mathbf{a} = \langle -2, -3, -5 \rangle$ and $\mathbf{b} = \mathbf{j}$, then $\operatorname{comp}_{\mathbf{b}}\mathbf{a} = \mathbf{a} \cdot \mathbf{j} = -3$.

### 1.4. **Matrices.**

**Definition 1.9.** An $m \times n$ matrix is a rectangular array of numbers with $m$ rows and $n$ columns.

Example:

$$A := \begin{pmatrix} 3 & 5 & \pi \\ 4 & 0 & 0 \\ 6 & 7 & 9 \\ 1 & -2 & 3 \end{pmatrix}$$

is a $4 \times 3$ matrix. (The dimensions are always given in the order "height $\times$ width".)

The notation $a_{ij}$ means the number in the $i^{\text{th}}$ row and $j^{\text{th}}$ column. In the example above, $a_{32} = 7$.

An $m \times 1$ matrix is also called a column vector. For example, $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ is the same as the vector $\langle 2, 3, 5 \rangle$.

1.5. **Determinants.** To each *square* matrix $A$ is associated a number called the determinant.

$$\begin{aligned}
\det \begin{pmatrix} a \end{pmatrix} &:= a \\
&= \pm \text{ length of segment in the real line } \mathbb{R} \text{ determined by } a \\
\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} &:= ad - bc \\
&= \pm \text{ area of parallelogram in } \mathbb{R}^2 \text{ spanned by } \langle a, b \rangle \text{ and } \langle c, d \rangle \\
&\qquad \text{(the sign is } + \text{ if } \langle c, d \rangle \text{ is counterclockwise from } \langle a, b \rangle) \\
\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} &:= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - c_1 b_2 a_3 - c_2 b_3 a_1 - c_3 b_1 a_2 \\
&= \pm \text{ volume of parallelepiped in } \mathbb{R}^3 \text{ spanned by the rows } \mathbf{a}, \mathbf{b}, \mathbf{c} \\
&\qquad \text{(the sign is } + \text{ if } \mathbf{a}, \mathbf{b}, \mathbf{c} \text{ agree with the right hand rule)}
\end{aligned}$$

(One could also use the columns instead of the rows.)

Alternative notation for determinant: $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$. (This is a scalar, not a matrix!)

## 2. February 7

Laplace expansion (along the first row) for a $3 \times 3$ determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

The general rule leading to the formula above is this:

(1) Move your finger along the entries in a row.
(2) At each position, compute the minor, defined as the smaller determinant obtained by crossing out the row and the column through your finger; then multiply the minor by

the number you are pointing at, and adjust the sign according to the checkerboard pattern

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

(the pattern always starts with + in the upper left corner).

(3) Add up the results.

There is a similar expansion for a determinant of any size, computed along any row or column.

Properties of determinants:

D-1: Interchanging two rows changes the sign of $\det A$.
D-2: If one of the rows is all 0, then $\det A = 0$.
D-3: Multiplying an entire row by a scalar $c$ multiples $\det A$ by $c$.
D-4: Adding a multiple of a row to another row does not change $\det A$.

These properties can all be interpreted geometrically. The same properties hold for *column* operations.

**Example 2.1.** Let

$$A := \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}.$$

If we add 10 times the first row to the second row (and leave the first row unchanged), we get a new matrix

$$B := \begin{pmatrix} 2 & 3 \\ 25 & 37 \end{pmatrix}.$$

Property D-4 says that $\det B = \det A$.

**Question 2.2.** Suppose that $A$ is a $3 \times 3$ matrix such that $\det A = 5$. Doubling every entry of $A$ gives a matrix $2A$. What is $\det(2A)$?

Solution: Each time we multiply a row by 2, the determinant gets multiplied by 2. We need to do this three times to double the whole matrix $A$, so the determinant gets multiplied by $2 \cdot 2 \cdot 2 = 8$. Thus $\det(2A) = 8 \det(A) = 40$.

2.1. **Cross product.**

**Definition 2.3** (Cross product, also called vector product). The cross product of vectors **a** and **b** in $\mathbb{R}^3$ is the vector

$$\mathbf{a}\times\mathbf{b} := \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$:= +\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$
$$= (a_2b_3 - a_3b_2)\,\mathbf{i} - (a_1b_3 - a_3b_1)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}.$$

Geometrically: If **a** and **b** are nonzero vectors forming an angle $\theta$, then $\mathbf{a}\times\mathbf{b}$ is the vector determined by the following three conditions:

- It is perpendicular to **a** and **b**.
- Its length is $|\mathbf{a}||\mathbf{b}|\sin\theta$ (which equals the area of the parallelogram spanned by **a** and **b**).
- Its direction is given by the right hand rule: if you point the fingers of your right hand in the direction of **a** so that bending your fingers makes them point in the direction of **b**, then your thumb shows the direction of $\mathbf{a}\times\mathbf{b}$.

**Example 2.4.**

$$\mathbf{i}\times\mathbf{k} = -\mathbf{j}.$$

(It is a vector perpendicular to **i** and **k**, and its length is $(1)(1)\sin\frac{\pi}{2} = 1$, so it must be **j** or $-\mathbf{j}$. The right hand rule says it is $-\mathbf{j}$.)

For any vectors **a** and **b**,

$$\mathbf{a}\times\mathbf{a} = \mathbf{0}, \qquad \mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}.$$

(We haven't explained why the determinant and cross product have the geometric interpretations claimed, but we might explain this later on.)

## 2.2. Matrix times a column vector.

### 2.2.1. *Example and definition.*

**Example 2.5.** $\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{pmatrix}}_{\text{matrix}} \underbrace{\begin{pmatrix} 10 \\ 1 \\ 100 \end{pmatrix}}_{\text{column vector}} = \underbrace{\begin{pmatrix} 312 \\ 523 \end{pmatrix}}_{\text{column vector}}$ because

$$\langle 1,2,3\rangle \cdot \langle 10,1,100\rangle = 312$$
$$\langle 2,3,5\rangle \cdot \langle 10,1,100\rangle = 523.$$

The example above illustrates the following general definition:

**Definition 2.6** (Matrix times a column vector)**.** If $A$ is a matrix and $\mathbf{x}$ is a column vector, then

$$\boxed{A\mathbf{x} := \begin{pmatrix} (\text{first row of } A) \cdot \mathbf{x} \\ (\text{second row of } A) \cdot \mathbf{x} \\ \vdots \end{pmatrix}}$$

($A\mathbf{x}$ is defined only if the width of $A$ equals the height of $\mathbf{x}$.)

2.2.2. *Recovering columns of a matrix from matrix-vector multiplications.* It is possible to recover the columns of a matrix $A$ just from knowing what $A$ times each standard basis vector ($\mathbf{i}$, $\mathbf{j}$, etc.) is.

**Example 2.7.** Recovering the first column: $\begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 & 7 & 8 \\ 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

In general:

$$\boxed{(\text{first column of } A) = A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \boxed{(\text{second column of } A) = A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \text{ and so on.}$$

2.2.3. *Matrix-vector multiplication as a linear combination of the columns.*

**Example 2.8.** $\begin{pmatrix} 6 & 7 & 8 \\ 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6x + 7y + 8z \\ 2x + 3y + 5z \end{pmatrix} = x \begin{pmatrix} 6 \\ 2 \end{pmatrix} + y \begin{pmatrix} 7 \\ 3 \end{pmatrix} + z \begin{pmatrix} 8 \\ 5 \end{pmatrix}$.

In general: $\boxed{A\mathbf{x} = (\text{some linear combination of the columns of } A)}$.

2.3. **Matrices as linear transformations.**

2.3.1. *Going from a matrix to a linear transformation.* Given a number, say 3, we can create a function $f(x) := 3x$ that takes as input a number $x$ and outputs the number $3x$.

Higher-dimensional analogue: Given a matrix $A$, we can create a function $\mathbf{f}(\mathbf{x}) := A\mathbf{x}$ that takes as input a column vector $\mathbf{x}$ and outputs the column vector $A\mathbf{x}$. Functions arising in this way are called linear transformations.

**Example 2.9.** The $2 \times 3$ matrix $\begin{pmatrix} 6 & 7 & 8 \\ 2 & 3 & 5 \end{pmatrix}$ gives rise to the linear transformation

$$\mathbf{f} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6x + 7y + 8z \\ 2x + 3y + 5z \end{pmatrix}.$$

One writes $\mathbf{f}\colon \mathbb{R}^3 \to \mathbb{R}^2$ since the possible inputs are the vectors in $\mathbb{R}^3$, and every output is a vector in $\mathbb{R}^2$. What does this function do to $\mathbf{j}$? Answer: $\mathbf{f}(\mathbf{j}) = \mathbf{f}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$.

*Remark* 2.10. The reason that matrix-vector multiplication is defined the way it is is so that linear transformations can be written as $A\mathbf{x}$!

Class on February 8 was cancelled because of winter storm Nemo.

## 3. February 12

### 3.1. Matrices as linear transformations, continued.

3.1.1. *Going from a linear transformation to a matrix.* Given a linear transformation, how do we reconstruct the matrix that gives rise to it?

Answer 1: If we know a formula for the function, we can just read off the entries of the matrix. For example, given the function $\mathbf{f}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6x + 7y & \\ 2x & + 5z \end{pmatrix}$, the matrix is $\begin{pmatrix} 6 & 7 & 0 \\ 2 & 0 & 5 \end{pmatrix}$.

Answer 2: If we know only *values* of the linear transformation $\mathbf{f}$, then we can recover the matrix one column at a time: the first column is $\mathbf{f}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, the second column is $\mathbf{f}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and so on.

**Question 3.1.** Given $\theta$, there is a $2 \times 2$ matrix $R$ whose associated linear transformation rotates each vector in $\mathbb{R}^2$ counterclockwise by the angle $\theta$. What is it?

The rotation maps $\langle 1, 0 \rangle$ to $\langle \cos\theta, \sin\theta \rangle$ and $\langle 0, 1 \rangle$ to $\langle -\sin\theta, \cos\theta \rangle$. Thus

$$(\text{first column of } R) = R\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$(\text{second column of } R) = R\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

so

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

3.1.2. $2 \times 2$ *matrices as linear transformations of the plane.* A linear transformation $\mathbf{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ can be visualized by drawing a subset of the input plane $\mathbb{R}^2$, evaluating $\mathbf{f}$ at every point of the subset, and plotting each value in the output plane $\mathbb{R}^2$.

**Example 3.2.** Let $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. Let $\mathbf{f}$ be the associated linear transformation, so $\mathbf{f}(x, y) = (3x, 3y)$. Then $\mathbf{f}$ maps a smiley face (say centered at the origin, of radius 1) to a larger smiley face. In particular, $\mathbf{f}$ maps the left ear at $(1, 0)$ to $(3, 0)$. (Such a linear transformation is called a dilation.)

**Question 3.3.** What is a matrix $A$ whose associated linear transformation $\mathbf{f}$ sends the smiley face to a fat smiley face, the same height as before but twice as wide?

Solution: The first column is $\mathbf{f}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and the second column is $\mathbf{f}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

3.1.3. *Area scaling factor.*

**Example 3.4.** What does the dilation given by $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ do to area?

(Hint: Consider what it does to a $1 \times 1$ square.)

Answer: It multiplies area by 9. Notice that $\det \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = 9$ too.

In general, for any $2 \times 2$ matrix $A$, the associated linear transformation has

$$\boxed{\text{area scaling factor} = |\det A|.}$$

(The area scaling factor is always nonnegative, while $\det A$ could be negative, so it is necessary to take the absolute value.)

*Why?* If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the square spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is transformed into the parallelogram spanned by $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$; the square has area 1 and the parallelogram has area $|\det A|$.

## 3.2. **Matrix operations.**

Addition, subtraction, and scalar multiplication are defined entrywise, just as for vectors. (For addition and subtraction, the two matrices have to have the same dimensions.)

Transpose:

$$\begin{pmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \end{pmatrix}^T = \begin{pmatrix} 2 & 7 \\ 3 & 11 \\ 5 & 13 \end{pmatrix}.$$

Each row of the original matrix corresponds to a column of the transpose matrix. The $ij$-entry of $A$ equals the $ji$-entry of $A^T$.

Multiplication: If $A$ is $m \times n$ and $B$ is $n \times p$, then $AB$ is the $m \times p$ matrix whose $ij$-entry is the dot product

$$(i^{\text{th}} \text{ row of } A) \cdot (j^{\text{th}} \text{ column of } B)$$

**Example 3.5.** Multiplying a $2 \times 3$ matrix by a $3 \times 3$ matrix gives a $2 \times 3$ matrix:

$$\begin{pmatrix} 2 & 3 & 7 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 & 9 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 23 & 13 & 18 \\ 5 & 11 & 0 \end{pmatrix}.$$

For example, the $2,1$ entry is computed as follows:

$$\underbrace{\langle 0,5,1\rangle}_{\text{2nd row}} \cdot \underbrace{\langle 10,1,0\rangle}_{\text{1st column}} = 0(10) + 5(1) + 1(0) = \underbrace{5}_{\text{2,1 entry}}.$$

Matrix multiplication corresponds to composition of the linear transformations. More explicitly, if $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ corresponds to a matrix $A$, and $\mathbf{g}\colon \mathbb{R}^p \to \mathbb{R}^n$ corresponds to a matrix $B$, then the composition $\mathbf{f} \circ \mathbf{g}$ (i.e., the function $\mathbf{f}(\mathbf{g}(\mathbf{x}))$) corresponds to $AB$. That's why matrix multiplication is defined the way it is.

*Warning* 3.6. Even when $AB$ and $BA$ both make sense, they might be unequal. (In other words, matrix multiplication is not commutative.)

3.3. **Identity matrix.** The identity function on $\mathbb{R}^3$ is $\mathbf{f}(x,y,z) := (x,y,z)$. In terms of column vectors:

$$\mathbf{f}\begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} 1x + 0y + 0z \\ 0x + 1y + 0z \\ 0x + 0y + 1z \end{pmatrix}.$$

The corresponding matrix

$$I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is called the $3 \times 3$ identity matrix. (You can guess what the $n \times n$ identity matrix looks like.) It acts like the number 1:

$$IA = A \qquad \text{and} \qquad AI = A$$

whenever the multiplication makes sense.

3.4. **Inverse matrices.**

3.4.1. *Motivation: solving linear systems.* To solve $3x = 5$, multiply both sides by $3^{-1}$.

Similarly, one way to solve

$$2x_1 + 3x_2 = 4$$
$$4x_1 + 5x_2 = 6,$$

is to rewrite as

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix},$$

which has the shape $A\mathbf{x} = \mathbf{b}$, and left multiply both sides by $A^{-1}$ to get $\mathbf{x} = A^{-1}\mathbf{b}$.

3.4.2. *Definition.*

**Definition 3.7.** The inverse of an $n \times n$ matrix $A$ is another $n \times n$ matrix $A^{-1}$ such that

$$AA^{-1} = I \qquad \text{and} \qquad A^{-1}A = I.$$

*It exists if and only if* $\det A \neq 0$.

If $A$ corresponds to the linear transformation $\mathbf{f}$, then $A^{-1}$ corresponds to the inverse function $\mathbf{f}^{-1}$ (if it exists).

3.4.3. *Calculation.*

(1) For each $i, j$, calculate the $i, j$ minor, i.e., the determinant obtained by crossing out the row and column containing $a_{ij}$. These minors are the entries of a matrix.
(2) Change the signs by the checkerboard pattern to get the cofactor matrix.
(3) Take the transpose to get the adjoint matrix.
(4) Multiply by the scalar $\frac{1}{\det A}$.

**Example 3.8.** Suppose that $A = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{pmatrix}$. Cross out row 1 and column 2 and take the determinant to get the $1, 2$ minor: $\begin{vmatrix} 0 & 1 \\ 2 & 5 \end{vmatrix} = 0 - 2 = -2$. Do the same for each $i, j$ to get *all* nine minors; put them in a matrix:

$$\text{matrix of minors:} \qquad \begin{pmatrix} 2 & -2 & -2 \\ 1 & -6 & -4 \\ -1 & 0 & 0 \end{pmatrix}.$$

Change signs to get

$$\text{cofactor matrix:} \qquad \begin{pmatrix} 2 & 2 & -2 \\ -1 & -6 & 4 \\ -1 & 0 & 0 \end{pmatrix}.$$

Take the transpose:

$$\text{adjoint:} \qquad \operatorname{adj} A = \begin{pmatrix} 2 & -1 & -1 \\ 2 & -6 & 0 \\ -2 & 4 & 0 \end{pmatrix}.$$

We find $\det A = -2$ (e.g., by Laplace expansion along first column). Multiply by $\frac{1}{\det A}$:

$$\text{inverse matrix:} \qquad A^{-1} = \frac{1}{-2} \begin{pmatrix} 2 & -1 & -1 \\ 2 & -6 & 0 \\ -2 & 4 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1/2 & 1/2 \\ -1 & 3 & 0 \\ 1 & -2 & 0 \end{pmatrix}.$$

3.5. **Equations of planes.**

**Question 3.9.** What does the set of vectors perpendicular to $\langle 1, 2, 3 \rangle$ look like?

Solution: It's a plane through the origin. Its equation is

$$\langle 1, 2, 3 \rangle \cdot \langle x, y, z \rangle = 0,$$

which is

$$x + 2y + 3z = 0.$$

The vector $\mathbf{n} := \langle 1, 2, 3 \rangle$ is called a normal vector to the plane. (Normal is another word for perpendicular.)

**Question 3.10.** What is the plane with normal vector $\langle 1, 2, 3 \rangle$ passing through $(4, 5, 6)$?

Solution: A point $(x, y, z)$ lies on this plane if the vector from $(4, 5, 6)$ to $(x, y, z)$ (not the position vector of $(x, y, z)$!) is perpendicular to $\langle 1, 2, 3 \rangle$, so its equation is

$$\langle 1, 2, 3 \rangle \cdot (\langle x, y, z \rangle - \langle 4, 5, 6 \rangle) = 0,$$

which is

$$(x - 4) + 2(y - 5) + 3(z - 6) = 0.$$

## 4. February 14

**Question 4.1.** (Followup question) What is the distance from $(2, 3, 5)$ to that plane?

Solution: If we choose any point on the plane, say $(4, 5, 6)$, and form the vector

$$\mathbf{v} := \langle 2, 3, 5 \rangle - \langle 4, 5, 6 \rangle = \langle -2, -2, -1 \rangle$$

between the two points, then the desired distance is *not* the length of $\mathbf{v}$, because the straight line path from $(2, 3, 5)$ to $(4, 5, 6)$ is not the shortest path from $(2, 3, 5)$ to the plane. Instead we want "the amount of $\mathbf{v}$ in the direction parallel to the normal vector $\mathbf{n} := \langle 1, 2, 3 \rangle$", taking the absolute value if necessary. Thus the desired distance is the *absolute value* of the scalar component of $\mathbf{v}$ in the direction of $\mathbf{n}$. That scalar component is

$$\begin{aligned}\operatorname{comp}_{\mathbf{n}} \mathbf{v} &= \mathbf{v} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \\ &= \frac{\langle -2,-2,-1\rangle \cdot \langle 1,2,3\rangle}{\sqrt{1^2+2^2+3^2}} \\ &= \frac{-2-4-3}{\sqrt{14}} \\ &= -\frac{9}{\sqrt{14}},\end{aligned}$$

so the distance is $9/\sqrt{14}$.

**Question 4.2.** Are the vector $\langle -5,1,1\rangle$ and the plane $x+2y+3z=6$

(1) parallel,
(2) perpendicular,
(3) both,
(4) or neither?

Hint: $\langle -5,1,1\rangle \cdot \langle 1,2,3\rangle = 0$.

Answer: The vector $\langle -5,1,1\rangle$ is perpendicular not to the plane, but to a normal vector of the plane. So it is parallel to the plane. The vectors perpendicular to the plane are the scalar multiples of the normal vector, so $\langle -5,1,1\rangle$ is not like this. So the answer is (1).

### 4.1. **Square systems of linear equations.**

#### 4.1.1. *Homogeneous systems.* The system

$$\begin{aligned} x + 2y + 3z &= 0 \\ 8x \qquad + 4z &= 0 \\ 7x + 6y + 5z &= 0 \end{aligned}$$

is of the form $A\mathbf{x} = \mathbf{0}$, where $A$ is a $3\times 3$ matrix, and $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. It is called homogeneous because the right sides are all 0.

Solving $A\mathbf{x} = \mathbf{0}$ for any square $A$:

- If $\det A \neq 0$, then $\mathbf{x} = \mathbf{0}$ is the only solution. (Proof: Multiply both sides by $A^{-1}$.)
- If $\det A = 0$, then there are infinitely many solutions.

Geometric interpretation, in the $3\times 3$ case: For any $3\times 3$ matrix $A$, the solution set to $A\mathbf{x} = \mathbf{0}$ is the intersection of 3 planes passing through $\mathbf{0}$ (assuming that no row of $A$ is all 0).

- If $\det A \neq 0$, the intersection is only $\mathbf{0}$ (this is what usually happens).
- If $\det A = 0$, the intersection is either a line or a plane (through $\mathbf{0}$).

4.1.2. *General systems.* A general square system has the form $A\mathbf{x} = \mathbf{b}$.

For any square $A$:

- If $\det A \neq 0$, then $\mathbf{x} = A^{-1}\mathbf{b}$ is the only solution. (Proof: Multiply both sides by $A^{-1}$.)
- If $\det A = 0$, there are either infinitely many solutions or no solutions.

If $A$ is $3 \times 3$, the solution set is still an intersection of 3 planes, but they may be shifted away from $\mathbf{0}$.

4.2. **Lines.** Two ways to describe lines in $\mathbb{R}^3$:

- intersection of two planes
- parametric equation (to be discussed today)

Think of the trajectory of an airplane moving at constant velocity. Let $\mathbf{r}_0$ be the position vector of the airplane at time $t = 0$. Let $\mathbf{v}$ be the velocity.

Where is the airplane at time $t = 1$? Answer: $\mathbf{r}_0 + \mathbf{v}$.

At time $t = 2$? Answer: $\mathbf{r}_0 + 2\mathbf{v}$.

In general, at time $t$ it is at

$$\mathbf{r}(t) := \mathbf{r}_0 + t\mathbf{v}.$$

This is a vector-valued function of $t$. Each real number $t$ gives one point on the line, and as $t$ varies, these points trace out the whole line.

**Example 4.3.** Line $L$ through $(1, 2, 3)$ and $(4, 1, 3)$? Use initial position vector $\mathbf{r}_0 := \langle 1, 2, 3\rangle$ and velocity vector $\mathbf{v} := \langle 4, 1, 3\rangle - \langle 1, 2, 3\rangle = \langle 3, -1, 0\rangle$ so that at time $t = 1$ the point reaches $(4, 1, 3)$. So $L$ is given by

$$\mathbf{r} = \langle 1, 2, 3\rangle + t\langle 3, -1, 0\rangle.$$

Parametric equations of $L$ in terms of coordinates:

$$x = 1 + 3t, \qquad y = 2 - t, \qquad z = 3.$$

**Question 4.4.** The lines

$$x = 1 + 3t, \qquad y = 2 - t, \qquad z = 3$$

and

$$x = 2t, \qquad y = -1 + t, \qquad z = 1 + t$$

(1) are the same,
(2) are parallel,
(3) intersect in one point, or
(4) are skew (i.e., do not intersect, but are not parallel either)?

Answer: The velocity vectors $\langle 3, -1, 0\rangle$ and $\langle 2, 1, 1\rangle$ are not scalar multiples of each other, so the lines are not the same, and are not parallel. They intersect if the system

$$\begin{aligned} 1 + 3t &= 2s \\ 2 - t &= -1 + s \\ 3 &= 1 + s. \end{aligned}$$

is solvable. Why did we use a different variable on the right side? Imagine airplanes moving along the lines. If we used the same $t$ on both sides, a solution would be a time when both airplanes are at the same place. If we use $t$ on the left and $s$ on the right, a solution would mean that airplane #1 at some time is at the same point as airplane #2 at a different time, meaning that their *paths* still cross, and this is what we're trying to test! OK, let's now solve the system. The last equation implies $s = 2$. The first equation then implies $t = 1$. These values make all three equations true. So the lines intersect at the point $\mathbf{r}_1(1) = \mathbf{r}_2(2) = \langle 4, 1, 3\rangle$.

4.3. **Parametric equations of curves.** As $t$ ranges through all real numbers,

$$x = 2\cos t, \qquad y = \sin t$$

describes

1) A circle
2) An ellipse
3) A line of slope 1/2
4) A point.

Answer: Without the 2, it would be the unit circle. The 2 stretches it in the $x$-direction, to make an ellipse. It is given by the implicit equation

$$x^2 + 4y^2 = 4.$$

To find the implicit equation, one must eliminate $t$ from the parametric equations; how to do this depends on the shape of the parametric equations, and may require some guesswork. In this problem, we know that $\sin^2 t + \cos^2 t = 1$, and this can be rewritten as $y^2 + (x/2)^2 = 1$ in which $t$ does not appear, which is equivalent to $x^2 + 4y^2 = 4$. The parametric equations and the implicit equation are completely different ways of describing the same curve: the parametric equations tell you the curve one point at a time, one point for each value of $t$; the implicit equation gives you a test for when a given point in $\mathbb{R}^2$ lies on the curve.

Slope of tangent line to this ellipse at a given time? Use the chain rule

$$\underbrace{\frac{dy}{dt}}_{\text{known}} = \underbrace{\frac{dy}{dx}}_{???} \underbrace{\frac{dx}{dt}}_{\text{known}}$$

to get

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-2\sin t} = -\frac{1}{2}\cot t.$$

The following will be explained in recitation on Monday.

What is $\frac{d^2y}{dx^2}$? It's

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{\frac{1}{2}\csc^2 t}{-2\sin t} = -\frac{1}{4\sin^3 t}.$$

## 5. February 15

### 5.1. **Parametric equations of curves: an example.**

**Example 5.1.** Did Exercise 1I-4 about the roll of tape centered at $(0,0)$ and with end initially at $(a,0)$. Parameter: the radian measure $\theta$ of the amount of tape unwound so far. Length of tape unwound so far is $a\theta$. Position of point where the unwound tape meets the roll: $(a\cos\theta, a\sin\theta)$. Final answer:

$$\begin{aligned}\mathbf{r} &= \langle a\cos\theta, a\sin\theta\rangle + a\theta\langle\cos\theta, \sin\theta\rangle \\ &= \langle a(1+\theta)\cos\theta, a(1+\theta)\sin\theta\rangle.\end{aligned}$$

### 5.2. **Derivative of a vector-valued function.**

5.2.1. *Definition and physical interpretation.*

**Definition 5.2.** The derivative of a vector-valued function $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) := \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

If $\mathbf{r}(t)$ is the position of a moving particle at time $t$, then $\mathbf{r}'(t)$ is its velocity at time $t$.

5.2.2. *Calculating the derivative.* Derivatives can be calculated coordinate-wise:

**Theorem 5.3.** *If*

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j},$$

*then*

$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}.$$

This holds because all the ingredients used in the definition of derivative (vector subtraction, scalar multiplication by $1/h$, and limits) can be calculated coordinate-wise. It's the same in 3D.

Advice: Use the definition to understand the derivative physically, but use the theorem to calculate it.

5.3. **Integration of vector-valued functions.**

Similarly, the definite integral

$$\int_a^b \mathbf{r}(t)\,dt := \lim_{\Delta t\to 0} \sum_{i=1}^{n} \mathbf{r}(t_i^*)\,\Delta t$$

can be computed coordinate-wise: if

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j},$$

then

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j}.$$

Also, there is a fundamental theorem of calculus for vector-valued functions: if $\mathbf{r}(t) = \mathbf{R}'(t)$, then

$$\underbrace{\int_a^b \mathbf{r}(t)\,dt}_{\text{integral of a rate of change}} = \underbrace{\mathbf{R}(b) - \mathbf{R}(a)}_{\text{total change}}.$$

It follows from the scalar function version.

5.4. **Acceleration.** Recall: If $\mathbf{r}(t)$ is the position of an object at time $t$, then its velocity at time $t$ is $\mathbf{v}(t) := \mathbf{r}'(t)$. Next, its acceleration at time $t$ is $\mathbf{a}(t) := \mathbf{v}'(t) = \mathbf{r}''(t)$. All of these are vector-valued functions.

On the other hand, its speed is $|\mathbf{v}(t)|$; this is a scalar function whose values are nonnegative.

**Question 5.4.** What does it mean if a particle's position vector $\mathbf{r}(t)$ satisfies $\frac{d|\mathbf{r}|}{dt} = 0$ for all $t$?

Answer: The *length* of the position vector is constant, so the particle is staying on a sphere centered at the origin.

5.5. **Arc length.** Let $s(t)$ be the scalar function that gives the arc length of the trajectory from a starting point $\mathbf{r}(a)$ to a variable end point $\mathbf{r}(t)$. Imagine that $dt$ is a tiny amount of time. If it is really tiny, then we can imagine that the speed $|\mathbf{v}|$ doesn't change much during that time interval from time $t$ to time $t+dt$, so the distance traveled is approximately $|\mathbf{v}|\,dt$. Imagine that $ds$ means the change in $s(t)$ during the same time interval. Then $ds$ is the distance traveled, so

$$ds = |\mathbf{v}|\,dt.$$

(The reason this explanation is in magenta is that $ds$ and $dt$ are not actually numbers. It would be more correct to say that $\Delta s \approx |\mathbf{v}|\,\Delta t$, and then to divide by $\Delta t$ and take a limit.)

In any case, mathematically what this means is that

$$\frac{ds}{dt} = |\mathbf{v}|.$$

So, by the fundamental theorem of calculus,

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du.$$

(The $u$ in the integration can be thought of as a variable number increasing from $a$ up to $t$; we couldn't just use $t$, because the name $t$ is already being used, as the upper limit of the integration.)

Think: total distance traveled equals sum of distance traveled over each time subinterval between $a$ and $t$.

5.6. **Unit tangent vector.** Suppose that $\mathbf{v}(t) \neq \mathbf{0}$. Then $\mathbf{v}(t)$ is a tangent vector to the curve.

**Definition 5.5.** The unit tangent vector at $\mathbf{r}(t)$ is the vector

$$\mathbf{T} := \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}.$$

Then

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \mathbf{T}|\mathbf{v}| = \mathbf{T}\,\frac{ds}{dt}.$$

One writes "$d\mathbf{r} = \mathbf{T}\, ds$".

5.7. **Foci of an ellipse.** (This section is just to help make sense of Kepler's first law, and to help with one of the homework problems.)

One way to write down an ellipse is to write an equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Here $a$ is half the width, and $b$ is half the height.

But if you were to ask an ancient Greek what an ellipse is, the answer would be:

> Fix two points $P$ and $Q$ and fix a number $\ell$ greater than $PQ$. Then the locus (possible positions) of a point $R$ such that $PR + RQ = \ell$ is an ellipse.

The points $P$ and $Q$ are the foci (plural of focus) of the ellipse.

Another property of the foci: Inside an elliptical room with mirror walls, if you place a light at one focus, the rays will reflect and meet again at the other focus. (Same for sound.)

5.8. **Kepler's second law.** In the early 1600s, Johannes Kepler noticed that Tycho Brahe's data on planetary motion was consistent with three laws:

(1) The orbit of a planet is an ellipse with the sun at one focus.
(2) A planet moves in a plane containing the sun, and the line segment connecting the sun to the planet sweeps out area at a constant rate.
(3) The square of the period of revolution of a planet about the sun is proportional to the cube of the major semiaxis of its elliptical orbit.

Another law: "Gravitational force $\mathbf{F}$ is central": In mathematical terms, the planet's acceleration vector $\mathbf{a}$ is always parallel to the vector from the sun to the planet.

**Theorem 5.6.** *Kepler's second law is equivalent to the law saying that the acceleration is central.*

*Proof.* Let the origin be where the sun is. Let $\mathbf{r} = \mathbf{r}(t)$ be the position vector of the planet. Let $A = A(t)$ be the area swept out from some starting time until $t$.

Between time $t$ and $t + \Delta t$,

$$\Delta A \approx \text{Area(triangle)} = \frac{1}{2}|\mathbf{r} \times \Delta\mathbf{r}|.$$

Divide by $\Delta t$ to get a rate, and take the limit as $\Delta t \to 0$:

$$\frac{dA}{dt} = \frac{1}{2}|\mathbf{r} \times \frac{d\mathbf{r}}{dt}| = \frac{1}{2}|\mathbf{r} \times \mathbf{v}|.$$

Suppose that Kepler's second law holds: $\frac{dA}{dt}$ is constant. Then $|\mathbf{r} \times \mathbf{v}|$ is constant. But the direction of $\mathbf{r} \times \mathbf{v}$ is also constant since it is perpendicular to the plane of motion (and by continuity cannot suddenly switch to the opposite direction). Thus $\mathbf{r} \times \mathbf{v}$ is constant. So

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{0}.$$

On the other hand, by a rule for differentiating a cross product (from the textbook reading for today),

$$\begin{aligned} \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) &= \frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mathbf{r} \times \frac{d\mathbf{v}}{dt} \qquad \text{(important: keep the } \mathbf{r} \text{ and } \mathbf{v} \text{ in order)} \\ &= \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} \\ &= \mathbf{r} \times \mathbf{a}. \end{aligned}$$

So

$$\mathbf{r} \times \mathbf{a} = \mathbf{0}.$$

This means that $\mathbf{a}$ is parallel to $\mathbf{r}$.

The converse, that the acceleration being central implies Kepler's second law, can be proved by reversing the steps of the previous paragraph. □

## 6. February 21

Midterm #1 on Tues. Feb. 26, 11am-12 in 50-340 (upstairs in Walker Memorial, not here!) It covers everything up to what we do today (level curves, partial derivatives, tangent plane, tangent plane approximation) but not Friday's topic (max/min). You don't need to know Kepler's laws. Practice midterms are available on Stellar; ignore the max/min problems in them for now.

Homework #3 due Thurs. Mar. 7 at 11 A.M. in room 2-255. But start the problems labelled Feb. 21 *now* (such problems may appear on the midterm), and come to office hours if you aren't sure that you're doing them correctly.

6.1. **Graphs and level curves of two-variable functions.** Three ways to depict a 2-variable function $f(x, y)$:

(1) Map of its values: At many points $(x, y)$ in the plane, write the value $f(x, y)$. For example, if $f(x, y) := y(y + 1)/2 - x + 10$, then the values at integer points are

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 30 | 29 | 28 | 27 | 26 | 25 | 24 | 23 | 22 | 21 | 20 |
| 25 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 |
| 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 |
| 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 |
| 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 |
| 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 |
| 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 |
| 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 |
| 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 |
| 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 |
| 25 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 |

(the colored entry is $f(0, 0)$).

(2) Graph: Above each point $(x, y)$ in the plane, plot a point in $\mathbb{R}^3$ whose $z$-coordinate is the value $f(x, y)$. Taken together, these points form a surface in $\mathbb{R}^3$ called the graph of $f$. It is the set of points in space satisfying the equation $z = f(x, y)$.

(3) Level curves: For each number $h$, the level curve at height $h$ is the set of points $(x, y)$ in the $xy$-plane such that $f(x, y) = h$. In the example above, each level curve is a parabola.

**Question 6.1.** Consider $f(x, y) = x^2 + y^2$. Its graph is a paraboloid. Draw level curves for equally spaced values of $h$; these are circles in $\mathbb{R}^2$ centered at $(0, 0)$, namely $x^2 + y^2 = h$. Then as one goes out, are the circles

(1) getting closer together
(2) occurring at equally spaced radii
(3) getting farther apart?

Answer: Getting closer together. The farther out you are, the steeper the paraboloid is, so the shorter you have to go horizontally to get a fixed increase in height.

6.2. **Partial derivatives.**

6.2.1. *Introduction via an example.*

Here is a map showing values of a function $f(x,y)$ at integer points, with the colored value at $(0,0)$:

| | | | | | |
|---|---|---|---|---|---|
| 20 | 19 | 18 | 17 | 16 | 15 |
| 16 | 16 | 16 | 16 | 16 | 16 |
| 12 | 13 | 14 | 15 | 16 | 17 |
| 8 | 10 | 12 | 14 | 16 | 18 |
| 4 | 7 | 10 | 13 | 16 | 19 |

If one starts at $(0,0)$ and moves to the right, the value increases by 2 for each increase of $x$ by 1; thus the rate of change, denoted $\frac{\partial f}{\partial x}(0,0)$, equals 2. (Note: $\frac{\partial f}{\partial x}$ is often pronounced "partial $f$ partial $x$".) Similarly, $\frac{\partial f}{\partial y}(0,0)=3$, and $\frac{\partial f}{\partial x}(2,-1)=3$. Since $\frac{\partial f}{\partial x}$ has potentially different values at different points, it is itself a function defined on the plane.

6.2.2. *Definition.*

**Definition 6.2.** The partial derivative of $f(x,y)$ with respect to $x$ is a function $\frac{\partial f}{\partial x}$ whose value at $(x_0,y_0)$ is

- the rate of change of $f(x,y)$ when $x$ is varying near $x=x_0$ and $y$ is held constant at the value $y_0$, or,
- more precisely,

$$\frac{\partial f}{\partial x}(x_0,y_0) := \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} \qquad \text{(if the limit exists).}$$

Other notations:

$$f_x(x_0,y_0), \qquad \left(\frac{\partial f}{\partial x}\right)_0.$$

The 0 subscript on the derivative means "evaluate me at $(x_0,y_0)$".

What is the difference between $\frac{df}{dx}$ and $\frac{\partial f}{\partial x}$? The first notation is used if $f$ is a function $f(x)$ of $x$ alone. The second notation is used if $f$ is a function of several variables but we are measuring the rate of change of $f$ arising from a change in only the variable $x$.

Define $\frac{\partial f}{\partial y}$ similarly.

6.2.3. *How to compute* $\frac{\partial f}{\partial x}$.

View $y$ as a constant and differentiate with respect to $x$. (And then evaluate at $(x_0,y_0)$ if desired.) Example: If $f(x,y)=x^3y^5$, then $f_x=3x^2y^5$ and $f_x(2,1)=12$.

**Question 6.3.** Let $f(x,y)=x^y$ for $(x,y)$ in the half-plane $x>0$. What is $\frac{\partial f}{\partial y}$?

Possible answers:

(1) $x^y \ln x$
(2) $yx^{y-1}$
(3) 0
(4) None of the above

Answer: (1). Computing $\dfrac{\partial f}{\partial y}$ is like computing

$$\frac{d}{dy}2^y = \frac{d}{dy}e^{y\ln 2} = e^{y\ln 2}\ln 2 = 2^y \ln 2$$

except with a "constant" $x$ in place of the 2. So the answer is $x^y \ln x$.

More notation: to calculate the second partial derivative

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right) = \frac{\partial^2 f}{\partial y\, \partial x},$$

first take the $x$-derivative of $f$, and then take the $y$-derivative of the result. The other second partial derivatives are $f_{xx}$, $f_{yx}$, and $f_{yy}$. For example,

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}.$$

For most functions arising in practice, including any function for which all the second partial derivatives are continuous, $f_{xy} = f_{yx}$. So usually you don't have to worry about the *order* in which you take derivatives.

6.3. **The tangent plane (to the graph of a 2-variable function).** (Below, numbers are colored green to distinguish them from variables.)

Recall: To find the tangent line at a point $(x_0, y_0)$ on the curve $y = f(x)$:

(1) compute the function $f'(x)$
(2) plug in $x = x_0$ to get a number $f'(x_0)$ (it's going to be the slope)
(3) write down the equation of the line through $(x_0, y_0)$ with slope $f'(x_0)$ (point-slope form):

$$\underset{\text{equation of tangent line}}{y - y_0 = f'(x_0)(x - x_0).}$$

Equivalently:

$$\underbrace{y = f(x_0) + \left(\frac{df}{dx}\right)_0 (x - x_0)}_{\text{call this } \ell(x)}.$$

Then $\ell(x)$ is a linear polynomial (of the form $mx + b$) such that $\ell(x_0) = f(x_0)$ and $\ell'(x_0) = f'(x_0)$ (same value and same derivative at $x_0$).

If $x$ is close to $x_0$, then

$$\underset{\text{height of graph of } f}{f(x)} \approx \underset{\text{height of tangent line}}{\ell(x)}$$

so

$$f(x) \approx \underbrace{f(x_0) + \left(\frac{df}{dx}\right)_0 (x - x_0)}_{\text{tangent line approximation}}.$$

If we move $f(x_0)$ to the left side, this can be rewritten as

$$\Delta f \approx \left(\frac{df}{dx}\right)_0 \Delta x.$$

Now let $f(x,y)$ be a 2-variable function. Let $(x_0, y_0, z_0)$ be a point on the graph of $f(x,y)$ (so $z_0 = f(x_0, y_0)$). To find the tangent plane there:

(1) compute the functions $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

(2) plug in $x = x_0$ and $y = y_0$ to get two numbers $\left(\frac{\partial f}{\partial x}\right)_0$ and $\left(\frac{\partial f}{\partial y}\right)_0$.

(3) write down the equation

$$z - z_0 = \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0)$$

Note: This plane is rigged so as to pass through $(x_0, y_0, z_0)$. It is also rigged to have the same rate of increase as the graph of $f(x,y)$ as one moves to the right or up starting from $(x_0, y_0)$.

**Question 6.4.** What function is the tangent plane the graph of?

Answer: Solve for $z$ to get the equation of the plane in the form $z = \ell(x,y)$; this gives

$$\ell(x,y) = f(x_0, y_0) + \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0).$$

(We also replaced $z_0$ by $f(x_0, y_0)$.)

The following example was not done in class.

**Question 6.5.** Suppose that $f(x,y) = xy^2 - 5y$. What is the tangent plane to the graph of $f$ at $(2,3,3)$?

Solution: We compute

$$\begin{aligned}\frac{\partial f}{\partial x} &= y^2 \\ \frac{\partial f}{\partial y} &= 2xy - 5 \\ \left(\frac{\partial f}{\partial x}\right)_0 &= 9 \\ \left(\frac{\partial f}{\partial y}\right)_0 &= 7, \\ z_0 &= f(2,3) = 3\end{aligned}$$

so the tangent plane is

$$z - 3 = 9(x-2) + 7(y-3).$$

6.4. **The tangent plane approximation.** If $(x, y)$ is close to $(x_0, y_0)$,

$$\text{height of graph} \approx \text{height of the tangent plane}$$

$$f(x,y) \approx f(x_0,y_0) + \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0).$$

Equivalent formulation:

$$\begin{aligned}\Delta f &\approx \left(\frac{\partial f}{\partial x}\right)_0 \Delta x + \left(\frac{\partial f}{\partial y}\right)_0 \Delta y \\ &\approx f_x\,\Delta x + f_y\,\Delta y\end{aligned}$$

(the partial derivatives $f_x$ and $f_y$ should be evaluated at $(x_0, y_0)$) The formula above tells you approximately *how much* $f$ changes in response to changes in $x$ and $y$.

The approximation can be expected to be reasonably good if $(x, y)$ is close to $(x_0, y_0)$, and the functions $f_x$ and $f_y$ are continuous in a neighborhood of $(x_0, y_0)$, so that they don't change too suddenly.

**Question 6.6.** A point $P$ in $\mathbb{R}^2$ is near $(-12, 5)$, but its coordinates could be off by as much as 0.1 each. How far is $P$ from the origin? Estimate the maximum error.

There was not enough time in class to complete the solution below.

Solution: Let $f(x,y) = \sqrt{x^2+y^2}$. Then $f_x = \frac{x}{\sqrt{x^2+y^2}}$ and $f_y = \frac{y}{\sqrt{x^2+y^2}}$, so

$$\begin{aligned}f(x,y) &\approx f(-12,5) + f_x(-12,5)\,\Delta x + f_y(-12,5)\,\Delta y \\ &\approx 13 - \frac{12}{13}\,\Delta x + \frac{5}{13}\,\Delta y,\end{aligned}$$

which is 13 with maximum error of absolute value

$$\approx \frac{12}{13}(0.1) + \frac{5}{13}(0.1) \approx 0.13.$$

(The worst error occurs when $(\Delta x, \Delta y) = (-0.1, 0.1)$ or $(\Delta x, \Delta y) = (0.1, -0.1)$.)

MIDTERM 1 COVERS UP TO HERE

## 7. February 22

### 7.1. Maximum and minimum.

#### 7.1.1. *Critical points.*

**Definition 7.1.** A critical point for $f(x, y)$ is a point $(a, b)$ such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

**Question 7.2.** Which of the following is true, if $f(x, y)$ has a critical point at $(a, b)$?

(1) $f(x, y)$ is a constant function
(2) The tangent plane to the graph of $f(x, y)$ at $(a, b, f(a, b))$ is horizontal.
(3) There is no tangent plane to the graph of $f(x, y)$ at $(a, b, f(a, b))$.

*Answer:* (2).

The function $f(x, y)$ doesn't have to be constant; for example $x^2 + y^2$ has a critical point at $(0, 0)$, but it isn't constant.

Tangent plane at the point above $(a, b)$:

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b),$$

If $(a, b)$ is a critical point, then $f_x(a, b)$ and $f_y(a, b)$ are 0, so the tangent plane is $z - f(a, b) = 0$, which is horizontal (its height $z$ is constant).

**Question 7.3.** What are the critical points of $f(x, y) := x^4 + y^3 - 3y$?

Solution: Compute the partial derivatives, and set them equal to 0:

$$4x^3 = 0$$
$$3y^2 - 3 = 0.$$

Solving this system gives $x = 0$ and $y = \pm 1$. So the critical points are $(0, 1)$ and $(0, -1)$.

#### 7.1.2. *Global and local extrema.*

**Definition 7.4.** The function $f(x, y)$ has a global maximum at the point $(a, b)$ if $f(x, y) \leq f(a, b)$ for all points $(x, y)$ in the domain of $f$ (i.e., for all points $(x, y)$ where $f(x, y)$ is defined).

Global minimum is similar.

**Example 7.5.** If $f(x, y) = (x - y)^2 + 5$, then every point along the line $x = y$ is a global minimum.

**Definition 7.6.** The function $f(x,y)$ has a local maximum at the point $(a,b)$ if $f(x,y) \le f(a,b)$ at all points $(x,y)$ *sufficiently near* $(a,b)$ (but maybe there are other points farther away from $(a,b)$ where the value of $f$ is even larger).

Local minimum is similar.

Every global max is automatically a local max.

Drew a level curve diagram, waved the chalk over it, and asked the students to yell whenever it crossed a local min or local max.

**Theorem 7.7.** *Every local max (or local min) is a critical point, assuming that the partial derivatives exist at the point being tested.*

*Proof.* The point has to be a local max for the partial functions obtained by plugging a number into one of the variables, so by one-variable calculus, the derivatives of the partial functions are 0 there if they exist. □

**Question 7.8.** True or false: Every critical point of a function $f(x,y)$ is either a local min or a local max. (Hint: What happens for functions of 1 variable?)

*Answer:* False. Here are two counterexamples.

(1) $f(x,y) := x^3$ has a critical point at $(0,0)$, but it is not a local max (because there are nearby points to the right where $f(x,y) > 0$), and not a local min (because there are nearby points to the left where $f(x,y) < 0$).

| | | | | | | |
|---|---|---|---|---|---|---|
| −27 | −8 | −1 | 0 | 1 | 8 | 27 |
| −27 | −8 | −1 | 0 | 1 | 8 | 27 |
| −27 | −8 | −1 | 0 | 1 | 8 | 27 |
| −27 | −8 | −1 | 0 | 1 | 8 | 27 |
| −27 | −8 | −1 | 0 | 1 | 8 | 27 |
| −27 | −8 | −1 | 0 | 1 | 8 | 27 |
| −27 | −8 | −1 | 0 | 1 | 8 | 27 |

(2) $f(x,y) := x^2 - y^2$ has a critical point at $(0,0)$, but it is neither a local max nor a local min, because there are nearby points (on one axis or the other) where the value is larger or smaller than 0.

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | −5 | −8 | −9 | −8 | −5 | 0 |
| 5 | 0 | −3 | −4 | −3 | 0 | 5 |
| 8 | 3 | 0 | −1 | 0 | 3 | 8 |
| 9 | 4 | 1 | 0 | 1 | 4 | 9 |
| 8 | 3 | 0 | −1 | 0 | 3 | 8 |
| 5 | 0 | −3 | −4 | −3 | 0 | 5 |
| 0 | −5 | −8 | −9 | −8 | −5 | 0 |

7.1.3. *Solving unconstrained max/min problems.*

As a warmup for the general method, consider the following 1-variable examples:

- The global min of $f(x) := x^2$ on $\mathbb{R}$ occurs at $x = 0$ where the derivative $f'(x) = 2x$ becomes zero.
- The function $f(x) := |x|$ on $\mathbb{R}$ has a global min at $x = 0$, where $f'(x)$ is undefined.
- Consider the function $f(x) := x^3 - 3x$ on $\mathbb{R}$. Its derivative $f'(x) = 3x^2 - 3$ becomes 0 at $x = \pm 1$. But neither $x = -1$ nor $x = 1$ is a global max; in fact, $f(x) \to +\infty$ as $x \to +\infty$, so a global max does not exist.
- Consider $f(x) := x^3 - 3x$ again, but this time *restricted to the interval* $[-10, 10]$. As before, $f'(x) = 0$ if and only if $x = \pm 1$. We have $f(-1) = 2$ and $f(1) = -2$, but the global max on $[-10, 10]$ actually occurs on the boundary of the interval, where $f(10) = 970$.

General method for solving a max/min problem (without constraint equations):

1. Identify the function $f(x, y)$ to be maximized.
2. Identify the domain $\mathcal{R}$ on which $f$ is to be maximized.
   - Easy case: only constraint inequalities or no constraints, so that the dimension of $\mathcal{R}$ equals the number of variables, which for this function is 2. (Example: constraint inequalities $x, y > 0$ defining the first quadrant.)
   - Hard case: constraint equation(s), so that the dimension of $\mathcal{R}$ is less than the number of variables. (Example: constraint equation $x^2 + y^2 = 1$ defining the unit circle, which is of dimension only 1.)

   Assume that we are in the easy case. (In the hard case, a more sophisticated method is needed: *Lagrange multipliers*.)
3. Check all of the following to find potential maxima:

   A. critical points in $\mathcal{R}$ (points where $f_x = 0$ and $f_y = 0$ simultaneously)

   B. points in $\mathcal{R}$ where $f_x$ or $f_y$ is undefined

   C. behavior at $\infty$ (what happens to $f$ as $(x, y) \to \infty$ in $\mathcal{R}$?)

   D. boundary behavior (if there are constraint inequalities) — this may lead to another, lower-dimensional max/min problem.

   The global max $(a, b)$ is to be found among these points, so evaluate $f$ at these points to find the maximum value. (*Warning*: If the values of $f$ become larger and larger as $(x, y)$ approaches the boundary or $\infty$, then the global max *does not exist.*)

The method also works for finding max/min of functions in 3 variables on a 3-dimensional region, and so on.

Terminology: the global maximum is the location $(a, b)$, but the maximum value is $f(a, b)$. In any max/min problem, you will need to determine which is being asked for.

**Example 7.9.** Find the point on the surface $xyz^2 = 2$ closest to the origin.

*Solution:* There is a constraint equation (3 variables, but only 2-dimensional domain), but fortunately we can eliminate $z$ by solving for $z$.

Exploit symmetry: The surface lies in the regions where $xy > 0$, which means that $x, y > 0$ or $x, y < 0$. Because of the symmetries

$$(x, y, z) \mapsto (-x, -y, z)$$
$$(x, y, z) \mapsto (x, y, -z)$$

that preserve the equation of the surface, it's enough to consider the parts with $x, y > 0$ and $z > 0$: this is the set of points of the form

$$\left(x, y, \sqrt{\frac{2}{xy}}\right)$$

for $x, y > 0$.

1. What function do we want to minimize? Shortcut: The point where distance is minimized is the same as the point where distance$^2$ is minimized; let's use distance$^2$ since it has a simpler formula, namely
$$f(x, y) := x^2 + y^2 + \frac{2}{xy}.$$
2. What is the region $\mathcal{R}$? Since we are considering $x, y > 0$ only, $\mathcal{R}$ is the first quadrant. (It's 2-dimensional, so we don't need Lagrange multipliers.)
3. A. Critical points: solve
$$f_x = 2x - \frac{2}{x^2 y} = 0$$
$$f_y = 2y - \frac{2}{xy^2} = 0.$$
These lead to
$$x^3 y = 1$$
$$xy^3 = 1,$$
which imply $x^3 y = xy^3$, and we may divide by $xy$ to get $x^2 = y^2$, so $x = y$ or $x = -y$. Since $x, y > 0$, we must have $x = y$. Then $x^4 = 1$, and $x > 0$ so $x = 1$. Thus the only critical point is $(x, y) = (1, 1)$.
B. Points in $\mathcal{R}$ where $f_x$ or $f_y$ is undefined: none. (The points where $x = 0$ or $y = 0$ are not part of $\mathcal{R}$.)
C. Behavior as $(x, y) \to \infty$: as $x$ or $y$ grows, the function $f(x, y)$ tends to $+\infty$ (because of the $x^2$ and $y^2$ terms in $f(x, y)$), so it's not approaching a minimum out there.
D. Boundary behavior: As $x \to 0$ (from the right, while $y$ is bounded), the function $f(x, y)$ tends to $+\infty$ (because of the $\frac{2}{xy}$ term in $f(x, y)$), so it's not approaching a minimum there. Same if $y \to 0$.

Conclusion: $f(x, y)$ is minimized at $(x, y) = (1, 1)$, and $z = \sqrt{2}$ there, so the point is $(1, 1, \sqrt{2})$. The other symmetric points are $(1, 1, -\sqrt{2})$ and $(-1, -1, \sqrt{2})$ and $(-1, -1, -\sqrt{2})$. (And the minimum *value* of the distance is $\sqrt{D(1, 1)} = \sqrt{1 + 1 + 2} = 2$.)

The following good question came from a student:

**Question 7.10.** The point $(1, 1, \sqrt{2})$ we computed seems to be the point with minimum height, closest to the $xy$-plane (among those on the piece of the surface above $\mathcal{R}$), but how do we know that there isn't another point on the surface that is even closer to the origin?

Answer: The function we minimized was not the $z$-coordinate of a point on the surface, but the squared distance to the origin, so the point we computed is truly the one closest to the origin.

Another way of saying this: we found the lowest point on the graph $z = f(x, y)$ above $\mathcal{R}$, but this graph is different from the piece of the original surface, which is given by $z = \sqrt{\frac{2}{xy}}$.

If we had wanted the point closest to the $xy$-plane, we would have instead minimized the $z$-coordinate, which is given by the function $\sqrt{\frac{2}{xy}}$; in that case, the minimum does not exist since $\sqrt{\frac{2}{xy}}$ can be made arbitrarily close to 0 by taking $x$ and $y$ to be large positive numbers.

## 8. February 28

8.1. **Least squares interpolation.** Problem: Given data points $(x_1, y_1), \ldots, (x_n, y_n)$ that approximately lie on an unknown line $y = ax + b$, find the line.

What are the unknowns here? $a$ and $b$!

Given a candidate line $y = ax + b$, how do we measure how good of an approximation it is? For each input $x_i$, the line predicts an output of $ax_i + b$, but the actual output was $y_i$, so the error in the prediction is $|y_i - (ax_i + b)|$. Then the total error from all the data points would be

$$\sum_{i=1}^{n} |y_i - (ax_i + b)|,$$

and we want to find $a, b$ that make this small. Because of the absolute values, the partial derivatives of this function do not exist everywhere, which complicates the minimization problem, so instead we try to minimize the sum of the *squares* of the errors.

**Definition 8.1.** The least squares line (the "best" line) is the one for which

$$D := \sum_{i=1}^{n} (y_i - (ax_i + b))^2$$

is minimum.

This $D$ is a function $D(a,b)$.

The minimum occurs where $\frac{\partial D}{\partial a}$ and $\frac{\partial D}{\partial b}$ are both 0. Instead of expanding out $D$, use the chain rule:

$$\frac{\partial D}{\partial a} = \sum_{i=1}^{n} 2(y_i - (ax_i + b))(-x_i) = 0$$
$$\frac{\partial D}{\partial b} = \sum_{i=1}^{n} 2(y_i - (ax_i + b))(-1) = 0.$$

This is a system of two linear equations in $a$ and $b$ (remember: the $x_i$ and $y_i$ are given numbers). Solving for $a$ and $b$ gives the best line.

The same method can be used to approximate data points by the graph of an unknown quadratic function $y = ax^2 + bx + c$.

How do you know what kind of function to use? Maybe the source of the data suggests that a particular shape of function is the right answer. Or maybe the data themselves look as if they can be fitted with a parabola, or ....

If you suspect that a function $y = cx^d$ is best, plot $\ln x$ versus $\ln y$ so that the relationship is linear again:

$$\ln y = C + d \ln x,$$

where $C := \ln c$ and $d$ are constants to be solved for. In other words, find the line that best approximates the points $(\ln x_i, \ln y_i)$; this gives $C$ and $d$.

8.2. **Second derivative test for two-variable functions.** Examples of critical point behavior at $(0,0)$:

- $x^2 + y^2$ has local min
- $-x^2 - y^2$ has local max
- $x^2 - y^2$ has saddle point
- $x^3$ has none of the above

**Example 8.2.** $f(x,y) = x^2 + 6xy + 11y^2$. Complete the square: it's $(x+3y)^2 + 2y^2$, so it has a local min at $(0,0)$.

More generally: consider $f(x,y) = \frac{1}{2}Ax^2 + Bxy + \frac{1}{2}Cy^2$. (Reason for the $\frac{1}{2}$: it makes

$$f_{xx} = A, \qquad f_{xy} = f_{yx} = B, \qquad f_{yy} = C$$

at $(0,0)$.) Let's suppose that $A \neq 0$. Completing the square rewrites the function as

$$f(x,y) = \frac{A}{2}\left(x + \frac{B}{A}y\right)^2 + \left(\frac{AC - B^2}{2A}\right) y^2.$$

| Case | Conclusion |
|---|---|
| $AC - B^2 > 0$ and $A > 0$ | local min |
| $AC - B^2 > 0$ and $A < 0$ | local max |
| $AC - B^2 < 0$ | saddle point |
| $AC - B^2 = 0$ | inconclusive |

"Inconclusive" means that it could be anything: a local min, a local max, a saddle point, or something weirder.

The formula $AC - B^2$ can be remembered as a determinant:

$$AC - B^2 = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \text{ evaluated at } (a, b).$$

Even more generally:

**Second derivative test:** Suppose that $f(x, y)$ has a critical point at $(a, b)$ (i.e., $f_x(a, b) = 0$ and $f_y(a, b) = 0$), and that all second derivatives exist and are continuous in a neighborhood of $(a, b)$. Define

$$A := f_{xx}(a, b), \qquad B := f_{xy}(a, b), \qquad C := f_{yy}(a, b).$$

Then the type of the critical point is given by the table above. This works even when $A = 0$.

*Warning* 8.3. It's not OK to use the second derivative test when there is a constraint equation!

8.2.1. *An example.*

**Question 8.4.** What kind of critical point does

$$f(x, y) := xy(x - y) = x^2y - xy^2$$

have at $(0, 0)$?

Calculate all the partial derivatives up to second order:

$$\begin{aligned} f_x &= 2xy - y^2 \\ f_y &= x^2 - 2xy \\ f_{xx} &= 2y \\ f_{xy} = f_{yx} &= 2x - 2y \\ f_{yy} &= -2x. \end{aligned}$$

Since $f_x$ and $f_y$ are 0 at $(0,0)$, there is a critical point there. Next,

$$\begin{aligned} A &:= f_{xx}(0,0) = 0 \\ B &:= f_{xy}(0,0) = 0 \\ C &:= f_{yy}(0,0) = 0. \end{aligned}$$

Since $AC - B^2 = 0$, the second derivative test is inconclusive.

So what next? The lines $x = 0$, $y = 0$ and $x - y = 0$ divide the plane into six regions on which $f(x,y)$ is alternately positive or negative. So $f(x,y)$ has neither a local min nor a local max at $(0,0)$. In fact, it has what is called a *monkey saddle*: there are three negative regions, two for the legs and one for the tail. (A monkey was drawn.)

Challenge: Find a surface that has an octopus saddle!

## 9. March 1

### 9.1. More on the tangent plane approximation.

**Question 9.1.** What is the linear polynomial in the coefficients $a, b$ that best approximates the $x$-coordinate of the solution to the system

$$\begin{aligned} ax + by &= 1 \\ 2x + y &= 0 \end{aligned}$$

for $(a,b)$ near $(7,3)$?

Solution: The variables that are changing are $a$ and $b$, and the function $f(a,b)$ to be approximated is the $x$-coordinate of the solution. We can find $f(a,b)$ explicitly by solving the system: solve the second equation for $y$ to get $y = -2x$, and plug back into the first equation to get $ax + b(-2x) = 1$, so $x = 1/(a-2b)$. Thus

$$\begin{aligned} f(a,b) &:= \frac{1}{a-2b} \\ f_a &= -\frac{1}{(a-2b)^2} \\ f_b &= \frac{2}{(a-2b)^2} \end{aligned}$$

The linear polynomial that best approximates this for $(a,b)$ near $(7,3)$ is

$$\begin{aligned} f(7,3) + f_a(7,3)(a-7) + f_b(7,3)(b-3) &= 1 + (-1)(a-7) + 2(b-3) \\ &= 2 - a + 2b. \end{aligned}$$

So, for example, the solution $x$ to the system

$$\begin{aligned} 7.1x + 3.2y &= 1 \\ 2x + y &= 0 \end{aligned}$$

is going to be approximately

$$1 + (-1)(0.1) + 2(0.2) = 1.3.$$

Approximation can be done for functions of more than two variables too.

**Question 9.2.** What is the best linear polynomial approximation to a 3-variable function $f(x, y, z)$ for inputs near a starting point $(x_0, y_0, z_0)$?

As one moves from $(x_0, y_0, z_0)$ to $(x, y, z)$, the approximation formula says that $\Delta f$ is caused by $\Delta x$, $\Delta y$, $\Delta z$, each magnified by the corresponding partial derivative:

$$\underbrace{\Delta f}_{\text{change in } f} \approx (f_x)_0 \, \Delta x + (f_y)_0 \, \Delta y + (f_z)_0 \, \Delta z.$$

Thus

$$\begin{aligned} f(x, y, z) &= \underbrace{f(x_0, y_0, z_0)}_{\text{starting value}} + \underbrace{\Delta f}_{\text{change in } f} \\ f(x, y, z) &\approx f(x_0, y_0, z_0) + (f_x)_0 \, (x - x_0) + (f_y)_0 \, (y - y_0) + (f_z)_0 \, (z - z_0). \end{aligned}$$

9.2. **Differentials.** Recall: if $u = f(x)$, then one writes $du = f'(x) \, dx$. Here $du$ and $dx$ are differentials: they are not numbers, vectors, or matrices, but instead a new kind of object.

If $f(x, y)$ is a 2-variable function, its total differential is

$$df := f_x \, dx + f_y \, dy.$$

It helps us remember two formulas:

(1) The approximation formula

$$\Delta f \approx f_x \Delta x + f_y \Delta y.$$

(2) The chain rule: if $x = x(t)$ and $y = y(t)$, then we "divide by $dt$" to get

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}.$$

This can also be written as

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

More complicated version: Suppose that $Q = Q(u, v, w)$ where $u = u(x, y)$, $v = v(x, y)$, and $w = w(x, y)$. (Drew dependency diagram as in book.) Then

$$\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial Q}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial Q}{\partial w}\frac{\partial w}{\partial x}$$

and $\dfrac{\partial Q}{\partial y}$ is similar.

9.3. **Using the chain rule to derive the product and quotient rules.**

**Example 9.3** (Product rule)**.** If $f = uv$ where $u = u(x)$ and $v = v(x)$, then

$$\frac{df}{dx} = f_u \frac{du}{dx} + f_v \frac{dv}{dx}.$$

In other words,

$$\frac{d}{dx}(uv) = vu' + uv'.$$

The quotient rule can also be obtained this way.

9.4. **Review: definitions of** cos **and** sin**.** What is the *meaning* of $\cos\theta$ and $\sin\theta$?

**Definition 9.4.** Draw a unit circle centered at the origin. Let $P$ be the point reached by going $\theta$ units counterclockwise from $(1, 0)$. (If $\theta$ is negative, this means going clockwise.) Then

$$\cos\theta := x\text{-coordinate of } P$$

$$\sin\theta := y\text{-coordinate of } P.$$

**Example 9.5.** $\cos\pi = (x\text{-coordinate of } (-1, 0)) = -1$.

9.5. **Review: polar coordinates.** A point $P$ in the plane can be specified in rectangular coordinates $x, y$ or in polar coordinates $r, \theta$. Here $r$ means the distance $OP$ to the origin, and $\theta$ means the angle measured counterclockwise from the positive $x$-axis to the ray $\overrightarrow{OP}$ (the value of $\theta$ is not completely determined by $P$, since one can $2\pi$ as many times as one wants; to make $\theta$ unique, require it to be in a certain range, usually $\theta \in [0, 2\pi)$ or $\theta \in (\pi, \pi])$.

Then

$$\mathbf{P} = \langle x, y \rangle = r\langle \cos\theta, \sin\theta \rangle,$$

and we get the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta.$$

for going from $r, \theta$ to $x, y$. To go the other way, from $x, y$ to $r, \theta$, use

$$r = \sqrt{x^2 + y^2}$$

and then solve for $\theta$ in one of the two equations above, making sure that it specifies a point in the correct quadrant so that the other equation is not off by a sign.

**Question 9.6.** At a time when a particle is at $(4, 3)$ and has velocity vector $\langle 3, -1\rangle$ (in rectangular coordinates), what is $\dfrac{dr}{dt}$?

Solution: View $r = r(x, y)$ and $x = x(t)$, $y = y(t)$. Namely, start with $r = \sqrt{x^2 + y^2}$, and apply the chain rule

$$\frac{dr}{dt} = \frac{\partial r}{\partial x}\frac{dx}{dt} + \frac{\partial r}{\partial y}\frac{dy}{dt}.$$

Here

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2}}\\ \frac{\partial r}{\partial y} &= \frac{1}{2}(x^2+y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2+y^2}},\end{aligned}$$

which at $(4, 3)$ are $4/5$ and $3/5$. Also, at the given time $\langle \frac{dx}{dt}, \frac{dy}{dt}\rangle = \langle 3, -1\rangle$, so

$$\begin{aligned}\frac{dr}{dt} &= \frac{\partial r}{\partial x}\frac{dx}{dt} + \frac{\partial r}{\partial y}\frac{dy}{dt}\\ &= \frac{4}{5}(3) + \frac{3}{5}(-1)\\ &= \frac{9}{5}.\end{aligned}$$

Alternative solution (avoiding square roots): Start with

$$r^2 = x^2 + y^2.$$

Apply $\dfrac{d}{dt}$:

$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

At the given time, $r = 5$, so this becomes

$$2(5)\frac{dr}{dt} = 2(4)(3) + 2(3)(-1),$$

which again leads to

$$\frac{dr}{dt} = \frac{9}{5}.$$

# 10. March 5

## 10.1. Gradient.

**Definition 10.1.** The gradient of a scalar function $f(x, y, z)$ is the vector-valued function

$$\nabla f := \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

It is kind of like a derivative. Similar definition for functions in any number of variables.

**Example 10.2.** Let $f(x, y, z) = x^2 y + 7z$, and let $P = (2, 3, 5)$. What is $\nabla f(P)$?

Solution: $\nabla f = \langle 2xy, x^2, 7\rangle$, so its value at $(2, 3, 5)$ is $\langle 12, 4, 7\rangle$.

We can restate certain theorems and definitions in terms of $\nabla$:

- Chain rule: If $f = f(x, y, z)$ and $\mathbf{r}(t) = (x(t), y(t), z(t))$, then

$$\begin{aligned}\frac{d}{dt} f(\mathbf{r}(t)) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).\end{aligned}$$

- Critical points of $f(x, y)$: These are points $P$ where $\nabla f(P) = \mathbf{0}$.

Geometric property of $\nabla$:

**Theorem 10.3.** *If $\mathbf{r}(t)$ is any parametric curve on which $f$ is constant, then at each time $t$, the vector $\nabla f(\mathbf{r}(t))$ is perpendicular to the tangent vector (velocity vector) $\mathbf{r}'(t)$.*

*Proof.* For some constant $c$,

$$f(\mathbf{r}(t)) = c.$$

Take $\frac{d}{dt}$:

$$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0.$$

So $\nabla f(\mathbf{r}(t))$ is perpendicular to $\mathbf{r}'(t)$. □

In 2D, this says that $\nabla f(P)$ is perpendicular to the level curve $f(x, y) = c$ through $P$.

In 3D, this says that $\nabla f(P)$ is perpendicular to the level surface $f(x, y, z) = c$ through $P$. Proof: $\nabla f(P)$ is perpendicular to the tangent vector to any curve in the level surface.

**Examples 10.4.**

- If $f(x, y, z) = 2x + 3y + 5z$, then at every point $P$ in $\mathbb{R}^3$, the vector $\nabla f = \langle 2, 3, 5\rangle$ is perpendicular to the level surface $2x + 3y + 5z = c$ through $P$ (which is a plane).
- If $f(x, y) = x^2 + y^2$, then $\nabla f = \langle 2x, 2y\rangle$, which points out from the origin, perpendicular to the level curves (which are circles).

10.2. **Tangent plane.** Before we talked about the tangent plane to a graph $z = f(x, y)$. Now we can talk about the tangent plane to almost any surface:

**Definition 10.5.** The tangent plane to the surface $f(x, y, z) = c$ at $P = (x_0, y_0, z_0)$ is the plane with normal vector $\nabla f(P)$ through $P$:

$$\frac{\partial f}{\partial x}(P)(x - x_0) + \frac{\partial f}{\partial y}(P)(y - y_0) + \frac{\partial f}{\partial z}(P)(z - z_0) = 0.$$

(This makes sense only if $\nabla f(P) \neq 0$.)

**Example 10.6.** If $f(x, y, z) = x^2+y^2-z^2$ then $\nabla f = \langle 2x, 2y, -2z\rangle$, so the cone $x^2+y^2-z^2 = 0$ has a tangent plane at each point except $(0, 0, 0)$. At $P = (3, 4, 5)$, the tangent plane is

$$6(x - 3) + 8(y - 4) - 10(z - 5) = 0.$$

10.3. **Directional derivatives.** Recall: Given $f(x, y)$, the partial derivatives $f_x$ and $f_y$ measure the rate of change of $f$ as one moves in the direction of $\mathbf{i}$ or $\mathbf{j}$.

More generally:

**Definition 10.7.** Let $\mathbf{u}$ be any unit vector. The directional derivative of $f(\mathbf{x})$ in the direction $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}} f(\mathbf{x}) &:= \left.\frac{d}{ds} f(\mathbf{x} + s\mathbf{u})\right|_{s=0} \\ &= \lim_{s\to 0} \frac{f(\mathbf{x} + s\mathbf{u}) - f(\mathbf{x})}{s}. \end{aligned}$$

This is because $\mathbf{r}(s) := \mathbf{x} + s\mathbf{u}$ describes the position of a point starting at $\mathbf{x}$ and moving at speed 1 in the direction $\mathbf{u}$.

Why $s$, and not $t$? Answer: speed is 1, so distance traveled $s$ equals time $t$.

Alternative notation: $\left.\frac{df}{ds}\right|_{\mathbf{u}}(P)$. Sometimes also $\left.\frac{df}{ds}\right|_P$ or just $\frac{df}{ds}$ (this presumes that $P$ and $\mathbf{u}$ are understood).

**Question 10.8.** If $f(x, y) := x^2 + y^2$ and $\mathbf{x} = \langle 6, 8\rangle$, in which directions $\mathbf{u}$ is the directional derivative maximum?

Possible answers:

(1) $\mathbf{i}$
(2) $\mathbf{j}$
(3) $(3/5, 4/5)$
(4) $(-3/5, -4/5)$
(5) $(4/5, -3/5)$ or $(-4/5, 3/5)$.
(6) None of these.

**Theorem 10.9** (Formula for calculating directional derivatives)**.**

$$D_{\mathbf{u}}f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}.$$

*Proof.*

$$\begin{aligned} D_{\mathbf{u}}f(\mathbf{x}) &= \frac{d}{ds} f(\mathbf{x}+s\mathbf{u})\bigg|_{s=0} \\ &= \nabla f(\mathbf{x}+s\mathbf{u}) \cdot \frac{d}{ds}(\mathbf{x}+s\mathbf{u})\bigg|_{s=0} \\ &= \nabla f(\mathbf{x}) \cdot \mathbf{u}. \end{aligned}$$

□

Given $f$ and $\mathbf{x}$, we maximize $D_{\mathbf{u}}f(\mathbf{x})$ by choosing $\mathbf{u}$ to point in the direction of $\nabla f(\mathbf{x})$. And if $\mathbf{u}$ is so chosen, then $D_{\mathbf{u}}f(\mathbf{x}) = |\nabla f(\mathbf{x})|$, by the geometric interpretation of the dot product of two vectors forming a zero angle. Hence:

Geometric interpretation of $\nabla f$:

- direction of $\nabla f$ = direction in which $f$ is increasing the fastest (perpendicular to level curve/surface)
- length of $\nabla f$ = directional derivative of $f$ in that direction

**Example 10.10.** Back to $f(x,y) := x^2 + y^2$ and $P = (6,8)$. We have $\nabla f(x,y) = \langle 2x, 2y\rangle$, which at $P$ is $\langle 12, 16\rangle$, which is 20 times the unit vector $\mathbf{u} := \langle 3/5, 4/5\rangle$. So the function is increasing the fastest in the direction of $\langle 3/5, 4/5\rangle$ (and its rate of increase in that direction is 20). So the answer to Question 10.8 is (3).

**Question 10.11.** In Question 10.8, in which directions is the directional derivative 0?

Solution: In the direction $\mathbf{v} := \langle 4/5, -3/5\rangle$ (tangent to the level curve), we have

$$\nabla f(\mathbf{x}) \cdot \mathbf{v} = 0,$$

so the directional derivative $D_{\mathbf{v}}f(P)$ is 0. The same thing happens in the direction $-\mathbf{v} = \langle -4/5, 3/5\rangle$. So the answer to Question 10.11 is (5).

## 11. March 7

### 11.1. Constraint equations vs. constraint inequalities.

(1) Find the minimum value of $f(x,y) := 2x^2 - 7xy$ subject to the constraint inequalities $x \geq 1$, $y \geq 1$, and $x + y \leq 4$.

This is a standard max/min problem with boundaries to check.

(2) Find the minimum value of $f(x,y,z) := x^2+y^2+z^2$ subject to the constraint equation $3x+5y+z=9$.

Eliminate $z$ to get an equivalent problem: Find the minimum value of $F(x,y) := x^2+y^2+(9-3x-5y)^2$ with no constraint equation.

(3) Find the minimum value of $f(x,y,z) := x^2+y^2$ subject to the constraint equation $e^{x+y} = xy+2$.

When there is a constraint equation (and you can't or don't want to eliminate it), you need Lagrange multipliers!

### 11.2. **Lagrange multipliers.**

Lagrange multipliers — a method for finding max/min of $f(x,y)$ when $x$ and $y$ are required to satisfy a constraint $g(x,y) = c$ (the "hard" case):

1. Identify the function $f(x,y)$ to be maximized (or minimized).
2. Identify the constraint equation $g(x,y) = c$ and constraint inequalities that define the domain $\mathcal{R}$ of $f$. Usually the presence of the constraint equation means that the dimension of $\mathcal{R}$ is one less than the number of variables (but constraint inequalities do not reduce the dimension).
3. Compute $\nabla f$ and $\nabla g$.
4. Solve the system

$$g = c$$
$$\nabla f = \lambda \, \nabla g$$

   in $(x,y,\lambda)$ to find the possible pairs $(x,y)$ (we don't care about the value of $\lambda$, so a good strategy is to eliminate it).
5. Check also
   A. points on $g = c$ where $\nabla g = 0$
   B. points on $g = c$ where $f$ or $g$ is not differentiable
   C. behavior at $\infty$ (what happens to $f$ as $(x,y)$ approaches $\infty$ *along the constraint curve* $g = c$?)
   D. boundary points (if there are constraint inequalities)

   The global max $(a,b)$ is to be found among these points from steps 5 and 5, so evaluate $f$ at these points to find the maximum value.

   (*Warning*: If the values of $f$ become larger and larger as $(x,y)$ approaches the boundary or $\infty$, then the global max *does not exist*.)

Sample problem: Find the highest point on the intersection of the cylinder $x^2+y^2 = 13$ and the plane $2x+3y-z = 8$ in $\mathbb{R}^3$.

*Steps 1 and 2: Identify the function $f(x, y)$ to be maximized/minimized, and the constraint equation $g(x, y) = c$. (Also find any constraint inequalities: these will be useful later, to determine what boundaries need to be checked.)*

We want to maximize $z = 2x + 3y - 8$ subject to the constraint $x^2 + y^2 = 13$. So take $f(x, y) := 2x + 3y - 8$ and $g(x, y) := x^2 + y^2$. No constraint inequalities.

*Step 3: Compute $\nabla f$ and $\nabla g$.*

We get $\nabla f = \langle 2, 3\rangle$ and $\nabla g = \langle 2x, 2y\rangle$.

*Step 4: Solve the system*

$$\begin{aligned} g &= c \\ \nabla f &= \lambda \nabla g \end{aligned}$$

*in $(x, y, \lambda)$ to find the possible pairs $(x, y)$ (we don't care about the value of $\lambda$, so a good strategy is to eliminate it).*

The system to be solved is

$$\begin{aligned} x^2 + y^2 &= 13 \\ 2 &= \lambda(2x) \\ 3 &= \lambda(2y). \end{aligned}$$

Multiply the second equation by $y$ and the third equation by $x$ and equate to get

$$2y = 3x.$$

Solve for $y$ and substitute into the first equation:

$$\begin{aligned} x^2 + \left(\frac{3}{2}x\right)^2 &= 13 \\ \frac{13}{4}x^2 &= 13 \\ x^2 &= 4 \\ x &= \pm 2. \end{aligned}$$

So we get $(x, y) = (2, 3)$ or $(-2, -3)$.

*Step 5: Check also*

A. *points on $g = c$ where $\nabla g = 0$*
   No such points, since $\nabla g = 0$ only at $(0, 0)$, which is not on $x^2 + y^2 = 13$.
B. *points on $g = c$ where $f$ or $g$ is not differentiable*
   No such points.
C. *behavior at $\infty$ (what happens to $f$ as $(x, y)$ approaches $\infty$ along the constraint curve $g = c$?)*
   Not applicable — the constraint curve is bounded.

D. *boundary points (if there are constraint inequalities)*
Not applicable.

So the only points to check are $(2, 3)$ and $(-2, -3)$. We have $f(2, 3) = 5$, and $f(-2, -3) = -21$, so the maximum is at $(x, y) = (2, 3)$. There, $z = f(2, 3) = 5$. So the highest point is $(2, 3, 5)$.

Why does this work? Why should $\nabla f$ be a multiple of $\nabla g$ at a maximum (when $\nabla g \neq 0$)?

Answer: First, the vector $\nabla g$ is perpendicular to the level curve $g = c$. Claim: $\nabla f$ is perpendicular to $g = c$ too. Reason: If not, then the directional derivative of $f$ in one of two directions along $g = c$ would be positive, so one could increase $f$ by moving in that direction, meaning that we weren't really at a maximum.

Since $\nabla f$ and $\nabla g$ are both perpendicular to the level curve, $\nabla f$ must be a multiple of $\nabla g$ (if $\nabla g \neq 0$).

Lagrange multipliers apply also to max/min of functions of more than 2 variables.

(Lagrange multipliers also can be used when there is more than one constraint — this is discussed in the textbook, but we won't study such problems in this course.)

The second derivative test cannot be used when there is a constraint equation.

Sample problem 2: Find the point on the surface $x^2 + y^2 = (z-1)^3$ closest to the origin.

Solution: (We are going to make some mistakes below in red, and then correct them in green.) We want to minimize $f(x, y, z) := x^2 + y^2 + z^2$ subject to the constraint $g = 0$ where $g(x, y, z) := x^2 + y^2 - (z-1)^3$. We need to solve the Lagrange multiplier system

$$\begin{aligned} x^2 + y^2 - (z-1)^3 &= 0 \\ \langle 2x, 2y, 2z \rangle &= \lambda \langle 2x, 2y, -3(z-1)^2 \rangle, \end{aligned}$$

which, when written out in components, says

$$\begin{aligned} x^2 + y^2 - (z-1)^3 &= 0 \\ 2x &= \lambda(2x) \\ 2y &= \lambda(2y) \\ 2z &= \lambda(-3(z-1)^2). \end{aligned}$$

Either the second or third equation implies $\lambda = 1$. Substituting $\lambda = 1$ in the last equation leads to

$$\begin{aligned} 2z &= -3(z-1)^2 \\ 3(z-1)^2 + 2z &= 0 \\ 3z^2 - 4z + 3 &= 0 \end{aligned}$$

but $4^2 - 4(3)(3) < 0$, so there are no solutions.

Oops, we divided by either $x$ or $y$ to obtain $\lambda = 1$, so that was valid only if $x \neq 0$ or $y \neq 0$. Thus to finish finding all solutions to the Lagrange multiplier system, We also need to consider the case in which $x = y = 0$. Then the constraint equation implies $z = 1$, and substituting this into the last of the four equations in the Lagrange multiplier system says

$$2 = \lambda(0)$$

which is impossible. So there is no minimum.

Oops, we forgot to check points where $\nabla g = 0$. This says

$$\langle 2x, 2y, -3(z-1)^2 \rangle = \langle 0, 0, 0 \rangle,$$

which leads to $(x, y, z) = (0, 0, 1)$. (The other parts of Step 5 do not give anything additional.) By the geometry, there has to be a minimum somewhere, and $(0, 0, 1)$ is the only candidate, so $(0, 0, 1)$ is the closest point.

The surface could be sketched by considering the slices obtained by intersecting with horizontal planes $z = c$.

Make sure that you read the new notes on non-independent variables.

## 12. March 8

Topic for today: non-independent variables (partial derivatives of functions of variables related by constraint equations).

### 12.1. Constraint equations and dimension.

Part of the specification of a function is its domain, the set of inputs on which it is being considered. Usually, each constraint equation reduces the dimension of the domain by 1. This leads to the following:

> Rule of thumb: Usually, if a domain defined by $e$ constraint equations in $n$ variables, it will be $(n - e)$-dimensional.

**Example 12.1.** The constraint equation

$$x + 2y + 3z = 5$$

defines a 2-dimensional domain (a plane).

**Example 12.2.** The constraint equations

$$x^2 + y^2 + z^2 = 100$$
$$x + 2y + 3z = 5$$

define a 1-dimensional domain (a circle, arising as the intersection of a sphere and a plane).

*Warning* 12.3. It is not *always* true that each constraint equation reduces the dimension by 1. (See the new notes on non-independent variables.)

Constraint inequalities usually do not affect the dimension.

### 12.2. Independent variables.

**Example 12.4.** On the upper half of the sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$ one can express $z$ in terms of $x, y$, namely

$$z = \sqrt{1 - x^2 - y^2}.$$

Here we think of $x, y$ as independent variables, and $z$ depends on $x, y$. On the lower half one would use

$$z = -\sqrt{1 - x^2 - y^2}.$$

instead.

Dimension can be understood as the number of independent variables.

> Rule of thumb, rewritten: If a domain is defined by $e$ constraint equations in $n$ variables, then
>
> $$\boxed{\text{dimension} = \#\text{ independent variables} \overset{\text{usually}}{=} n - e.}$$
>
> In this situation, then one can locally choose $n - e$ variables to be the independent ones so that each of the remaining $e$ variables can be expressed as a function of the $n - e$ independent variables.

The precise mathematical statement along these lines is called the implicit function theorem; it is discussed in more advanced math courses.

### 12.3. Derivatives.

**Question 12.5.** Consider the function $f(x, y) := xy$ where $(x, y)$ is constrained to lie on the line $2x + y = 7$. What is $\dfrac{df}{dx}$ at the point $(3, 1)$?

(We wrote $\dfrac{df}{dx}$ instead of $\dfrac{\partial f}{\partial x}$ because there is only one independent variable: $y$ depends on $x$.)

*Incorrect solution:* The derivative of $xy$ with respect to $x$ is $y$, whose value at $(3, 1)$ is 1.

What makes this wrong? It is true that if $f(x, y) := xy$ on $\mathbb{R}^2$, then $\dfrac{\partial f}{\partial x} = y$. But the definition of partial derivative assumes that it makes sense to hold $y$ constant while varying $x$, which is impossible if $(x, y)$ is required to satisfy the constraint $2x+y = 7$. If $x$ is varying, then $y$ is not constant.

*Correct solution 1 (elimination of dependent variable):* How many independent variables? Answer: $2-1=1$. To compute $\frac{df}{dx}$, we need $x$ to be the independent variable. Now use the constraint equation to eliminate the dependent variable $y$ and express everything in terms of $x$:

$$\begin{aligned} y &= 7-2x \\ f &= x(7-2x) = 7x-2x^2 \\ \frac{df}{dx} &= 7-4x, \end{aligned}$$

and the value of $\frac{df}{dx}$ at $(x,y)=(3,1)$ is $7-4(3)=-5$. □

In Correct Solution 1, we were lucky that it was easy to solve for $y$ in terms of $x$. In more complicated situations, this might not be possible, but one can still determine how quickly $y$ changes as $x$ changes, by taking the differential of the constraint equation. To see how this works, let's solve the same problem again.

*Correct solution 2 (differentials):* If $f=xy$ is viewed as a function on $\mathbb{R}^2$, the definition of $df$ gives

$$df = y\,dx + x\,dy. \tag{1}$$

This expresses how $f$ changes as $x$ and $y$ change.

If $f$ is restricted to a function on the domain defined by the constraint equation, then (1) still holds, but now any change in $x$ causes a change in $y$, so $dx$ and $dy$ are related. To find the relation, *take the differential of the constraint equation* $2x+y=7$; this gives

$$2\,dx + dy = 0$$

so

$$dy = -2\,dx$$

(which makes sense since $(x,y)$ is constrained to lie on the line $2x+y=7$ of slope $-2$). To compute $\frac{df}{dx}$, we want to consider $f$ as a function of the independent variable $x$ alone, so we should express $df$ in terms of $dx$ alone. To eliminate the $dy$ term, substitute $dy=-2\,dx$ into (1) to get

$$\begin{aligned} df &= y\,dx + x(-2\,dx) \\ &= (y-2x)\,dx. \end{aligned}$$

This means that

$$\frac{df}{dx} = y-2x.$$

At $(3,1)$, this is $1-2(3)=-5$. □

12.4. **Partial derivatives.** Now let's consider what happens in a question like Question 12.5 when there is more than one independent variable.

**Question 12.6.** Consider the function $f(x, y, z) := x + y + x^2 z$ where $(x, y, z)$ is constrained to lie on the surface $xyz = 6$. What is $\dfrac{\partial f}{\partial x}$ at the point $(1, 2, 3)$?

Answer: In the presence of the constraint equation $xyz = 6$, the notation $\dfrac{\partial f}{\partial x}$ is meaningless, so the question does not make sense!

Here is why: Usually $\dfrac{\partial f}{\partial x}$ means the rate of change of $f$ as $x$ varies while all the other variables are held constant. But we can't hold both $y$ and $z$ constant while varying $x$, if we want the constraint equation $xyz = 6$ to remain true.

Conclusions:

1. We are allowed to talk about partial derivatives of $f$ only if $f$ is expressed as a function of *independent* variables (independence guarantees that we can vary one variable while holding the others constant).
2. If $f$ is initially expressed in terms of variables satisfying constraint equations, *we must choose some of the variables to be the independent ones*, and view $f$ and all other variables as functions of the independent variables before talking about partial derivatives of $f$. The notation for the partial derivatives must indicate which variables are being used as the independent ones.

The notational convention is that all the independent variables are listed at the bottom of the partial derivative notation, with the variables being held constant listed as subscripts outside parentheses:

**Definition 12.7.** The notation

$$\left(\frac{\partial f}{\partial x}\right)_y$$

means that we are viewing $f$ as a function of independent variables $x$ and $y$, and measuring the rate of change of $f$ as $x$ varies while holding $y$ constant.

**Question 12.8.** Suppose that $f(x, y, z) := x + y + x^2 z$, where $x, y, z$ are constrained to lie on the surface $xyz = 6$. What is $\left(\dfrac{\partial f}{\partial x}\right)_y$ at the point $(1, 2, 3)$?

*Correct solution 1 (elimination of dependent variable):* We can use the constraint equation to eliminate $z$ and express $f$ in terms of independent variables $x$ and $y$:

$$f = x + y + x^2\left(\frac{6}{xy}\right) = x + y + \frac{6x}{y}.$$

Then

$$\left(\frac{\partial f}{\partial x}\right)_y = 1 + \frac{6}{y},$$

so $\left(\frac{\partial f}{\partial x}\right)_y$ at $(1, 2, 3)$ is

$$1 + \frac{6}{2} = 4.$$

12.5. **Partial derivatives and differentials.** Now let's solve the same problem using differentials.

*Correct solution 2 (differentials):* To compute $\left(\frac{\partial f}{\partial x}\right)_y$, in which the independent variables are $x$ and $y$, we need to express $df$ in the form

$$df = ?\,dx + ?\,dy,$$

where each ? represents a function; then $\left(\frac{\partial f}{\partial x}\right)_y$ is the first ? (and $\left(\frac{\partial f}{\partial y}\right)_x$ is the second ?).

But $f$ is initially given as a function of dependent variables $x, y, z$. If $f = x + y + x^2 z$ is viewed as a function on $\mathbb{R}^3$, the definition of $df$ gives

$$df = (1 + 2xz)\,dx + dy + x^2\,dz. \tag{2}$$

If $f$ is restricted to a function on the domain defined by the constraint equation, then (2) still holds, but taking the differential of the constraint equation $xyz = 6$ gives a relation between $dx$, $dy$, $dz$:

$$yz\,dx + xz\,dy + xy\,dz = 0. \tag{3}$$

Because we want $df$ in terms of $dx$ and $dy$ only, we solve (3) for $dz$,

$$\begin{aligned} xy\,dz &= -yz\,dx - xz\,dy \\ dz &= -\frac{z}{x}\,dx - \frac{z}{y}\,dy, \end{aligned}$$

and substitute into (2):

$$\begin{aligned} df &= (1 + 2xz)\,dx + dy + x^2\,dz \\ &= (1 + 2xz)\,dx + dy + x^2\left(-\frac{z}{x}\,dx - \frac{z}{y}\,dy\right) \\ &= (1 + 2xz)\,dx + dy - xz\,dx - \frac{x^2 z}{y}\,dy \\ &= (1 + xz)\,dx + \left(1 - \frac{x^2 z}{y}\right)\,dy. \end{aligned}$$

This means that

$$\left(\frac{\partial f}{\partial x}\right)_y = 1 + xz,$$

and the value of $\left(\frac{\partial f}{\partial x}\right)_y$ at $(1,2,3)$ is $1+1(3)=4$.

12.6. **Proving rules concerning partial derivatives.** There are many rules relating different partial derivatives, but they all follow from the method of differentials, so there is no need to memorize the rules. The purpose of this section is not to list rules to be memorized, but to give practice in using the method of differentials.

**Problem 12.9.** The cyclic rule states that if variables $x, y, z$ are related by a constraint equation such that (as expected) any two of the variables may be taken as the independent variables, then

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$

Prove this rule.

*Proof.* Let $f(x,y,z)=0$ be the constraint equation, where $f$ is a function that makes sense on $\mathbb{R}^3$. Taking the differential of the constraint equation gives

$$f_x\,dx + f_y\,dy + f_z\,dz = 0,$$

where $f_x$, $f_y$, $f_z$ are the partial derivatives of $f$ viewed as a function on $\mathbb{R}^3$ (or at least a 3-dimensional part of $\mathbb{R}^3$). Solving for $dx$ gives

$$dx = -\frac{f_y}{f_x}\,dy - \frac{f_z}{f_x}\,dz, \tag{4}$$

which means that when $x$ is viewed as a function of independent variables $y, z$, then

$$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{f_y}{f_x},$$

the coefficient of $dy$ in (4). A similar argument shows that

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{f_z}{f_y}$$
$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{f_x}{f_z},$$

and multiplying all three gives

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = \left(-\frac{f_y}{f_x}\right)\left(-\frac{f_z}{f_y}\right)\left(-\frac{f_x}{f_z}\right) = -1. \qquad \square$$

## 13. March 12

13.1. **Proving rules concerning partial derivatives, continued.** Another example is the two-Jacobian rule. To state it, we need a definition:

**Definition 13.1.** If $u = u(x, y)$ and $v = v(x, y)$, then the Jacobian of $(u, v)$ with respect to $(x, y)$ is the function

$$\frac{\partial(u, v)}{\partial(x, y)} := \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x v_y - v_x u_y.$$

Here $u_x$ means $\left(\frac{\partial u}{\partial x}\right)_y$, and so on.

Two-Jacobian rule: If $u, v, w, x, y$ are related by constraint equations such that any two of the variables may be taken as the independent variables, then

$$\left(\frac{\partial u}{\partial v}\right)_w = \frac{\partial(u, w)/\partial(x, y)}{\partial(v, w)/\partial(x, y)}.$$

(The right side could also be written out as

$$\frac{u_x w_y - w_x u_y}{v_x w_y - w_x v_y},$$

a ratio of determinants.)

How is the two-Jacobian rule used? It says that if one knows the partial derivatives of all the variables with respect to independent variables $x, y$, then one can calculate the partial derivatives using any other variables as the independent ones.

The two-Jacobian rule can be proved using differentials.

13.2. **Non-independent variables, continued: gradient in terms of new variables.**

**Problem 13.2.** Suppose that $u = u(x, y)$, and $v = v(x, y)$. If $u$ is viewed as a function of $x$ and $v$, what is

$$\nabla u := \left\langle \left(\frac{\partial u}{\partial x}\right)_v, \left(\frac{\partial u}{\partial v}\right)_x \right\rangle$$

in terms of $u_x$, $u_y$, $v_x$, $v_y$?

Solution: We need to write

$$du =?\, dx +?\, dv.$$

Given:

$$du = u_x\, dx + u_y\, dy$$

$$dv = v_x\, dx + v_y\, dy.$$

Eliminate $dy$ by solving for it in the second equation, and substituting it into the first:

$$\begin{aligned} du &= u_x\,dx + u_y\left(\frac{dv - v_x\,dx}{v_y}\right) \\ &= \frac{u_x v_y - u_y v_x}{v_y}\,dx + \frac{u_y}{v_y}\,dv. \end{aligned}$$

Thus the gradient of $u$ viewed as function of $x$ and $v$ is

$$\nabla_{x,v} u = \left\langle \frac{u_x v_y - u_y v_x}{v_y}, \frac{u_y}{v_y} \right\rangle.$$

13.3. **Partial differential equations.**

Ordinary differential equation (ODE): An equation involving derivatives of an unknown function of *one* variable.

Example:

$$\frac{d^2 f}{dx^2} - 5\frac{df}{dx} + 6f = 0.$$

Here, $f(x) = e^{2x}$ is one solution.

Partial differential equation (PDE): An equation involving partial derivatives of an unknown function of *several* variables.

Example:

$$\frac{\partial f}{\partial t} = t^2 \frac{\partial^2 f}{\partial x^2} + y\frac{\partial f}{\partial z} + f.$$

(You are supposed to solve for $f = f(x, y, z, t)$.)

PDEs are generally very hard to solve. E.g., Navier-Stokes equations governing fluid flow — there is a \$1,000,000 prize for understanding the solutions.

**Example 13.3** (An easy PDE)**.** Find all $f = f(x, y)$ satisfying

$$\frac{\partial^2 f}{\partial x^2} = 0.$$

(You don't get \$1,000,000 for solving this one.)

Solution:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 0,$$

so the function $\dfrac{\partial f}{\partial x}$ is constant along each horizontal line; i.e., $\dfrac{\partial f}{\partial x} = g(y)$ for some function $g$. So along each horizontal line, $f$ grows at a constant rate (depending on the line). Thus $f(x, y) = g(y)x + h(y)$ for some functions $g$ and $h$.

Conversely, given any functions $g(y)$ and $h(y)$, the function $f(x, y) = g(y)x + h(y)$ is a solution.

To have a unique solution, more information about $f$ would need to be given.

**Example 13.4** (Laplace equation)**.** 1-dimensional case: Imagine an insulated metal rod with ends held at different temperatures. Let $x$ = position and $w(x)$ = steady-state (equilibrium) temperature at position $x$.

Physical heuristic: Once equilibrium is reached, the temperature of each tiny bit of rod should equal the average of the temperatures of the neighboring bits, so $w(x)$ should be a linear polynomial $ax + b$. So $\frac{d^2w}{dx^2} = 0$. (If there were a point where the graph were concave up, the temperature would be increasing there; the reverse for concave down.)

2-dimensional case: Now imagine a metal plate with each point on the boundary held at some temperature.

$$\boxed{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0}.$$

Notation:

$$\begin{aligned}\nabla &:= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \\ \nabla^2 &:= \nabla \cdot \nabla \\ &:= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \text{(Laplace operator)}\end{aligned}$$

Then the Laplace equation can be rewritten as $\nabla^2 w = 0$.

**Example 13.5** (Heat equation)**.** Same rod, not in equilibrium yet. Now $w = w(x, t)$. Then

$$\boxed{\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2}}.$$

for some constant $k$ (depending on how well the rod conducts heat). E.g., if at a given time, the temperature is concave up at a point, then that point will be warmed up by its neighbors.

In 2 dimensions:

$$\boxed{\frac{\partial w}{\partial t} = k\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)}$$

**Example 13.6** (Wave equation)**.** Imagine a rubber band whose shape is the graph of a function $w(x, t)$ (viewed as function of $x$ at a fixed time $t$). Then

$$\boxed{\frac{\partial^2 w}{\partial t^2} = c^2\frac{\partial^2 w}{\partial x^2}}$$

for a constant $c$, which represents the speed at which waves travel. E.g., if at a given time, the graph is concave up at a point, then that piece of rubber band will be *accelerated* upward by the pull of the neighboring pieces.

For this class, you are not required to memorize the named PDEs above, but you should understand what it means for a function to be a solution to a PDE, and you should know how to find all the solutions $f(x, y)$ to simple PDEs such as $\frac{\partial^2 f}{\partial x^2} = 0$. The named PDEs were presented just to give a few examples of how they come up in physics.

## 14. March 14

14.1. **Double integrals.** Let $R$ be a region in $\mathbb{R}^2$, cut into tiny regions $R_1, \ldots, R_n$. Choose $(x_1, y_1)$ in $R_1$, ..., $(x_n, y_n)$ in $R_n$. Then

$$\iint\limits_R f(x, y)\, dA \approx f(x_1, y_1)\, \mathrm{Area}(R_1) + \cdots + f(x_n, y_n)\, \mathrm{Area}(R_n).$$

It's a number. (The actual definition involves a limit as the maximum size of the $R_i$ goes to 0.)

If $f \geq 0$ everywhere on $R$, then $\iint_R f(x, y)\, dA$ can be interpreted as the volume under the graph of $f$ (above $R$).

14.1.1. *Computing double integrals as iterated integrals.*

Suppose that $R$ is a rectangle $[a, b] \times [c, d]$. Partition $[a, b]$ into tiny subintervals, so $R$ gets sliced into thin rectangles, and the volume above it gets sliced into slabs. Then

$$\begin{aligned}\iint\limits_R f(x, y)\, dA &= \int_{x=a}^{x=b} \left( \int_{y=c}^{y=d} f(x, y)\, dy \right) dx \\ &=: \int_a^b \int_c^d f(x, y)\, dy\, dx.\end{aligned}$$

Think of the inner integral (with $x$ treated as a constant) as the area of a slab; multiplying it by the width "$dx$" of a slab gives the volume of the slab and we sum these ("integrate") to get the total volume. (In class, we formed a $5 \times 4$ grid of students, and integrated the height function.)

Similarly,

$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

14.1.2. *Non-rectangular regions.*

**Problem 14.1.** Let $R$ be the bounded region between $y - x = 2$ and $y = x^2$. Find $\iint_R (2x + 4y)\, dA$.

Solution: We'll compute it as $\iint (2x + 4y)\, dy\, dx$. What are the limits of integration?

*Step 1: Sketch the region.* Solving $y - x = 2$ and $y = x^2$ shows that the line and the parabola intersect at $(-1, 1)$ and $(2, 4)$.

*Step 2: The outer integral goes from the smallest $x$-coordinate of a point in $R$ to the largest $x$-coordinate.*

The smallest $x$-coordinate is $-1$ and the largest $x$-coordinate is 2. So the outer integral will look like

$$\int_{x=-1}^{x=2}$$

*Step 3: Hold $x$ fixed, and increase $y$; look at the $y$-values where the line enters and leaves $R$ — usually these depend on $x$. The inner integral goes from the smaller $y$-value to the larger $y$-value.*

It enters at $y = x^2$ and leaves at $y = x + 2$, so the iterated integral is

$$\int_{x=-1}^{x=2} \int_{y=x^2}^{y=x+2} (2x + 4y)\, dy\, dx.$$

Then how do you evaluate the iterated integral?

*Step 4: Evaluate the* inner *integral first, treating $x$ as constant. The result will be a function of $x$.*

It's

$$\begin{aligned} \int_{y=x^2}^{y=x+2} (2x + 4y)\, dy &= 2xy + 2y^2 \Big|_{y=x^2}^{y=x+2} \\ &= (2x(x+2) + 2(x+2)^2) - (2x^3 + 2x^4) \\ &= -2x^4 - 2x^3 + 4x^2 + 12x + 8. \end{aligned}$$

*Step 5: Then evaluate the outer integral, which is just a definite integral of a 1-variable function of $x$.*

Answer:

$$\begin{aligned} \int_{x=-1}^{x=2} \int_{y=x^2}^{y=x+2} (2x + 4y)\, dy\, dx &= \int_{-1}^{2} (-2x^4 - 2x^3 + 4x^2 + 12x + 8)\, dx \\ &= \frac{333}{10}. \end{aligned}$$

14.1.3. *Dividing the region into pieces.* Sometimes to compute a double integral, the region needs to divided into two or more pieces. The integral is then the sum of the integrals over the pieces.

Example: rectangle with a smaller rectangle removed from its center.

14.1.4. *Volume between two surfaces.* If $f(x,y)$ and $g(x,y)$ are functions on a region $R$ and $f(x,y) \geq g(x,y)$ for all $(x,y)$ in $R$, then the volume of the solid between the graphs of $f$ and $g$ is $\iint_R (f(x,y) - g(x,y))\, dA$.

Problem (p. 960, #44): Set up an iterated integral that computes the volume of the region bounded by the surfaces $z = x^2 + 3y^2$ and $z = 4 - y^2$.

Solution: The intersection of the two surfaces is defined by the system

$$z = x^2 + 3y^2$$
$$z = 4 - y^2.$$

The projection to the $xy$-plane of this is obtained by eliminating $z$:

$$x^2 + 3y^2 = 4 - y^2.$$

So $R$ is the region bounded by the ellipse

$$x^2 + 4y^2 = 4.$$

Inside this ellipse the function $4 - y^2$ is larger than $x^2 + 3y^2$ (as you can see by comparing their values at $(0, 0)$). So we want

$$\iint_R ((4 - y^2) - (x^2 + 3y^2))\, dA = \int_{-1}^{1} \int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} (4 - x^2 - 4y^2)\, dx\, dy.$$

(Note: it is OK to write $\int_{-1}^{1}$ instead of $\int_{y=-1}^{y=1}$, since the variable is determined by the matching differential: the integrals are *nested*, so the leftmost $\int$ corresponds to the rightmost differential $dy$, and so on.)

## 15. March 15

**15.1. Double integrals, continued.** What's wrong with the following?

$$\int_2^5 \int_{x+3}^{x+7} (x^2 + 2y)\, dx\, dy$$

Since the outer differential on the right is $dy$, this would mean

$$\int_{y=2}^{y=5} \int_{x=x+3}^{x=x+7} (x^2 + 2y)\, dx\, dy,$$

but the inner limits make no sense. In the inner integral, $y$ is constant, and the range for $x$ could depend on $y$, but the range for $x$ cannot be expressed in terms of $x$!

**15.2. Area in polar coordinates.** Question: The area of the region described by $r \in [3, 3.2]$, $\theta \in [\pi/4, \pi/4 + 0.1]$ is closest to $x/100$ for which integer $x$?

Answer: 6.

Why? In general, the region with polar coordinates in $[r, r + \Delta r]$ and $[\theta, \theta + \Delta\theta]$ is approximately a rectangle with sides $\Delta r$ and $r\Delta\theta$ (the latter is the length of an arc of radius $r$ and measure $\Delta\theta$, by definition of radian). So its area is approximately $r\,\Delta r\,\Delta\theta$, which in our case is $3(0.2)(0.1) = 0.06$.

To remember:

$$\boxed{dA = dx\, dy = r\, dr\, d\theta.}$$

15.3. **Double integrals in polar coordinates.**

**Problem 15.1.** Re-express

$$\int_1^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} x\sqrt{x^2+y^2}\,dx\,dy$$

as an iterated integral in polar coordinates.

Solution: Substitute

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ dx\,dy &= r\,dr\,d\theta \end{aligned}$$

but also re-compute the limits of integration.

The inequalities

$$-\sqrt{2-y^2} \le x \le \sqrt{2-y^2}$$

are equivalent to

$$x^2 \le 2 - y^2,$$

and to

$$x^2 + y^2 \le 2.$$

Thus the region is defined by

$$\begin{aligned} x^2 + y^2 &\le 2 \\ 1 \le y &\le \sqrt{2}. \end{aligned}$$

In the circle of radius $\sqrt{2}$ centered at $(0,0)$, our region is the upper segment obtained by cutting the circle with the chord from $(-1,1)$ to $(1,1)$. (Draw it!)

So $\theta \in [\pi/4, 3\pi/4]$, and the upper limit for $r$ is $\sqrt{2}$, but what is the lower limit for $r$? The inequality $y \ge 1$ becomes $r\sin\theta \ge 1$ in polar coordinates, so $r \ge 1/\sin\theta$.

Final answer:

$$\int_{\pi/4}^{3\pi/4} \int_{1/\sin\theta}^{\sqrt{2}} (r\cos\theta) r\; r\,dr\,d\theta.$$

15.4. **Applications of double integrals.**

15.4.1. *Average value.* Warmup: The average of numbers $x_1, \ldots, x_n$ is

$$\frac{x_1 + \cdots + x_n}{n}.$$

**Definition 15.2.** The average value of $f(x,y)$ on a region $R$ is

$$\frac{\iint_R f\,dA}{\text{Area}(R)}$$

15.4.2. *Mass and centroid.* For a metal plate in the $xy$-plane,

$$\text{mass} = (\underbrace{\text{mass per unit area}}_{\text{density}})(\text{area})$$

But if its density is not constant, $\delta = \delta(x, y)$ (in g/cm$^2$, say), then each piece of area must be multiplied by the density *there*.

**Definition 15.3.** The mass of a 2-dimensional object is

$$m := \iint_R \underbrace{\delta(x, y)\, dA}_{dm}.$$

The $x$-coordinate of the centroid is the average of the $x$-coordinates of the points in the region, *weighted by density*. Same for the $y$-coordinate. So:

**Definition 15.4.** The centroid of a 2-dimensional object is the point $(\bar{x}, \bar{y})$ where

$$\bar{x} := \frac{\iint_R x\, dm}{m} = \frac{\iint_R x\delta(x, y)\, dA}{\iint_R \delta(x, y)\, dA}$$
$$\bar{y} := \frac{\iint_R y\, dm}{m} = \frac{\iint_R y\delta(x, y)\, dA}{\iint_R \delta(x, y)\, dA}.$$

(If density is not specified, assume $\delta(x, y) \equiv 1$.) The centroid is also called the center of mass or the center of gravity.

Sometimes symmetry gives a shortcut. For example, the centroid of an equilateral triangle (of constant density) has reflectional symmetry in each of its three altitudes, so the centroid must lie on all three altitudes, hence at the center. Another example: A parallelogram has 180° rotational symmetry around the point where the two diagonals meet, so that point must be the centroid.

15.4.3. *Moment of inertia.* The moment of inertia of an object with respect to an axis measures how difficult it is to rotate it (around that axis).

For a point mass:

$$I = (\text{distance to axis})^2\, m$$

In general (since different pieces are at different distances to the axis):

$$I = \iint_R (\text{distance to axis})^2\, dm.$$

Special case: The polar moment of inertia of an object in the $xy$-plane is its moment of inertia around the $z$-axis. The distance from $(x, y, z)$ to the nearest point on the $z$-axis (namely, $(0, 0, z)$) is $\sqrt{x^2 + y^2}$, so

$$(\text{distance to axis})^2 = x^2 + y^2$$

in this special case.

**Example 15.5.** Polar moment of inertia of a triangle $R$ of constant density 1 with vertices at $(\pm 3, 0)$ and $(0, 2)$?

Answer: Since density is 1, we have $dm = dA$. The polar moment of inertia is

$$\iint_R (x^2 + y^2)\, dA = \int_0^2 \int_{-(3-3y/2)}^{3-3y/2} (x^2 + y^2)\, dx\, dy = \cdots$$

(We didn't have time to explain how to get the limits of integration, but this is the same as in earlier examples: draw the region, find the range for the outer integration variable $y$, and find the range for $x$ for each fixed value of $y$.)

## 16. March 19

### 16.1. Applications of double integrals, continued.

#### 16.1.1. *Volume of revolution.*

**Theorem 16.1** (First theorem of Pappus)**.** *A plane region $R$ lies on one side of an axis, and is then rotated $360^\circ$ around the axis. Let $A = \text{Area}(R)$, and let $D$ be the distance traveled by the centroid of $R$. Then the solid of revolution has volume $V = AD$.*

Why is this true? Set up a coordinate system so that the axis is the $y$-axis, and $R$ is in the half-plane $x \geq 0$. A little rectangle of area $dA$ sweeps out a ring of volume approximately $2\pi x\, \Delta A$ since the ring can be cut into approximate rectangular parallelepipeds with base $dA$ and heights summing to the length of the ring $(2\pi x)$. Summing over all rectangles and taking a limit as their size goes to 0 yields

$$V = \iint_R 2\pi x\, dA = (2\pi A)\frac{\iint_R x\, dA}{A} = A(2\pi \bar{x}) = AD.$$

### 16.2. Review: gradient and directional derivative.

Recall the geometric interpretation of $\nabla f$:

- direction of $\nabla f$ = direction in which $f$ is increasing the fastest (perpendicular to level curve/surface)
- length of $\nabla f$ = directional derivative of $f$ in that direction

Also recall the formula for computing a directional derivative (starting at $\mathbf{x}$ and moving in the direction $\mathbf{u}$):

$$D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}.$$

**Example 16.2.** Suppose that $f(x, y) = x^2 + 5y$.

*What is $(\nabla f)(6, 1)$ and what does it say geometrically?* Well, $\nabla f = \langle 2x, 5\rangle$, so $(\nabla f)(6, 1) = \langle 12, 5\rangle$. This has length $\sqrt{12^2 + 5^2} = 13$ and direction $\left\langle \frac{12}{13}, \frac{5}{13}\right\rangle$. This says that $f$ is increasing the fastest in the direction of $\langle \frac{12}{13}, \frac{5}{13} \rangle$, and that the directional derivative (rate of change) in that direction is 13.

We have $f(6,1) = 41$. *What is the nearest point to* $(6,1)$ *where* $f$ *takes the value* $41.26$*, approximately?* If from $(6,1)$ we move a distance $\epsilon$ in the direction of fastest increase, the value of $f$ increases by $13\epsilon$, so we need $13\epsilon = 0.26$, so $\epsilon = 0.02$. The point reached from $(6,1)$ by moving 0.02 units in the direction of $\langle \frac{12}{13}, \frac{5}{13} \rangle$ is

$$\langle 6,1 \rangle + 0.02 \left\langle \frac{12}{13}, \frac{5}{13} \right\rangle \approx \langle 6,1 \rangle + 0.02 \langle 1, 1/2 \rangle = \langle 6.02, 1.01 \rangle.$$

(The approximation of $5/13$ by $1/2$ was pretty crude, but good enough for blackboard work.)

*What is the equation of the level curve through* $(6,1)$*?*
The value of $f$ at $(6,1)$ is 41, so the level curve has equation $x^2 + 5y = 41$.

*What is the slope of the level curve at* $(6,1)$*?*
Solution 1: It is perpendicular to $\nabla f$, so the slope is $-12/5$.
Solution 2: The level curve is $y = -\frac{1}{5}x^2 + \frac{41}{5}$, i.e., the graph of $\ell(x) := -\frac{1}{5}x^2 + \frac{41}{5}$, and $\ell'(x) = -\frac{2}{5}x$, so the slope is $\ell'(6) = -\frac{12}{5}$.

*In which direction(s)* $\mathbf{u}$ *is the directional derivative of* $f$ *at* $(6,1)$ *equal to* $56/5$*?*
If $\mathbf{u} = \langle a, b \rangle$, then we need $a^2+b^2 = 1$ (so that $\mathbf{u}$ is a unit vector), and $(\nabla f)(6,1)\cdot\mathbf{u} = 56/5$, which says $12a+5b = 56/5$. Solving this system (many algebra details skipped) leads to two possibilities

$$\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle, \qquad \left\langle \frac{837}{845}, -\frac{116}{845} \right\rangle.$$

*In which direction(s)* $\mathbf{u}$ *is the directional derivative of* $f$ *at* $(6,1)$ *equal to* $14$*?*
Solution: None, because the maximum directional derivative was 13!

*In which direction(s)* $\mathbf{u}$ *is the directional derivative of* $f$ *at* $(6,1)$ *equal to* $-13$*?*
Solution: Since $(\nabla f)(6,1)$ had length 13, the only unit vector $\mathbf{u}$ that dots with it to give $-13$ is the unit vector in the opposite direction, namely $\langle -\frac{12}{13}, -\frac{5}{13} \rangle$.
*What does the graph of* $f$ *look like?*
The biggest slide ever.

16.3. **Review: differentials.** When solving problems involving non-independent variables, one usually needs to take the differential of each constraint equation.

**Problem 16.3.** What is the result of taking the differential of both sides of

$$\ln w + e^{uv} = 7\ ?$$

Solution: Let $L := \ln w + e^{uv}$ (the left side). Then

$$\begin{aligned} L_u &= e^{uv} v \\ L_v &= e^{uv} u \\ L_w &= \frac{1}{w}, \end{aligned}$$

so by definition,

$$dL = ve^{uv}\, du + ue^{uv}\, dv + \frac{1}{w}\, dw.$$

Similarly (but much more easily), applying $d$ to the right side gives

$$0\, du + 0\, dv + 0\, dw = 0.$$

Thus the answer is the equation

$$ve^{uv}\, du + ue^{uv}\, dv + \frac{1}{w}\, dw = 0.$$

16.4. **Review: non-independent variables.**

**Problem 16.4.** Express $\left(\frac{\partial u}{\partial t}\right)_x$ in terms of partial derivatives with respect to independent variables $u$ and $v$.

Solution using differentials: For independent variables $u$ and $v$, everything else is a function of $u$ and $v$, and

$$\begin{aligned} dt &:= t_u\, du + t_v\, dv \\ dx &:= x_u\, du + x_v\, dv. \end{aligned}$$

To get $\left(\frac{\partial u}{\partial t}\right)_x$, re-express $du$ in terms of $dt$ and $dx$:

$$du = ?\, dt + ?\, dx$$

and take the coefficient of $dt$. Lecture actually ended here, because time was up. To do this, pretend that $dt$ and $dx$ are known and $du$ and $dv$ are unknowns, and solve for $du$ by eliminating $dv$. Namely, multiply the first equation by $x_v$ and the second equation by $t_v$, and subtract:

$$\begin{aligned} x_v\,dt &= x_v t_u\,du + x_v t_v\,dv \\ t_v\,dx &= t_v x_u\,du + t_v x_v\,dv \\ x_v\,dt - t_v\,dx &= (x_v t_u - t_v x_u)\,du \\ du &= \frac{x_v}{x_v t_u - x_u t_v}\,dt + (\text{who cares})\;dx \end{aligned}$$

so

$$\left(\frac{\partial u}{\partial t}\right)_x = \frac{x_v}{t_u x_v - t_v x_u}.$$

If the problem gave the function $t(u, v)$ explicitly, then $t_u$ and $t_v$ could be computed as explicit functions of $u$ and $v$. If, moreover, the problem asked for the *value* of $\left(\frac{\partial u}{\partial t}\right)_x$ at a certain numerical point like $(t, u, v, x) = (2, 3, 5, 7)$, then those could be plugged in at the end.

MIDTERM #2 COVERS EVERYTHING UP TO HERE

## 17. March 22

### 17.1. Some quick clarifications.

**Question 17.1.** Which of the following need to be checked when maximizing $f(x, y)$ subject to $g(x, y) = c$ using Lagrange multipliers?

(1) solutions to the system

$$\begin{aligned} g(x, y) &= c \\ \nabla f &= \lambda \nabla g \end{aligned}$$

(2) points where $x = 0$ or $y = 0$
(3) points on $g = c$ where $\nabla f = \mathbf{0}$
(4) points on $g = c$ where $\nabla g = \mathbf{0}$
(5) points where $\nabla f$ does not exist
(6) points where $\nabla g$ does not exist
(7) behavior of $f$ as $(x, y) \to \infty$ along $g = c$
(8) boundary behavior

Answer: All of them except (2) and (3). (The only reason for handling $x = 0$ or $y = 0$ as separate cases would be if you wanted to divide by $x$ or $y$ while solving.)

**Question 17.2.** What are the boundaries of the following domains?

- The disk $x^2 + y^2 \leq 1$.
  Answer: The circle $x^2 + y^2 = 1$.
- The first quadrant, where $x \geq 0$ and $y \geq 0$.
  Answer: The nonnegative parts of the $x$- and $y$-axes.
- The interval consisting of numbers $x$ such that $3 \leq x \leq 5$.
  Answer: The two points $x = 3$ and $x = 5$.
- The circle $x^2 + y^2 = 1$.
  Answer: No constraint inequalities, so no boundary!
- The arc of the circle $x^2 + y^2 = 1$ given by $0 \leq \theta \leq \pi/4$.
  Answer: The points $(1, 0)$ (where $\theta = 0$) and $(\sqrt{2}/2, \sqrt{2}/2)$ (where $\theta = \pi/4$).

General rule of thumb: To find the boundary, take one of the constraint inequalities and change it to $=$; this gives one piece of the boundary. Then do this for each constraint inequality to get all the pieces of the boundary.

In particular, *if there are no constraint inequalities, there is no boundary.*

17.2. **Change of variables in double integrals.** Recall how to do change of variables (substitution) in one-variable integrals; e.g.,

$$\int_{1/2}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = \int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{1-\sin^2 u}}\, (\cos u\, du)\,.$$

where

$$x = \sin u$$
$$dx = \cos u\, du.$$

The boundary points were re-expressed in terms of $u$:

$$x = 1 \quad \text{corresponds to} \quad u = \pi/2$$
$$x = 1/2 \quad \text{corresponds to} \quad u = \pi/6.$$

Today: The 2-variable analogue.

17.2.1. *Transformations.* $(x, y) = \mathbf{f}(u, v)$

Example (from long ago): $\mathbf{f}(u, v) = (2u, v)$ (i.e., $x = 2u$, $y = v$) stretches in the horizontal direction. Mr. Smiley becomes wider.

The transformation $x = u$, $y = v - 3u^2$ does something worse. (Mr. Smiley is not smiling anymore.) But if you had to do an integral over this region, you could convert it into an integral over the original circle.

One way to visualize $\mathbf{f}$: Plug in various input points $(u, v)$ and see where they get mapped to.

Another way: Draw images of the lines $u = c$ and $v = c$.

*Important:* From now on, $\mathbf{f}$ should give a one-to-one transformation from a region $S$ in the $uv$-plane to a region $R$ in the $xy$-plane. One-to-one means that different points inside $S$ get mapped to different points inside $R$. (Without this condition, the $uv$-integral might double-count some parts of the $xy$-integral.)

17.2.2. *Steps for changing variables.*

**Goal:** Re-express

$$\iint_R f(x,y)\,dx\,dy$$

as a double integral in new variables $u, v$.

(1) Choose a transformation $x = x(u,v)$, $y = y(u,v)$ in order to make the integrand simpler or the region simpler.
(2) Find the equations of the boundary curves of $R$ in terms of $x, y$.
(3) Rewrite these in terms of $u, v$ to find the corresponding region in the $uv$-plane.
(4) Compute the Jacobian determinant

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

(*Note:* This assumes that $x, y$ are given as functions of $u, v$. If for some reason you have $u, v$ as functions of $x, y$, and don't feel like solving for $x, y$ in terms of $u, v$, then first compute

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

and then use the identity

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}$$

to compute the Jacobian.)

(5) *Take the absolute value* of the Jacobian determinant to get the area scaling factor:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|.$$

Area is always nonnegative!

(6) Substitute

$$x = x(u,v)$$
$$y = y(u,v)$$
$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv.$$

into the integral, and remember to use the equations of the boundary curves in terms of $u$ and $v$ to find the new region, and hence the new limits of integration.

*Remark* 17.3. In the case of the change of variable

$$x = r\cos\theta, \qquad y = r\sin\theta$$

the area scaling formula

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv.$$

turns out to be

$$dx\,dy = r\,dr\,d\theta.$$

**Problem 17.4.** Let $R$ be the square with corners $(\pm 1, 0)$ and $(0, \pm 1)$. Evaluate

$$I := \iint_R \frac{\sin^2(x-y)}{x+y+2}\,dx\,dy.$$

Solution:

*It's often a good idea to choose $u, v$ so that the sides of $R$ correspond to $u =$ constant, $v =$ constant.*

In this problem: Try $u = x + y$, $v = x - y$, so $x = (u+v)/2$, $y = (u-v)/2$.

Boundary curves: $u = 1$, $u = -1$, $v = 1$, $v = -1$.

Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = -1/2$$

so

$$dx\,dy = \left|-\frac{1}{2}\right|\,du\,dv = \frac{1}{2}\,du\,dv.$$

So

$$I = \int_{-1}^{1}\int_{-1}^{1} \frac{\sin^2 v}{u+2}\,\frac{1}{2}\,du\,dv.$$

To finish, one should evaluate the inside integral (with variable $u$), and then the outside integral (with variable $v$).

17.2.3. *Why is the scale factor equal to* $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$*?* The rectangle $[u, u+du] \times [v, v+dv]$ maps to (approximately) a parallelogram whose sides are given by the vectors $\langle x_u\,du, y_u\,du\rangle$ and $\langle x_v\,dv, y_v\,dv\rangle$. In other words, as the input moves $du$ in the $u$-direction, the $x$-coordinate of the output changes by approximately $x_u\,du$, and so on, because $x_u$ is the rate of change of $x$ with respect to $u$ (as $v$ is held constant).

The area of the parallelogram spanned by those two vectors (which we now write as columns of a matrix) is the absolute value of

$$\det \begin{pmatrix} x_u \, du & x_v \, dv \\ y_u \, du & y_v \, dv \end{pmatrix} = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} du \, dv.$$

(Here we used twice that if one multiplies an entire column of a matrix by a number, the determinant gets multiplied by that number.) This explains why

$$dx \, dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du \, dv.$$

## 18. April 2

### 18.1. Review: change of variable in double integrals.

**Problem 18.1.** Let $R$ be the region in the first quadrant inside the ellipse $x^2 + 9y^2 = 12$ but outside the circle $x^2 + y^2 = 4$. Evaluate

$$\iint_R \frac{xye^{y^2}}{1-y^2} \, dx \, dy.$$

Solution: Since $x^2$ and $y^2$ appear in the equations of the boundary curves, and since $y^2$ appears in $e^{y^2}$ in the integrand, let's use the change of variables $u = x^2$ and $v = y^2$, so $x = \sqrt{u}$ and $y = \sqrt{v}$. The original region $R$ is described by $x^2 + 9y^2 \leq 12$ and $x^2 + y^2 \geq 4$ and $x, y \geq 0$. The corresponding region $S$ in the $uv$-plane is described by $u + 9v \leq 12$ and $u + v \geq 4$ and $u, v \geq 0$; this is a triangle with vertices at $(4,0)$, $(12,0)$, and $(3,1)$. We have

$$\frac{\partial(u,v)}{\partial(x,y)} = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \det \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix} = 4xy,$$

so

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4xy},$$

which is positive, so its absolute value is also $\frac{1}{4xy}$, which is the area scaling factor:

$$dx \, dy = \frac{1}{4xy} \, du \, dv.$$

Thus the integral becomes

$$\iint_S \frac{xye^{y^2}}{1-y^2} \frac{1}{4xy} \, du \, dv = \frac{1}{4} \iint_S \frac{e^v}{1-v} \, du \, dv.$$

(We re-expressed everything in terms of $u$ and $v$.) At this point, we have the option of doing the integration in the order $dv \, du$ instead of $du \, dv$, but the order $dv \, du$ would require dividing the triangle into two triangles, so it is easier to keep $v$ as the outer variable. The range for $v$ is $[0, 1]$. For each $v$, we get the lower and upper limits for $u$ by solving $u + v = 4$ and $u + 9v = 12$ for $u$ in terms of $v$. This leads to

$$\begin{aligned}
\frac{1}{4}\int_{v=0}^{v=1}\int_{u=4-v}^{u=12-9v}\frac{e^v}{1-v}\,du\,dv &= \frac{1}{4}\int_{v=0}^{v=1}\frac{e^v}{1-v}u\Big|_{u=4-v}^{u=12-9v}\,dv \\
&= \frac{1}{4}\int_{v=0}^{v=1}\frac{e^v}{1-v}(8-8v)\,dv \\
&= 2\int_{v=0}^{v=1} e^v\,dv \\
&= 2\,e^v\big|_{v=0}^{v=1} \\
&= 2e - 2.
\end{aligned}$$

18.2. **Vector fields.**

**Definition 18.2.** A vector field is a function whose value at each point of a region is a vector.

It could be showing, for example,

- the wind velocity at each point,
- the velocity of a fluid at each point,
- the strength and direction of an electromagnetic force,
- or the gradient of a function at each point.

Mathematically, a 2-dimensional vector field has the form

$$\begin{aligned}
\mathbf{F}(x, y) &= \langle P(x, y), Q(x, y)\rangle \\
&= P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j},
\end{aligned}$$

where $P$ and $Q$ are functions.

**Example 18.3.** Consider $\mathbf{F}(x, y) = -y\mathbf{i}$. Its values in the upper half plane are vectors pointing to the left, and its values in the lower half plane are vectors pointing to the right. (The students in class demonstrated this by forming one big vector field with their hands.)

**Example 18.4.** The gradient field of $f(x, y) := x^2 + y^2$ is $\mathbf{F}(x, y) := 2x\mathbf{i} + 2y\mathbf{j}$, which is always pointing outward. Think of the direction of fastest increase, looking at a top view of a paraboloid. The gradient field is everywhere perpendicular to the level curves, which are circles.

$$\mathbf{F} \text{ is continuous} \iff P \text{ and } Q \text{ are continuous.}$$

$$\mathbf{F} \text{ is differentiable} \iff P \text{ and } Q \text{ are differentiable.}$$

18.3. **Line integrals.** Let $C$ be an oriented curve in $\mathbb{R}^2$. (Oriented means that one of the two directions along the curve has been chosen.)

Three kinds of line integrals:

- Line integral with respect to arc length: $\int_C f(x, y)\, ds$. (Remember: $s$ = distance traveled = arc length.)
- Line integral with respect to coordinate variables: $\int_C f(x, y)\, dx$ or $\int_C f(x, y)\, dy$ or $\int_C P(x, y)\, dx + Q(x, y)\, dy$
- Line integral of a vector field: $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Which are vectors and which are scalars? They are all scalars!

18.3.1. *What do these mean?* Divide $C$ into $n$ pieces by labelling points $P_0, P_1, \ldots, P_n$ in order along $C$ where $P_0$ and $P_n$ are the endpoints. Choose a sample point $P_i^*$ on the arc from $P_{i-1}$ to $P_i$. Let $\Delta s_i$ be the arc length of that arc. Then

$$\int_C f(x, y)\, ds \approx f(P_1^*)\Delta s_1 + \cdots + f(P_n^*)\Delta s_n.$$

Actual definition:

$$\int_C f(x, y)\, ds := \lim_{\max \Delta s_i \to 0} \left(f(P_1^*)\Delta s_1 + \cdots + f(P_n^*)\Delta s_n\right).$$

Similarly,

$$\int_C f(x, y)\, dx := \lim_{\max \Delta s_i \to 0} \left(f(P_1^*)(x(P_1) - x(P_0)) + \cdots + f(P_n^*)(x(P_n) - x(P_{n-1}))\right)$$

and

$$\int_C \mathbf{F} \cdot d\mathbf{r} := \lim_{\max \Delta s_i \to 0} \left(\mathbf{F}(P_1^*) \cdot \overrightarrow{P_0 P_1} + \cdots + \mathbf{F}(P_n^*) \cdot \overrightarrow{P_{n-1} P_n}\right).$$

**Example 18.5.**

$$\int_C 1\, ds = \text{ arc length of } C.$$

18.3.2. *How do you compute these?*

(1) Choose a parametrization of $C$, say $(x(t), y(t))$ for $t \in [a, b]$. Now $x, y, s, \mathbf{r}$ are all functions of $t$.

(2) Make the substitution $x = x(t)$ and $y = y(t)$ in the integrand.

(3) Also substitute

$$\begin{aligned} dx &= x'(t)\, dt \\ dy &= y'(t)\, dt \\ ds &= \sqrt{dx^2 + dy^2} = \sqrt{x'(t)^2 + y'(t)^2}\, dt \\ d\mathbf{r} &= \langle dx, dy \rangle = \langle x'(t), y'(t) \rangle\, dt. \end{aligned}$$

(4) Change $\int_C$ to $\int_a^b$.

(5) Evaluate the resulting 1-variable integral in $t$.

**Problem 18.6.** Let $C$ be the upper half of the circle $x^2+y^2=4$, oriented counterclockwise. Let $\mathbf{F}(x,y) = -y\mathbf{i}$. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$.

Is the answer going to be positive or negative? Positive, because $\mathbf{F}$ and $\Delta\mathbf{r}$ form an angle less than 90°.

Solution: Choose the parametrization $(2\cos t, 2\sin t)$ for $t \in [0,\pi]$. Then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \langle -y, 0\rangle \cdot \langle dx, dy\rangle \\ &= \int_0^\pi \langle -2\sin t, 0\rangle \cdot \langle -2\sin t, 2\cos t\rangle\, dt \\ &= \int_0^\pi 4\sin^2 t\, dt \\ &= \int_0^\pi (2-2\cos 2t)\, dt \\ &= 2\pi.\end{aligned}$$

18.3.3. *Geometric interpretation.* Recall that $d\mathbf{r} = \mathbf{T}\,ds = \langle dx, dy\rangle$, where $\mathbf{T}$ is the unit tangent vector. So if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C (\mathbf{F}\cdot\mathbf{T})\, ds \\ &= \int_C P\,dx + Q\,dy.\end{aligned}$$

18.3.4. *Applications.* If $C$ is a wire whose density per unit length is $\delta(x,y)$, then

$$\begin{aligned}\text{mass } m &:= \int_C dm = \int_C \delta\, ds \\ \text{mass-weighted average } \bar{f} &:= \frac{\int_C f\, dm}{m} \\ \text{centroid} &:= (\bar{x}, \bar{y}) \\ \text{moment of inertia } I &:= \int_C (\text{distance to axis})^2\, dm.\end{aligned}$$

All of these involve line integrals with respect to *arc length*.

Also, given a force field that is constant (independent of position),

$$\text{work} := \overrightarrow{\text{force}} \cdot \overrightarrow{\text{displacement}} \qquad := \mathbf{F}\cdot\Delta\mathbf{r},$$

More generally, for an object moving along $C$, the work done by a not necessarily constant force field $\mathbf{F}$ is

$$W := \int_C \mathbf{F}\cdot d\mathbf{r}.$$

(Think of adding up the work done over each little piece of $C$.)

18.4. **Adding and reversing paths.** If the endpoint of path $C_1$ equals the start point of $C_2$, then concatenating gives a path $C_1 + C_2$, and

$$\int_{C_1+C_2} \cdots = \int_{C_1} \cdots + \int_{C_2} \cdots .$$

Let $C$ be a piecewise differentiable oriented curve. Then $-C$ denotes the same curve with the reverse orientation. Intuitively, for each piece of $C$, the "length" $ds$ stays the same, but the "changes" $dx$, $dy$, and $d\mathbf{r}$ change sign, so

$$\int_{-C} f\,ds = \int_C f\,ds$$
$$\int_{-C} P\,dx + Q\,dy = -\int_C P\,dx + Q\,dy$$
$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}.$$

## 19. April 4

19.1. **Fundamental theorem of calculus for line integrals.**

$1^{\text{st}}$ FTC: If $f(x)$ is a continuous function on $[a, b]$, and

$$F(x) := \int_a^x f(t)\,dt$$

then $F'(x) = f(x)$.

$2^{\text{nd}}$ FTC: If $G(x)$ is such that $G'(x)$ is continuous on an interval $[a, b]$, then

$$\int_a^b G'(t)\,dt = G(b) - G(a).$$

It's the $2^{\text{nd}}$ FTC that is usually used each time you compute a definite integral.

FTC for line integrals: If $C$ starts at $A$ and ends at $B$, and $f(x, y)$ is a function that is continuously differentiable at each point of $C$, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

The left side is adding up the change in $f$ along each piece of the curve, and the right side is the total change in $f$.

*Proof.* Let $\mathbf{r}(t) = (x(t), y(t))$ for $t \in [a, b]$ be a parametrization of $C$. Let $G(t) = f(\mathbf{r}(t))$. Then

$$\begin{aligned}
\int_C \nabla f \cdot d\mathbf{r} &= \int_C \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle dx, dy \rangle \\
&= \int_C \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy \\
&= \int_a^b \left( \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right) dt \\
&= \int_a^b G'(t)\, dt \qquad \text{(by the chain rule)} \\
&= G(b) - G(a) \qquad \text{(by the 2}^{\text{nd}}\text{ FTC)} \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \\
&= f(B) - f(A).
\end{aligned}$$

□

Consequences:

- $\int_A^B \nabla f \cdot d\mathbf{r}$ is path independent: depends only on the endpoints $A$ and $B$, not on the path taken from $A$ to $B$!
- If $C$ is a closed curve (starts and ends at the same point), then $\oint_C \nabla f \cdot d\mathbf{r} = 0$. (There is no difference between $\oint$ and $\int$: the former is just a notation to help remind you that the curve is closed.)

These are properties special to *gradient* vector fields.

**Example 19.1.** Let $\mathbf{F} = y\,\mathbf{i}$. Let $C$ be the upper half of the unit circle, and let $C'$ be the lower half, both oriented from $(1, 0)$ to $(-1, 0)$. The geometry shows that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is negative, but $\int_{C'} \mathbf{F} \cdot d\mathbf{r}$ is positive. So line integrals of $\mathbf{F}$ are *not* path independent.

19.2. **Equivalent conditions.**

**Theorem 19.2.** *For a continuous vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ on a region $D$, the following are equivalent:*

(1) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for every closed curve $C$ inside $D$.*
(2) $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ *is path independent for all points $A$ and $B$ in $D$.*
(3) $\mathbf{F}$ *is a* gradient *vector field (i.e., $\mathbf{F} = \nabla f$ for some differentiable function $f(x, y)$, which is then called a potential function for $\mathbf{F}$ — the multivariable analogue of an antiderivative).*
(4) $P\,dx + Q\,dy$ *is an exact differential (i.e., has the form $f_x\,dx + f_y\,dy = df$ for some $f(x, y)$).*

*If any of these hold, $\mathbf{F}$ is called conservative.*

The theorem is saying that if any one of the four conditions holds for $\mathbf{F}$, then all of the other conditions hold too.

*Proof.* (1) $\Longrightarrow$ (2): Suppose that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve $C$. We need to prove path independence. So suppose that $C_1$ and $C_2$ are two paths from $A$ to $B$. Then $C_1 - C_2$ is a closed loop, so

$$0 = \oint_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

So $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

(2) $\Longrightarrow$ (3): Suppose that line integrals of $\mathbf{F}$ are independent of path. Choose a start point $(a, b)$ in $D$. Define

$$f(x, y) := \int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}.$$

(This makes sense since the value is independent of the path.) We will show that $\nabla f = \mathbf{F}$. First, let's calculate $f_x(x_0, y_0)$. Choose a path from $(a, b)$ to $(x_0, y_0)$ and continue it horizontally to $(x, y_0)$. Then

$$\begin{aligned}
\int_{(a,b)}^{(x,y_0)} \mathbf{F} \cdot d\mathbf{r} &= \int_{(a,b)}^{(x_0,y_0)} \mathbf{F} \cdot d\mathbf{r} + \int_{(x_0,y_0)}^{(x,y_0)} \mathbf{F} \cdot d\mathbf{r} \\
f(x, y_0) &= f(x_0, y_0) + \int_{t=x_0}^{t=x} \mathbf{F}(t, y_0) \cdot \mathbf{i}\, dt \qquad (\text{using } \mathbf{r}(t) = (t, y_0) \text{ so } d\mathbf{r} = \langle dt, 0 \rangle = \mathbf{i}\, dt) \\
&= f(x_0, y_0) + \int_{t=x_0}^{t=x} P(t, y_0)\, dt \\
f_x(x_0, y_0) &= P(x_0, y_0) \qquad (\text{by the single-variable } 1^{\text{st}} \text{ FTC}).
\end{aligned}$$

So $f_x = P$. Similarly $f_y = Q$. So $\nabla f = \mathbf{F}$.

Class actually ended here; we'll finish the proof next time.

(3) $\Longrightarrow$ (1): Suppose that $\mathbf{F} = \nabla f$. Then for any closed curve $C$ starting and ending at $A$, the FTC implies that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = f(A) - f(A) = 0.$$

(3) $\Longleftrightarrow$ (4): These are just different ways of saying that $P = f_x$ and $Q = f_y$ for some differentiable $f(x, y)$. □

Using the implications proved above, one can get from the truth of any one of the four conditions to the truth of any other.

## 20. April 5

Last time:

*FTC for line integral of a gradient*: For a path $C$ from $A$ to $B$, and continuously differentiable

$f$,

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

20.1. **Conservative force fields in physics.** In physics, if $\mathbf{F}$ is a force field (e.g., gravitational force), and $\mathbf{F} = -\nabla V$ then $V$ is called potential energy. If an object moves from $A$ to $B$ along a path $C$,

$$\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\Big|_A^B = \int m\mathbf{v}'(t)\cdot\mathbf{v}(t)\,dt = \int m\mathbf{a}(t)\cdot\mathbf{r}'(t)\,dt = \int_C \mathbf{F}\cdot d\mathbf{r} \overset{\text{FTC}}{=} -V|_A^B$$

increase in kinetic energy = work done by the force on the object = *decrease* in potential energy,

so

$$\text{kinetic energy} + \text{potential energy} \qquad \text{is } \textit{constant}:$$

conservation of energy! That's why $\mathbf{F}$ is called conservative.

20.2. **Test for gradient field.** Recall: A continuous vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ on a region $D$ is *conservative* if one of the following equivalent conditions holds:

(1) $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed curve $C$.
(2) $\int_A^B \mathbf{F}\cdot d\mathbf{r}$ is path independent for all points $A$ and $B$.
(3) $\mathbf{F} = \nabla f$ for some $f$.
(4) $P\,dx + Q\,dy$ is exact (equal to $df$ for some $f$).

**Example 20.1.** Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} := \langle 2xy, x^2\rangle$, so $P(x,y) = 2xy$ and $Q(x,y) = x^2$. Then $\mathbf{F} = \nabla f$, where $f(x,y) := x^2y$. In other words, condition (3) holds for $\mathbf{F}$. The theorem now says that all four conditions hold for $\mathbf{F}$.

For example, (4) says that $P\,dx + Q\,dy$ is exact. This is true: it is $df$, where $f(x,y) := x^2y$.

Suppose that $C$ is the unit circle, traversed counterclockwise. Is $\oint_C \langle 2xy, x^2\rangle \cdot d\mathbf{r} = 0$? Yes, since (1) is true for $\mathbf{F}$.

*Problem, part (a):* Suppose that $\mathbf{F}(x,y) = (6x^2 + 8xy)\mathbf{i} + (4x^2 + 3y^2)\mathbf{j}$. Is $\mathbf{F}$ conservative?

To help with this, we introduce a new condition, assuming that $\mathbf{F}$ is continuously differentiable:

(5) $Q_x = P_y$.

Conditions (1)–(4) imply (5):

Proof: If $P = f_x$ and $Q = f_y$, then $Q_x = f_{yx} = f_{xy} = P_y$.

Conversely, (5) implies (1)–(4) *if $D$ is simply connected.* We'll talk about this later on. For now: $\mathbb{R}^2$ is simply connected, as is any rectangle.

Back to the problem:

$$\begin{aligned} P &= 6x^2 + 8xy \\ P_y &= 8x \\ Q &= 4x^2 + 3y^2 \\ Q_x &= 8x. \end{aligned}$$

Since $Q_x = P_y$, $\mathbf{F}$ is conservative.

### 20.3. Finding the potential.

*Problem, part (b):* Find a function $f$ such that $\nabla f = \mathbf{F}$.

Solution 1: use line integrals (FTC backwards).

Choose a path $C$ from $(0, 0)$ to $(a, b)$. Then FTC says

$$f(a, b) - f(0, 0) = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

The $f(0, 0)$ is a constant of integration (if $f$ is a potential for $\mathbf{F}$, then so is $f + c$ for any constant $c$). Let's take $C = C_1 + C_2$ where $C_1$ goes from $(0, 0)$ to $(a, 0)$ and $C_2$ goes from $(a, 0)$ to $(a, b)$. Parametrize $C_1$ by $(t, 0)$ for $t \in [0, a]$:

$$\begin{aligned} \int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^a \langle 6t^2, 4t^2 \rangle \cdot \langle dt, 0 \rangle \\ &= \int_0^a 6t^2 \, dt \\ &= 2t^3 \Big|_0^a \\ &= 2a^3 \end{aligned}$$

and parametrize $C_2$ by $(a, t)$ for $t \in [0, b]$:

$$\begin{aligned} \int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^b \langle 6a^2 + 8at, 4a^2 + 3t^2 \rangle \cdot \langle 0, dt \rangle \\ &= \int_0^a 4a^2 + 3t^2 \, dt \\ &= 4a^2 t + t^3 \Big|_0^b \\ &= 4a^2 b + b^3 \end{aligned}$$

so

$$\begin{aligned} f(a,b) - f(0,0) &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \\ &= 2a^3 + 4a^2 b + b^3, \end{aligned}$$

so

$$f(x,y) = 2x^3 + 4x^2 y + y^3 \ (+\text{constant}).$$

There are infinitely many possible answers $f(x,y)$, one potential function for each value of the constant.

Solution 2: use antiderivatives.

We want $f(x,y)$ such that

$$\begin{aligned} f_x &= 6x^2 + 8xy \\ f_y &= 4x^2 + 3y^2. \end{aligned}$$

Using the first equation, treating $y$ as constant, we get

$$f(x,y) = 2x^3 + 4x^2 y + g(y)$$

for some "constant" $g(y)$ that does not depend on $x$. Take $\frac{d}{dy}$: get

$$f_y = 4x^2 + g'(y).$$

Comparing with the second requirement shows that $g'(y) = 3y^2$, so $g(y) = y^3 + c$. Substituting this back in yields

$$f(x,y) = 2x^3 + 4x^2 y + y^3 + c.$$

Whichever method was used, it is easy to check the answer: just compute $\nabla f$ and make sure that it equals $\mathbf{F}$.

## 21. April 9

### 21.1. **2D Curl.**

**Definition 21.1.** The (scalar) curl of a 2-dimensional vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is

$$\operatorname{curl} \mathbf{F} := \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} = Q_x - P_y.$$

It is a scalar function of $(x,y)$.

So the following are equivalent:

(5) $Q_x = P_y$
(6) $\operatorname{curl} \mathbf{F} = 0$.

Roughly, failure of conservativeness is measured by $\operatorname{curl} \mathbf{F}$.

21.1.1. *Physical interpretations of curl.* If $\mathbf{F}$ is the velocity field of a fluid, and you nail the center of a tiny paddlewheel at a point (see p. 1068 in the textbook, or V4.3 in the supplementary notes), then

$$\operatorname{curl} \mathbf{F} = 2(\text{angular velocity of paddlewheel}).$$

**Examples 21.2.**

- $\mathbf{F} = \langle 2, 3 \rangle$ (constant flow) has $\operatorname{curl} \mathbf{F} = 0$
- $\mathbf{F} = \langle x, y \rangle$ (expansion) has $\operatorname{curl} \mathbf{F} = 0$
- $\mathbf{F} = \langle -y, x \rangle$ (rotation with angular velocity 1) has $\operatorname{curl} \mathbf{F} = 2$

If instead $\mathbf{F}$ is a force field, $\operatorname{curl} \mathbf{F}$ is torque exerted on a tiny dumbbell (rotational analogue of force). An analogy:

$$\frac{\overrightarrow{\text{force}}}{\text{mass}} = \overrightarrow{\text{acceleration}} := \frac{d}{dt}(\overrightarrow{\text{velocity}})$$

$$\frac{\text{torque}}{\text{moment of inertia}} = \text{angular acceleration} := \frac{d}{dt}(\text{angular velocity}).$$

21.2. **Green's theorem.** Suppose $C$ is a simple closed curve bounding a region $R$. (Simple means that $C$ does not cross itself; for example, a figure-eight is a closed curve, but not a simple closed curve.) Assume that $C$ is positively-oriented: this means that as you walk along the curve, the region is to the *left*.

**Green's theorem:** If $\mathbf{F}$ is a continuously differentiable vector field, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} \, dA$$

Equivalent formulation: For any continuously differentiable functions $P$ and $Q$,

$$\oint_C P \, dx + Q \, dy = \iint_R (Q_x - P_y) \, dA.$$

*Sketch of proof.* Two comments:

- It's really two identities in one:

$$\oint_C P \, dx = -\iint_R P_y \, dA \qquad \text{and} \qquad \oint_C Q \, dy = \iint_R Q_x \, dA.$$

Let's prove the first one (the proof of the other one is similar).

- Additivity: If it's true for $R_1$ and $R_2$, it's true for $R := R_1 \cup R_2$ since if $C_1, C_2, C$ are the curves bounding $R_1, R_2, R$, then

$$\oint_C = \oint_{C_1} + \oint_{C_2}$$
$$\iint_R = \iint_{R_1} + \iint_{R_2}.$$

(The overlapping parts of $C_1$ and $C_2$ are in opposite directions, so those parts of the integral cancel.)

*Case 1: $R$ is "vertically simple", i.e., the area between two graphs $y = g_1(x)$ and $y = g_2(x)$ for $x \in [a, b]$.*

Then $C = C_1 + C_2 + C_3 + C_4$: the bottom, right, top, and left boundaries. To compute the line integrals on these four curves, use the following parametrizations:

$$\begin{aligned}
C_1 &: \quad \langle t, g_1(t) \rangle \quad && \text{for } t \text{ from } a \text{ to } b \\
C_2 &: \quad \langle b, t \rangle \quad && \text{for } t \text{ from } g_1(b) \text{ to } g_2(b) \\
C_3 &: \quad \langle t, g_2(t) \rangle \quad && \text{for } t \text{ from } b \text{ to } a \\
C_4 &: \quad \langle a, t \rangle \quad && \text{for } t \text{ from } g_2(a) \text{ to } g_1(a).
\end{aligned}$$

Substituting these in yields

$$\begin{aligned}
&\int_{C_1} P\,dx = \int_a^b P(t, g_1(t))\,dt = \int_a^b P(x, g_1(x))\,dx \qquad \text{(it's OK to rename } t \text{ as } x) \\
&\int_{C_2} P\,dx = 0 \qquad (\text{since } x = b,\ dx = 0) \\
&\int_{C_3} P\,dx = \int_b^a P(t, g_2(t))\,dt = -\int_a^b P(x, g_2(x))\,dx \\
&\int_{C_4} P\,dx = 0 \qquad (\text{same as for } C_2),
\end{aligned}$$

so

$$\begin{aligned}
\oint_C P\,dx &= \int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx \\
&= -\int_a^b \left(P(x, g_2(x)) - P(x, g_1(x))\right)\,dx \\
&= -\int_a^b \left( \int_{g_1(x)}^{g_2(x)} P_y\,dy \right)\,dx \\
&= -\iint_R P_y\,dA.
\end{aligned}$$

*Case 2: $R$ can be sliced with vertical lines into finitely many vertically simple pieces.* Use Case 1 and additivity.

*Case 3: Any region $R$.* Approximate $R$ better and better with regions as in Case 2 and take the limit of both sides. □

### 21.3. **Review: using line integrals.**

**Question 21.3.** In line integrals, how do you know which of $ds$, $dx$, $d\mathbf{r}$ to use?

It depends on what you want the line integral to represent. Some guidelines:

- If integrating a vector field, usually $d\mathbf{r}$ is what you want. For example, work done by a force field $\mathbf{F}$ is $\int_C \mathbf{F} \cdot d\mathbf{r}$.
- For mass, average value, centroid, or moment of inertia of a wire, use $dm = \delta\, ds$. (Here $ds$ is the length of a small piece of wire, $\delta$ is the mass per unit length, and $dm$ is the mass of the small piece of wire.)

## 22. April 11

### 22.1. **Applications of Green's theorem.**

#### 22.1.1. *Computing area.*

**Example 22.1.** Taking $P = 0$ and $Q = x$ in Green's theorem yields

$$\oint_C x\, dy = \iint_R (1 - 0)\, dA = \text{Area}(R).$$

Similarly, taking $P = -y$ and $Q = 0$ yields

$$-\oint_C y\, dx = \iint_R (0 - (-1))\, dA = \text{Area}(R).$$

In the 19th century, a mechanical device called a *planometer* was invented that could compute such line integrals, so that areas could be computed by just having the arm of the device trace the boundary.

22.1.2. *Vector fields with curl equal to 0 everywhere.* If $C$ is a simple closed curve, and $\mathbf{F}$ is a vector field such that $\operatorname{curl} \mathbf{F} = 0$ not only on $C$ but also *everywhere in the region $R$ bounded by $C$*, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Proof:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \pm \iint_R \operatorname{curl} \mathbf{F}\, dA = 0.$$

(The sign is there in case $C$ is not positively-oriented.)

In fact, this works even if $C$ is a closed curve that is not a simple closed curve. For example, a figure-eight can be decomposed into two simple closed curves. (Even more general closed curves can be approximated by finite unions of simple closed curves.)

This explains why if $\operatorname{curl} \mathbf{F} = 0$ on a region without holes (like $\mathbb{R}^2$ or a rectangle), then $\mathbf{F}$ is conservative. In terms of our numbering of conditions, this is (6) $\implies$ (1).

22.2. **Flux.** Recall:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

where $\mathbf{T}$ is the unit tangent vector at each point of $C$. Rotate $\mathbf{T}$ by $90^\circ$ clockwise to get the (outward) unit normal vector $\mathbf{n}$ (at each point of $C$).

**Definition 22.2.** The flux of $\mathbf{F}$ across a plane curve $C$ is

$$\boxed{\int_C \mathbf{F} \cdot \mathbf{n}\, ds.}$$

22.2.1. *Physical interpretation of flux.* Suppose that the velocity of a 2-dimensional fluid of unit density is given by a constant vector field $\mathbf{F}$.

How much fluid crosses a tiny line segment of length $ds$ perpendicular to $\mathbf{F}$ in one second? A rectangle's worth of fluid crosses the segment, so the answer is $|\mathbf{F}|\, ds$.

What if the segment is a piece of $C$, but $\mathbf{F}$ is not perpendicular to $C$? Then the fluid that crosses the segment in one second is a parallelogram of height $\mathbf{F} \cdot \mathbf{n}$ and base $ds$, so the answer is $\mathbf{F} \cdot \mathbf{n}\, ds$.

How much fluid crosses the whole closed curve $C$ per second? Answer: $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$. That's flux! This works even if $\mathbf{F}$ is not constant, but just continuous (which means that $\mathbf{F}$ is almost constant in small regions).

Note: Flux is measuring the rate of *outward* flow. At places along $C$ where $\mathbf{F}$ is pointing inwards, the contribution to flux will be negative.

22.2.2. *Computing flux.* Rotating

$$d\mathbf{r} = \mathbf{T}\, ds = \langle dx, dy \rangle$$

$90^\circ$ clockwise yields

$$\mathbf{n}\, ds = \langle dy, -dx \rangle.$$

So if $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, then

$$\begin{aligned}
\text{flux} &= \int_C \mathbf{F} \cdot \mathbf{n}\, ds \\
&= \int_C \langle M, N \rangle \cdot \langle dy, -dx \rangle \\
&= \int_C -N\, dx + M\, dy,
\end{aligned}$$

which can be computed as usual, by choosing a parametrization of $C$, and so on.

22.3. **Divergence.**

**Definition 22.3.** The (2-dimensional) divergence of $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is

$$\boxed{\operatorname{div} \mathbf{F} := M_x + N_y.}$$

Vector or scalar? Scalar.

Another way to remember the definition of $\operatorname{div} \mathbf{F}$:

$$\begin{aligned}
\operatorname{div} \mathbf{F} &= \nabla \cdot \mathbf{F} \\
&= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \cdot \langle M, N \rangle \\
&= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.
\end{aligned}$$

In contrast,

$$\nabla f := \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle.$$

22.3.1. *Physical interpretation of divergence.* If $\mathbf{F}$ is the velocity field of an incompressible fluid, then $\operatorname{div} \mathbf{F}$ measures how much fluid is being added to the system per unit area and per unit time.

**Example 22.4.** Suppose that $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$. At every point, there is more fluid going out than coming in, so every point is a source: $\operatorname{div} \mathbf{F} > 0$. (In fact, $\operatorname{div} \mathbf{F} = 1 + 1 = 2$ everywhere.)

**Example 22.5.** Suppose that $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$. This time, $\operatorname{div} \mathbf{F} = 1 + (-1) = 0$: the fluid is just flowing along, with nothing added or removed.

22.4. **Green's theorem for flux.** Suppose as usual that $C$ is a positively-oriented simple closed curve enclosing a region $R$.

The original Green's theorem: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F}\, dA$$

$$\oint_C P\, dx + Q\, dy = \iint_R (Q_x - P_y)\, dA.$$

Green's theorem for flux (also known as Green's theorem in normal form): If $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA$$
$$\oint_C -N\, dx + M\, dy = \iint_R (M_x + N_y)\, dA.$$

*Proof of Green's theorem for flux.* In the original Green's theorem, take $P = -N$ and $Q = M$. Done! □

Physically, Green's theorem for flux makes a lot of sense: For an incompressible fluid, if you add up all the fluid that is being added to the system inside the region $R$, the same amount must be flowing outward across $C$ (by conservation of mass, it has to go somewhere!)

**Problem 22.6.** Verify Green's theorem for flux when $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ and $C$ is the circle of radius $r$ centered at $(0, 0)$.

Solution: The flux of $\mathbf{F}$ across $C$ is

$$\begin{aligned}\oint_C \mathbf{F} \cdot \mathbf{n}\, ds &= \oint_C |\mathbf{F}|\, ds \\ &= \oint_C r\, ds \\ &= r \oint_C ds \\ &= r(2\pi r) \\ &= 2\pi r^2.\end{aligned}$$

On the other hand, $\operatorname{div} \mathbf{F} = 2$ at every point of the interior $R$ of the circle, so

$$\iint_R \operatorname{div} \mathbf{F}\, dA = \iint_R 2\, dA = 2\pi r^2.$$

Green's theorem for flux says that these two quantities should be equal, and they are!

## 23. April 12

### 23.1. Green's theorem for flux, continued.

**Problem 23.1.** What is the flux of $\mathbf{F} := x\mathbf{i} - y\mathbf{j}$ across the circle $C$ of radius $r$ centered at $(0, 0)$?

Solution 1: The flux is

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C \langle x,-y\rangle\cdot\langle dy,-dx\rangle$$
$$= \oint_C y\,dx + x\,dy.$$

Normally one would have to parametrize $C$ to compute this line integral, but in this case, there is a shortcut, because $y\,dx + x\,dy$ is exact, namely $df$ where $f = xy$. Another way of thinking about this: the integral is the same as

$$\oint_C \langle y,x\rangle\cdot\langle dx,dy\rangle = \oint_C \langle y,x\rangle\cdot d\mathbf{r},$$

but $\langle y,x\rangle$ is a conservative vector field (it's the gradient of $xy$), so the integral is 0.

Solution 2: Use Green's theorem for flux: the flux is

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \operatorname{div}\mathbf{F}\,dA = 0$$

since

$$\operatorname{div}\mathbf{F} = M_x + N_y = 1 + (-1) = 0$$

at every point of $R$.

Solution 3: Use symmetry: the value of $\mathbf{F}\cdot\mathbf{n}$ at $(x,y)$ is the opposite of $\mathbf{F}\cdot\mathbf{n}$ at $(y,x)$, so

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = 0.$$

23.2. **Extended Green's theorem.** Given a region $R$, the notation $\partial R$ is sometimes used to denote its boundary, positively oriented. If $\partial R$ consists of several curves $C_1$, $C_2$, $\ldots$, $C_n$, then

$$\int_{\partial R} \mathbf{F}\cdot d\mathbf{r} = \iint_R \operatorname{curl}\mathbf{F}\,dA$$

is still true, but the left side is now

$$\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} + \cdots + \oint_{C_n} \mathbf{F}\cdot d\mathbf{r}.$$

Just make sure that each $C_i$ is positively-oriented, so that $R$ lies to the left as one is traversing the curve.

Why is this true? Cut $R$ into smaller regions, apply the original Green's theorem to each piece, and add the results.

23.3. **Simply connected regions.** Suppose that $D$ is a connected region in $\mathbb{R}^2$. Let's say that a curve $C$ inside $D$ is shrinkable in $D$ if it is possible to continuously shrink $C$ to a point *within* $D$.

**Example 23.2.** If $D$ is $\mathbb{R}^2$ with the unit disk removed, then a circle of radius 2 centered at the origin is contained in $D$ but is not shrinkable in $D$. (To shrink it, the curve would have to enter the unit disk, at least temporarily, and that is not allowed.)

**Definition 23.3.** Call $D$ simply connected if every closed curve $C$ in $D$ is shrinkable in $D$.

Examples of simply connected regions:

- a solid rectangle
- a disk
- a solid triangle
- the whole plane $\mathbb{R}^2$
- a weird spiral-shaped region drawn on the board

23.4. **Test for gradient, again.** Let $\mathbf{F}$ be a continuous vector field on a region $D$.

6 conditions:

(1) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve $C$ inside $D$
(2) $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is path independent
(3) $\mathbf{F} = \nabla f$ for some $f$
(4) $P\,dx + Q\,dy = df$ for some $f$
(5) $Q_x = P_y$ at every point of $D$
(6) $\operatorname{curl} \mathbf{F} = 0$ at every point of $D$.

1,2,3,4 below are equivalent (if satisfied, $\mathbf{F}$ is *conservative*).
5,6 are equivalent (by definition of curl)
1,2,3,4 imply 5,6.

*Fake proof that* $6 \Rightarrow 1$*:* Suppose that $\operatorname{curl} \mathbf{F} = 0$ at every point of $D$. Then for any (simple) closed curve $C$ in $D$, Green's theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{\text{interior of } C} \operatorname{curl} \mathbf{F}\, dA = 0,$$

since the integrand is 0.

Problem: If the interior of $C$ is not entirely contained in $D$, $\operatorname{curl} \mathbf{F}$ might not even be defined on that interior, so the invocation of Green's theorem is not valid.

This is why $6 \Rightarrow 1$ works only when $D$ is simply connected.

Flowchart for testing whether $\mathbf{F}$ is a gradient:

Is $\operatorname{curl} \mathbf{F} = 0$ everywhere in the region? If no, then $\mathbf{F}$ is not a gradient. If yes, check: is the domain simply connected? If yes, then $\mathbf{F}$ is a gradient. If no, further testing is required.

If you think it is a gradient, maybe you can prove it by guessing $f$ such that $\nabla f = \mathbf{F}$.

If you think it is not a gradient, maybe you can prove it by finding a closed curve $C$ such that $\oint_C \mathbf{F} \cdot d\mathbf{r} \neq 0$.

**Example 23.4.** Is $\mathbf{F} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$ conservative on its domain?

Solution: It turns out that $\operatorname{curl} \mathbf{F} = 0$. But the region on which $\mathbf{F}$ is defined is the punctured plane, which is not simply connected. In fact, if $C$ is the unit circle, parametrized by $x = \cos t$ and $y = \sin t$ for $t \in [0, 2\pi]$, then

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle \, dt \\
&= \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, dt \\
&= \int_0^{2\pi} 1 \, dt \\
&= 2\pi.
\end{aligned}$$

If $\mathbf{F}$ really *were* a gradient, the integral would have been 0. Thus $\mathbf{F}$ is not a gradient.

On the other hand, if $\mathbf{F}$ is considered only on the region $x > 0$, then it turns out that $\mathbf{F} = \nabla f$ where $f(x, y) = \tan^{-1}(y/x)$. (This is the "$\theta$" of polar coordinates.)

MIDTERM #3 COVERS EVERYTHING UP TO HERE

## 24. April 18

### 24.1. Review: parametrizing a line segment.

**Problem 24.1.** Find a parametrization of the line segment from $(2, 3)$ to $(9, 4)$.

Solution 1: Imagine a particle moving at constant speed along the segment from $t = 0$ to $t = 1$. The initial position vector is $\langle 2, 3\rangle$, and the velocity vector is

$$\langle 9, 4\rangle - \langle 2, 3\rangle = \langle 7, 1\rangle,$$

so the parametrization of the line is

$$\begin{aligned}
\mathbf{r}(t) &:= \langle 2, 3\rangle + t\langle 7, 1\rangle \\
&= \langle 2 + 7t, 3 + t\rangle.
\end{aligned}$$

Answer: $\langle 2 + 7t, 3 + t\rangle$ for $t \in [0, 1]$.

Solution 2: Use the $x$-coordinate as the parameter: $t = x$. Next, find $y$ in terms of $x$: the line through $(2, 3)$ to $(9, 4)$ in point-slope form is

$$y - 3 = \frac{4-3}{9-2}(x - 2),$$

so

$$y = \frac{1}{7}(x - 2) + 3 = \frac{x + 19}{7}.$$

Thus

$$\mathbf{r}(t) := \langle t, \frac{t+19}{7} \rangle$$

for $t \in [2, 9]$ is a parametrization. (Warning: If the line segment had been vertical, then we could not have used $x$ as the parameter, but we could have used $y$ as the parameter instead.)

24.2. **Review: parametrizing an arc of a circle.**

**Problem 24.2.** Find a parametrization of the right half of the circle of radius 2 centered at $(5, 3)$.

Solution 1: The unit circle centered at $(0, 0)$ has parametrization

$$\langle \cos t, \sin t \rangle.$$

Multiplying each position vector by 2 gives a parametrization of the circle of radius 2 centered at $(0, 0)$:

$$\langle 2 \cos t, 2 \sin t \rangle.$$

Adding $\langle 5, 3 \rangle$ to each position vector gives a parametrization of the circle of radius 2 centered at $(5, 3)$:

$$\mathbf{r}(t) := \langle 5 + 2 \cos t, 3 + 2 \sin t \rangle,$$

and we want this for $t \in [-\pi/2, \pi/2]$ to get only the right half.

Solution 2: Can we use the $x$-coordinate as the parameter? Not directly, because for most values of $x$ there are *two* points on the curve, so $y$ cannot be expressed as a function of $x$. If we cut the semicircle in half, then we could use a different parametrization on each half using the $x$-coordinate as parameter.

But it's easier to use the $y$-coordinate as the parameter: $t = y$. To find $x$ in terms of $y$, write the equation of the circle

$$\begin{aligned}
(\text{distance from } (x, y) \text{ to } (5, 3)) &= 2 \\
\sqrt{(x-5)^2 + (y-3)^2} &= 2 \\
(x-5)^2 + (y-3)^2 &= 4
\end{aligned}$$

and solve for $x$:

$$(x-5)^2 = 4-(y-3)^2$$
$$x-5 = +\sqrt{4-(y-3)^2}$$

($x-5$ is positive on the right half of the circle, so we took the plus sign in the square root)

$$x = \sqrt{4-(y-3)^2}+5.$$

Thus the parametrization is

$$\mathbf{r}(t) := \langle \sqrt{4-(t-3)^2}+5, t\rangle$$

for $t \in [1,5]$ (the minimum and maximum values of $y$ are $3 \pm 2$).

24.3. **Review: antiderivatives of functions of more than one variable.**

**Problem 24.3.** What are all functions $f(x,y)$ on the right half plane $x > 0$ satisfying $f_x = x^2 + y^3 + \frac{y}{x}$?

Solution: By integration with respect to $x$, treating $y$ as a constant, compute one such antiderivative:

$$\frac{x^3}{3} + y^3x + y\ln x$$

(we could write $\ln x$ instead of $\ln|x|$ since the function is on the right half plane $x > 0$ only). Then the general antiderivative is this plus a constant *but the constant can depend on* $y$:

$$\frac{x^3}{3} + y^3x + y\ln x + g(y)$$

for any function $g(y)$.

It's easy to check the answer: just take the derivative with respect to $x$.

24.4. **Review: computing a potential function using antiderivatives.**

**Problem 24.4.** Explain why the vector field

$$\mathbf{F}(x,y) = \langle 3\cos(x+2y)+2x+y, 6\cos(x+2y)+x+3y^2\rangle$$

on $\mathbb{R}^2$ is conservative, and find a potential function.

Solution: Since $\mathbb{R}^2$ is simply connected, to say that $\mathbf{F}$ is conservative is the same as saying that $\operatorname{curl}\mathbf{F} = 0$. As usual, let $P$ and $Q$ be the component functions of $\mathbf{F}$, so $P := 3\cos(x+2y)+2x+y$ and $Q := 6\cos(x+2y)+x+3y^2$. Then

$$Q_x = -6\sin(x+2y)+1$$
$$P_y = -6\sin(x+2y)+1$$
$$\operatorname{curl}\mathbf{F} = Q_x - P_y = 0,$$

so $\mathbf{F}$ is conservative. (Note: another way to show that $\mathbf{F}$ is conservative would be to wait until after we find the potential function; at that point, we would know that $\mathbf{F}$ is a gradient vector field and hence conservative.)

Now let's find a potential function. One way would be to compute $\int_{(0,0)}^{(a,b)} \mathbf{F} \cdot d\mathbf{r}$, but instead let's use the method of antiderivatives. We seek a function $f(x, y)$ such that

$$f_x = 3\cos(x + 2y) + 2x + y$$
$$f_y = 6\cos(x + 2y) + x + 3y^2.$$

Integrating the first equation with respect to $x$ gives

$$f = 3\sin(x + 2y) + x^2 + xy + g(y)$$

for some function $g(y)$. Taking the derivative with respect to $y$ shows that

$$f_y = 6\cos(x + 2y) + x + g'(y),$$

which must match the earlier condition on $f_y$, so $g'(y) = 3y^2$. Integrating with respect to $y$ gives $g(y) = y^3 + c$ for some constant $c$. (Note that $g$ is a function of the variable $y$ only, so $c$ cannot depend on $x$.) Plugging this back into the formula for $f$ gives the most general potential function

$$f = 3\sin(x + 2y) + x^2 + xy + y^3 + c.$$

In other words, there are infinitely many potential functions, one for each number $c$. In particular, if we want just one potential function, we can choose a specific value for $c$; for instance,

$$f = 3\sin(x + 2y) + x^2 + xy + y^3 + 17$$

is one potential function.

The answer can be checked by computing $\nabla f$.

24.5. **Review: computing the centroid of a curve.**

**Problem 24.5.** What is the centroid of the upper half $C$ of the circle $x^2 + y^2 = 4$? (Assume constant density.)

Note: the problem is about the half-circle (a 1-dimensional object), not the half-disk (a 2-dimensional object).

Solution: The curve $C$ is symmetric under reflection in the $y$-axis, so the centroid $(\bar{x}, \bar{y})$ must lie on the $y$-axis. In other words, $\bar{x} = 0$. On the other hand, $\bar{y}$ is the average value of the $y$-coordinate:

$$\bar{y} := \frac{\int_C y\,dm}{\int_C dm} = \frac{\int_C y\delta\,ds}{\int_C \delta\,ds} = \frac{\delta \int_C y\,ds}{\delta \int_C ds} = \frac{\int_C y\,ds}{\text{length}(C)}.$$

To compute the numerator, we choose a parametrization of $C$, namely

$$\mathbf{r}(t) := \langle 2\cos t, 2\sin t\rangle$$

for $t \in [0, \pi]$ (we want only the upper half of the circle). Then

$$\begin{aligned}
d\mathbf{r} &= \langle -2\sin t, 2\cos t\rangle\, dt \\
ds = |d\mathbf{r}| &= \sqrt{(-2\sin t)^2 + (2\cos t)^2}\, dt = 2\, dt \\
\int_C y\, ds &= \int_{t=0}^{t=\pi} 2\sin t\,(2\, dt) \\
&= 4\int_0^\pi \sin t\, dt \\
&= -4\cos t|_0^\pi \\
&= -4((-1) - 1) \\
&= 8.
\end{aligned}$$

On the other hand, the length of $C$ is

$$\frac{1}{2}(2\pi 2) = 2\pi.$$

Dividing gives

$$\bar{y} = \frac{8}{2\pi} = \frac{4}{\pi},$$

so the centroid is $(0, 4/\pi)$. (Plotting this point suggests that this is a reasonable answer, always a good sign!)

## 25. April 23

### 25.1. Triple integrals.

Triple integral:

$$\iiint_T f(x,y,z)\, dV$$

where $f(x,y,z)$ is continuous on a 3-dimensional region $T$. (Think of dividing $T$ into tiny blocks, multiply the value of $f$ at a sample point in each block by the volume of the block, and add the results to get an approximation.)

**Problem 25.1.** Let $T$ be the region in $\mathbb{R}^3$ where $x^2 + y^2 + z^2 \leq 1$ and $z \geq 0$. So $T$ is the upper half of the unit ball in $\mathbb{R}^3$. Find the centroid of $T$. (Assume constant density.)

Solution: By symmetry, it must be of the form $(0, 0, \bar{z})$, where $\bar{z}$ is the average value of $z$ on the half-ball:

$$\bar{z} := \frac{\iiint_T z\, dV}{\frac{1}{2}\left(\frac{4}{3}\pi 1^3\right)}.$$

At a given height $z$, the cross-section is the disk $D_z$ defined by $x^2 + y^2 \leq 1 - z^2$ (with $z$ viewed as constant), a disk of radius $\sqrt{1-z^2}$. So

$$\iiint_T z\,dV = \int_{z=0}^{z=1} \left( \iint_{D_z} z\,dx\,dy \right) dz.$$

At this point we *could* write the inner double integral as an iterated integral and obtain the following:

$$\int_{z=0}^{z=1} \int_{y=-\sqrt{1-z^2}}^{y=\sqrt{1-z^2}} \int_{x=-\sqrt{1-y^2-z^2}}^{x=\sqrt{1-y^2-z^2}} z\,dx\,dy\,dz.$$

But it is easier instead to factor the $z$ out of the inner double integral ($z$ is a constant for the inner integral, in which $x$ and $y$ are the variables), and to recognize what remains as an area:

$$\begin{aligned}
\iiint_T z\,dV &= \int_{z=0}^{z=1} z \;\text{Area}(D_z)\,dz \\
&= \int_{z=0}^{z=1} z \left( \pi \sqrt{1-z^2}^{\,2} \right) dz \\
&= \pi \int_0^1 (z - z^3)\,dz \\
&= \frac{\pi}{4},
\end{aligned}$$

so

$$\begin{aligned}
\bar{z} &= \frac{\pi/4}{\frac{1}{2}\left(\frac{4}{3}\pi 1^3\right)} \\
&= \frac{3}{8}.
\end{aligned}$$

25.2. **Cylindrical coordinates.** In $\mathbb{R}^3$, replace $x, y$ by polar coordinates, but keep $z$: this leads to cylindrical coordinates.

To convert from cylindrical coordinates $(r, \theta, z)$ to rectangular coordinates $(x, y, z)$:

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z.
\end{aligned}$$

The other way:

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \tan\theta &= \frac{y}{x} \\ z &= z. \end{aligned}$$

What happens to volume?

$$dV = dx\,dy\,dz = dz\,r\,dr\,d\theta.$$

This can also be visualized geometrically.

Let's redo the half-ball integral: In cylindrical coordinates, the half-ball is given by $r^2 + z^2 \le 1$ and $z \ge 0$. For each fixed point $(r, \theta)$ in the plane, look at the vertical segment above/below it contained in $T$ to get the range of integration for $z$:

$$\begin{aligned} \iiint_T z\,dV &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=\sqrt{1-r^2}} z\,dz\,r\,dr\,d\theta \\ &= \int_0^{2\pi} \int_0^1 \frac{1-r^2}{2}\,r\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{1}{8}\,d\theta \\ &= \frac{\pi}{4}, \end{aligned}$$

which agrees with what we got before.

25.3. **Spherical coordinates.** Spherical coordinates are $(\rho, \phi, \theta)$ (pronounced "rho, phi, theta"), where

$\rho :=$ distance to $(0,0,0)$

$\phi :=$ angle down from $z$-axis (angle between position vector and positive $z$-axis)

$\theta :=$ same $\theta$ as in cylindrical coordinates, "longitude", depends only on $x, y$.

Warning: Some books use different Greek letters here, so when reading another book, check the definitions of the variables.

It may help to imagine an $r \times z$ swinging door hinged along the $z$-axis, with one corner at $(0,0,0)$ and one corner at $(x,y,z)$, so $\rho$ is length of the diagonal of the door, and $\phi$ is the angle the diagonal forms with the $z$-axis; by trigonometry in the upper half of the door (above the diagonal), $r = \rho\sin\phi$ is the width, and $z = \rho\cos\phi$ is the height (at least if $z > 0$). Finally, $\theta$ controls how far counterclockwise the door has swung.

Range of possible values:

$$\begin{aligned} \rho &\in [0, \infty) \\ \phi &\in [0, \pi] \\ \theta &\in [0, 2\pi] \qquad \text{(or } [-\pi, \pi] \text{ or } \ldots) \end{aligned}$$

Conversion: Given $(\rho, \phi, \theta)$, we have $r = \rho \sin \phi$, so

$$\begin{aligned} x &= r\cos\theta = \rho \sin\phi \cos\theta \\ y &= r\sin\theta = \rho \sin\phi \sin\theta \\ z &= \qquad\quad\ \rho\cos\phi. \end{aligned}$$

To convert backwards: Given $(x, y, z)$, compute

$$\rho := \sqrt{x^2 + y^2 + z^2},$$

then convert $(x, y)$ to polar coordinates $(r, \theta)$ to get $\theta$, and finally get $\phi$ from

$$\cos\phi = \frac{z}{\rho}.$$

(That is better than using

$$\sin\phi = \frac{r}{\rho}$$

since the latter cannot distinguish between $\phi$ and $\pi - \phi$.)

## 26. April 25

### 26.1. **Spherical coordinates, continued.**

26.1.1. *Relationship to latitude and longitude.* A great circle on a sphere $S$ is the intersection of $S$ with a plane through the center. If $P, Q$ are in $S$, the shortest path *along the sphere* connecting $P$ and $Q$ is an arc of a great circle. (To visualize this, rotate the sphere so that $P$ and $Q$ are on the equator.)

Set up an $xyz$-coordinate system with the origin at the center of the earth, with the positive $z$-axis passing through the North Pole, with the $xy$-plane containing the Equator, and with Greenwich (near London) contained in the $xz$-plane. Then $\theta°$ (i.e., $\theta$ measured in degrees) is longitude (east if $\theta > 0$). And $90° - \phi°$ is latitude (number of degrees up from the Equator).

26.1.2. *Volume in spherical coordinates.* The region with spherical coordinates in the intervals $[\rho, \rho + d\rho]$, $[\phi, \phi + d\phi]$, $[\theta, \theta + d\theta]$ defines a tiny box of volume approximately $d\rho$ times the surface area of a little "rectangle" on the sphere of sides $\rho\, d\phi$ and $r\, d\theta$ (these "sides" are actually tiny arcs of circles, and the formula for the length of such an arc is (radius)(arc measure)). The volume of the box is approximately

$$(d\rho)(\rho\, d\phi)(r\, d\theta) = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta,$$

so

$$\boxed{dV = dx\, dy\, dz = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.}$$

Another way to find this formula: use the change of variable formula

$$dx\, dy\, dz = \left| \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} \right| d\rho\, d\phi\, d\theta.$$

(The volume scaling factor is the absolute value of the $3 \times 3$ Jacobian determinant.)

26.2. **Gravitation.** Let $\mathbf{F}$ be the gravitational force of a point of mass $M$ with position vector $\mathbf{r}$ acting on a point of mass $m$ at $(0, 0, 0)$. Let $\rho = |\mathbf{r}|$. Then Newton says

$$\mathbf{F} = \frac{GmM}{\rho^2} \frac{\mathbf{r}}{\rho}.$$

If instead of a point of mass $M$, there is a solid body $T$ whose density is given by $\delta = \delta(x, y, z)$, so $dM = \delta\, dV$, then the $z$-component of the gravitational force is

$$\begin{aligned}
\iiint\limits_T \frac{Gm\, dM}{\rho^2} \frac{\mathbf{r}}{\rho} \cdot \mathbf{k} &= \iiint\limits_T \frac{Gm\, dM}{\rho^2} \frac{z}{\rho} \\
&= Gm \iiint\limits_T \frac{\cos\phi}{\rho^2} dM \\
&= Gm \iiint\limits_T \frac{\cos\phi}{\rho^2} \delta\, dV \\
&= Gm \iiint\limits_T \delta \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta.
\end{aligned}$$

(We took the $z$-component so that we would be integrating a scalar function, since we haven't talked about triple integrals of vector-valued functions.)

**Example 26.1.** Suppose that $T$ is a solid sphere of radius $a$ centered at $(0, 0, a)$, of constant density $\delta = 1$, and suppose that $m = 1$. Then by symmetry, the gravitational force $\mathbf{F}$ is pointing straight up, so $|\mathbf{F}|$ equals the $z$-component of $\mathbf{F}$, which is

$$\begin{aligned} G\iiint_T \cos\phi\sin\phi\,d\rho\,d\phi\,d\theta &= G\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2a\cos\phi}\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta \\ &= G\int_0^{2\pi}\int_0^{\pi/2} 2a\cos^2\phi\sin\phi\,d\phi\,d\theta \\ &= 2Ga\int_0^{2\pi} -\frac{1}{3}\cos^3\phi\Big|_0^{\pi/2}\,d\theta \\ &= 2Ga\int_0^{2\pi}\frac{1}{3}\,d\theta \\ &= \frac{4\pi Ga}{3}. \end{aligned}$$

On the other hand, since $\delta = 1$, the mass of $T$ is $M = \text{Volume}(T) = \frac{4}{3}\pi a^3$, so *if all the mass of $T$ were concentrated at its center*, then Newton's law of gravitation for point masses would give

$$|\mathbf{F}| = \frac{GM}{a^2} = \frac{4\pi Ga}{3}.$$

Same answer!

*Remark* 26.2. Newton proved more generally that on *any* external mass, a solid sphere of constant density exerts the same force as would a point mass $M$ placed at its center.

## 27. April 26

27.1. **Parametrized surfaces.** A region $R$ in $\mathbb{R}^2$ can be mapped to a curved surface $S$ in $\mathbb{R}^3$. Example: A disk can be mapped to a Salvador Dalí watch.

In general, a parametrized surface $S$ is the image of a region in the $uv$-plane under a function

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle.$$

It is smooth at a point corresponding to $(u,v) \in R$ if the vectors

$$\begin{aligned} \mathbf{r}_u &:= \langle x_u, y_u, z_u\rangle \\ \mathbf{r}_v &:= \langle x_v, y_v, z_v\rangle \end{aligned}$$

are independent (i.e., nonzero and nonparallel). This guarantees that tiny rectangles in $R$ get mapped to tiny "parallelograms" in $S$ (instead of being compressed into a curve, for example). The quotation marks are there because the "parallelograms" in $S$ could be slightly warped and hence not actual parallelograms.

**Problem 27.1.** Let $S$ be the lateral surface of the cone $x^2 + y^2 = z^2$ for $0 \le z \le 5$. (Lateral means "on the side". In cylindrical coordinates, the equation would be $r = z$; that's why it's a cone.) What is a parametrization of $S$? Is $S$ smooth?

Solution: We need to describe a point on the cone using two numbers $u, v$. Let $u$ be the $r$ of cylindrical coordinates, measuring the distance of a point is from the $z$-axis. Let $v$ be the $\theta$ of cylindrical coordinates. Then for a point $(x, y, z)$ on the cone,

$$\begin{aligned} x &= r\cos\theta = u\cos v \\ y &= r\sin\theta = u\sin v \\ z &= r = u. \end{aligned}$$

Also, the range for $u$ is $[0, 5]$ (since we want the part with $0 \le z \le 5$) and the range for $v$ is $[0, 2\pi]$. In other words, the parametrization is

$$\mathbf{r}(u, v) := \langle u\cos v, u\sin v, u\rangle,$$

defined on the rectangle $[0, 5] \times [0, 2\pi]$ in the $uv$-plane. The function $\mathbf{r}$ maps this rectangle onto the cone.

Now let's check smoothness. We have

$$\begin{aligned} \mathbf{r}_u &= \langle \cos v, \sin v, 1\rangle \\ \mathbf{r}_v &= \langle -u\sin v, u\cos v, 0\rangle. \end{aligned}$$

At a point where $u = 0$, we get $\mathbf{r}_v = 0$. But at a point where $u \neq 0$, we have that $\mathbf{r}_v$ is a nonzero vector in the $xy$-plane while $\mathbf{r}_u$ has a nonzero vertical component so $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and nonparallel. Thus $S$ is smooth except at the points where $u = 0$, which corresponds to $\mathbf{r} = \mathbf{0}$. This makes sense: the cone is smooth except at its vertex.

27.2. **Surface area.** Suppose that $S$ is a surface parametrized by $\mathbf{r}\colon R \to S$. To approximate the surface area of $S$, we cut $R$ into tiny rectangles, look at their images under $\mathbf{r}$, and add up the areas of these tiny "parallelograms" covering $S$.

Imagine a tiny rectangle $[u, u + du] \times [v, v + dv]$ in the $uv$-plane. It will be mapped to a "parallelogram" whose sides are the vectors $\mathbf{r}_u\, du$ and $\mathbf{r}_v\, dv$. Define

$$\begin{aligned} d\mathbf{S} &:= (\mathbf{r}_u\, du) \times (\mathbf{r}_v\, dv) \\ &= \mathbf{r}_u \times \mathbf{r}_v\, du\, dv. \end{aligned}$$

What is the geometric meaning of $d\mathbf{S}$?

- The length of $d\mathbf{S}$ is the *area* of the parallelogram, which should be thought of as a piece of surface area, denoted

$$dS := |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv.$$

- The direction of $d\mathbf{S}$ is the same as the unit normal vector $\mathbf{n}$ to the surface.

So

$$d\mathbf{S} = \mathbf{n}\, dS.$$

Define the surface area of $S$ to be

$$\text{Area}(S) := \iint_R dS = \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\; du\, dv.$$

27.2.1. *Surface area of a hemisphere.*

**Problem 27.2.** Let $S$ be the upper half of the unit sphere centered at the origin. What is the surface area?

(There are many ways to do this; our purpose here is to use the general method for calculating surface area via a parametrization — this will also serve as a review of spherical coordinates.)

Solution: First we choose a parametrization of $S$. To describe a point on $S$, use the spherical coordinates $\phi$ and $\theta$. (We don't need $\rho$, since it equals 1 at every point of $S$.) The region in the $\phi\theta$-plane is the rectangle $R$ where $0 \le \phi \le \pi/2$ and $0 \le \theta \le 2\pi$. The map that sends $R$ onto $S$ is

$$\mathbf{r}(\phi, \theta) := \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle.$$

Then

$$\begin{aligned}
\mathbf{r}_\phi &= \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi \rangle \\
\mathbf{r}_\theta &= \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0 \rangle \\
\mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} \\
&= (\sin^2\phi\cos\theta)\mathbf{i} + (\sin^2\phi\sin\theta)\mathbf{j} + (\sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta)\mathbf{k} \\
&= (\sin^2\phi\cos\theta)\mathbf{i} + (\sin^2\phi\sin\theta)\mathbf{j} + (\sin\phi\cos\phi)\mathbf{k} \\
&= (\sin\phi)\mathbf{r} \\
|\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sin\phi \\
dS &= \sin\phi\, d\phi\, d\theta \\
\text{Area}(S) &= \int_0^{2\pi} \int_0^{\pi/2} \sin\phi\, d\phi\, d\theta \\
&= \int_0^{2\pi} d\theta \\
&= 2\pi.
\end{aligned}$$

Note: The geometric interpretation of the cross product shows that the vector $\mathbf{r}_\phi \times \mathbf{r}_\theta$ is perpendicular to the sphere at each point; this explains why it turned out to be a scalar multiple of $\mathbf{r}$.

27.3. **3D vector fields.** A 3-dimensional vector field has the form

$$\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}.$$

**Definition 27.3.** The divergence of $\mathbf{F}$ is

$$\begin{aligned} \operatorname{div}\mathbf{F} &:= \nabla \cdot \mathbf{F} \\ &= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle \\ &= P_x + Q_y + R_z. \end{aligned}$$

Vector or scalar? It's a scalar, just as in the 2-dimensional case.

27.4. **Surface integrals.** Recall: For a curve $C$,

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt.$$

(We compute the left side by choosing a parametrization $\mathbf{r}(t)$ of $C$, for $t \in [a,b]$.

Similarly: If $S$ is a surface, and $f = f(x,y,z)$, then

$$\iint_S f\,dS = \iint_R f(\mathbf{r}(u,v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv,$$

where $\mathbf{r}(u,v)$ is a parametrization of $S$, mapping the points $(u,v) \in R$ to the points of $S$.

Important reminder: To compute a 1-dimensional integral on a curve $C$, you must choose a parametrization $\mathbf{r}(t)$ to convert it to an integral on a straight interval $[a,b]$ in the real line. Generally, to compute a 2-dimensional integral on a curved surface $S$, you must choose a parametrization $\mathbf{r}(u,v)$ to convert it to an integral over a flat region $R$ in the $uv$-plane. The only exceptions to this are:

- If the integrand is a constant $c$, then

$$\int_C c\,ds = c\,\text{Length}(C)$$

$$\int_S c\,dS = c\,\text{Area}(S),$$

  and you might know Length($C$) or Area($S$) already by geometry.
- Sometimes you can use Green's theorem or some other theorem to convert the integral to some other kind of integral.

27.4.1. *Applications of surface integrals.*

**Definition 27.4.** The average value of $f$ on a surface $S$ is

$$\frac{\iint_S f\,dS}{\text{Area}(S)}$$

The definitions of mass, centroid, and moment of inertia extend to surfaces as well.

27.5. **3D flux.** Recall: In 2D,

$$\text{flux of } \mathbf{F} \text{ across } C \ := \int_C \mathbf{F}\cdot\mathbf{n}\,ds.$$

In 3D, the flux of a continuous vector field $\mathbf{F}$ across a *surface* $S$ is a special kind of surface integral:

$$\boxed{\text{flux of } \mathbf{F} \text{ across } S \ := \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S \mathbf{F}\cdot d\mathbf{S},}$$

where $\mathbf{n}$ is a unit normal vector to the surface, and

$$d\mathbf{S} := \mathbf{n}\,dS.$$

There are actually two choices for $\mathbf{n}$ at each point. If $S$ bounds a 3D region, then one usually chooses $\mathbf{n}$ to be the *outward* unit normal.

27.5.1. *Physical meaning of 3D flux.* Intuitive explanation: Imagine a 3D fluid with constant velocity field $\mathbf{F}$. The amount of fluid that flows across a tiny parallelogram of area $dS$ in unit time is the fluid in a parallelepiped of base $dS$ and height $\mathbf{F}\cdot\mathbf{n}$. Summing over all tiny parallelepipeds comprising a surface $S$ gives

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS.$$

That's flux!

This intuition works more generally for any continuous $\mathbf{F}$ and smooth $S$, since $\mathbf{F}$ is almost constant on small regions, and $S$ is well approximated by tiny parallelograms. Flux measures the rate of flow (volume per unit time, measured in $\text{m}^3/\text{s}$, say).

27.5.2. *Computing 3D flux.* To compute a flux integral

$$\iint_S \mathbf{F}\cdot d\mathbf{S}:$$

(1) Choose a parametrization of $S$, say $\mathbf{r}(u,v)$ for $(u,v)$ in the flat $2D$ region $R$. (The whole point is to convert the surface integral over the curved surface $S$ into a double integral over the flat region $R$.)
(2) Compute the partial derivatives $\mathbf{r}_u$, $\mathbf{r}_v$, and their cross product $\mathbf{r}_u\times\mathbf{r}_v$.
(3) Make sure that $\mathbf{r}_u\times\mathbf{r}_v$ is in the same direction as the desired outward unit normal $\mathbf{n}$. (If it is in the opposite direction, negate your answer at the end.)

(4) Substitute

$$d\mathbf{S} = \mathbf{r}_u \times \mathbf{r}_v \, du \, dv.$$

to get

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_R \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv.$$

(5) Evaluate the double integral over the flat 2-dimensional region $R$; usually this will be done by converting it to an iterated integral.

Next time: an example of computing 3D flux.

## 28. April 30

**Example 28.1.** Let $S$ in $\mathbb{R}^3$ be defined by $x^2 + y^2 = 4$ and $0 \le z \le 3$. (So $S$ is the lateral surface of a cylinder.) Let $\mathbf{F}(x, y, z) = \langle yz, x^2 + 2, 5 \rangle$. What is the outward flux of $\mathbf{F}$ across $S$?

Solution: Choose the parameterization where $u$ is the $\theta$ of cylindrical coordinates, and $v = z$. So $\mathbf{r}(u, v) = \langle 2\cos u, 2\sin u, v \rangle$, for $0 \le u \le 2\pi$ and $0 \le v \le 3$. Then

$$\begin{aligned}
\mathbf{r}_u &= \langle -2\sin u, 2\cos u, 0 \rangle \\
\mathbf{r}_v &= \langle 0, 0, 1 \rangle \\
\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \langle 2\cos u, 2\sin u, 0 \rangle.
\end{aligned}$$

Notice that $\mathbf{r}_u \times \mathbf{r}_v$ is in the same direction as the outward unit normal $\mathbf{n}$. Next

$$\begin{aligned}
d\mathbf{S} &= \mathbf{r}_u \times \mathbf{r}_v \, du \, dv \\
&= \langle 2\cos u, 2\sin u, 0 \rangle \, du \, dv.
\end{aligned}$$

So the flux is

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^3 \int_0^{2\pi} \langle yz, x^2 + 2, 5 \rangle \cdot \langle 2\cos u, 2\sin u, 0 \rangle \, du \, dv \\
&= \int_0^3 \int_0^{2\pi} \big( yz(2\cos u) + (x^2 + 2)(2\sin u) \big) \, du \, dv \\
&= \int_0^3 \int_0^{2\pi} \big( (2\sin u)v(2\cos u) + ((2\cos u)^2 + 2)(2\sin u) \big) \, du \, dv \\
&= \int_0^3 \int_0^{2\pi} \big( 2v\sin 2u + 8\cos^2 u \sin u + 4\sin u \big) \, du \, dv \\
&= 0,
\end{aligned}$$

because $\int_0^{2\pi} \sin 2u\, du$ is the integral over two complete cycles, which is 0, and the last two terms in the integrand are negated by $u \mapsto u + \pi$ so their integrals over $[0, \pi]$ cancel with the integrals over $[\pi, 2\pi]$.

28.1. **Divergence theorem.** Recall Green's theorem for flux:

Let $C$ be a positively-oriented simple closed (piecewise smooth) curve bounding a region $R$ in $\mathbb{R}^2$. Let $\mathbf{F}$ be a vector field that is continuously differentiable not only on $C$ but also on $R$. Then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA$$

The divergence theorem is the 3D analogue.

**Definition 28.2.** A region $T$ in $\mathbb{R}^3$ is called bounded if there is a sphere that contains it.

**Definition 28.3.** A closed surface $S$ is a (piecewise smooth) surface that is the entire boundary of a bounded region $T$ in $\mathbb{R}^3$. To say that it is positively-oriented means that at each (smooth) point of $S$ we choose the *outward* unit normal $\mathbf{n}$. An integral over a closed surface $S$ is sometimes written using the notation $\oiint_S$, although $\iint_S$ is also correct.

Examples: A sphere, a cube, a torus, a finite cylinder (including the top and bottom disks).

**Theorem 28.4** (Divergence theorem, also known as Gauss's theorem or Ostrogradsky's theorem — actually discovered earlier by Lagrange)**.** *Let $S$ be a positively-oriented closed surface bounding a region $T$ in $\mathbb{R}^3$. Let $\mathbf{F}$ be a vector field that is continuously differentiable not only on $S$ but also on $T$. Then*

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_T \operatorname{div} \mathbf{F}\, dV.$$

We'll prove this later. There is also an extended version in which the boundary of $T$ consists of several surfaces. For instance, if $T$ is a spherical shell, its boundary $S$ consists of an outer sphere and an inner sphere, oriented appropriately.

28.1.1. *Physical interpretation.* The physical interpretation of the divergence theorem is the same as that of Green's theorem for flux. Suppose that $\mathbf{F}$ is the velocity field of an incompressible 3D fluid. Then $\operatorname{div} \mathbf{F}$ is the source rate (the rate at which fluid is being created per unit volume) so the right side is total rate of fluid creation inside $T$. The left side is the outward flux across $S$, the rate at which fluid is crossing $S$. The divergence theorem is saying that the rate at which fluid is created inside $T$ must equal the rate at which it is overflowing and leaking across $S$.

Why is it true that for $\mathbf{F} = \langle P, Q, R \rangle$ the formula $\operatorname{div} \mathbf{F} = P_x + Q_y + R_z$ gives the source rate?

Imagine a tiny rectangular box with coordinates in $[x, x + \Delta x]$, $[y, y + \Delta y]$, $[z, z + \Delta z]$.

Parametrize the top face by $\mathbf{r}(u,v) = \langle x+u, y+v, z+\Delta z\rangle$, so $dS = du\,dv$. The flux across the top face is

$$\int_{\text{top face}} \mathbf{F}\cdot\mathbf{n}\,dS = \int_{v=0}^{\Delta y}\int_{u=0}^{\Delta x} \mathbf{F}(x+u, y+v, z+\Delta z)\cdot\mathbf{k}\,du\,dv$$
$$= \int_{v=0}^{\Delta y}\int_{u=0}^{\Delta x} R(x+u, y+v, z+\Delta z)\,du\,dv.$$

Similarly, the flux across the bottom face is

$$\int_{v=0}^{\Delta y}\int_{u=0}^{\Delta x} \mathbf{F}(x+u, y+v, z)\cdot(-\mathbf{k})\,du\,dv = -\int_{v=0}^{\Delta y}\int_{u=0}^{\Delta x} R(x+u, y+v, z)\,du\,dv.$$

Adding gives the net flux across the top and bottom faces:

$$\int_{v=0}^{\Delta y}\int_{u=0}^{\Delta x} \left(R(x+u, y+v, z+\Delta z) - R(x+u, y+v, z)\right)\,du\,dv \approx \int_{v=0}^{\Delta y}\int_{u=0}^{\Delta x} \frac{\partial R}{\partial z}(x+u, y+v, z)\,\Delta z\,du\,dv$$
$$\approx R_z\,\Delta x\,\Delta y\,\Delta z$$

by the approximation formula, because $R_z$ is approximately constant on the tiny box.

A similar computation shows that the other pairs of opposite sides contribute a net flux of approximately

$$P_x\,\Delta x\,\Delta y\,\Delta z \qquad \text{and} \qquad Q_y\,\Delta x\,\Delta y\,\Delta z$$

The total source rate *for the box* is the net flux across all six sides, which is

$$(P_x + Q_y + R_z)\,\Delta x\,\Delta y\,\Delta z = (\operatorname{div}\mathbf{F})\,\Delta x\,\Delta y\,\Delta z.$$

So the source rate *per unit volume* is

$$\frac{(\operatorname{div}\mathbf{F})\,\Delta x\,\Delta y\,\Delta z}{\Delta x\,\Delta y\,\Delta z} = \operatorname{div}\mathbf{F}.$$

## 29. May 2

29.1. **Proof of the divergence theorem.** Last time we introduced the divergence theorem:

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_T \operatorname{div}\mathbf{F}\,dV.$$

Why is this true?

Suppose that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. We need to show

$$\oiint_S (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k})\cdot d\mathbf{S} = \iiint_T (P_x + Q_y + R_z)\,dV.$$

It's enough to show

$$\oiint_S R\,\mathbf{k}\cdot d\mathbf{S} = \iiint_T R_z\,dV \tag{5}$$

since this together with the similar equalities for $P$ and $Q$ will sum to give the whole theorem.

We'll prove (5) in the case of a "vertically simple" region $T$ whose bottom and top surfaces are graphs, say $z = f_1(x, y)$ and $z = f_2(x, y)$ above a region $D$ in the plane.

Right side of (5):

$$\iiint_T R_z \, dV = \iint_D \int_{z=f_1(x,y)}^{z=f_2(x,y)} R_z \, dz \, dA$$
$$= \iint_D R(x, y, z)\Big|_{z=f_1(x,y)}^{z=f_2(x,y)} \, dA$$
$$= \iint_D (R(x, y, f_2(x, y)) - R(x, y, f_1(x, y)) \, dx \, dy.$$

Flux through top? Use the parametrization $\mathbf{r}(x, y) = \langle x, y, f_2(x, y)\rangle$ for $(x, y) \in D$, so

$$d\mathbf{S} = \mathbf{r}_x \times \mathbf{r}_y \, dx \, dy$$
$$= \left\langle 1, 0, \frac{\partial f_2}{\partial x} \right\rangle \times \left\langle 0, 1, \frac{\partial f_2}{\partial y} \right\rangle dx \, dy$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f_2}{\partial x} \\ 0 & 1 & \frac{\partial f_2}{\partial y} \end{vmatrix} dx \, dy$$
$$= \left\langle -\frac{\partial f_2}{\partial x}, -\frac{\partial f_2}{\partial y}, 1 \right\rangle dx \, dy,$$
$$\mathbf{k} \cdot d\mathbf{S} = dx \, dy.$$

so

$$\iint_{\text{top}} R\mathbf{k} \cdot d\mathbf{S} = \iint_D R(x, y, f_2(x, y)) \, dx \, dy.$$

Flux through bottom? Similar, except $\mathbf{r}_u \times \mathbf{r}_v$ is in the opposite direction of the outward $\mathbf{n}$, so

$$\iint_{\text{bottom}} R\mathbf{k} \cdot d\mathbf{S} = -\iint_D R(x, y, f_1(x, y)) \, dx \, dy.$$

Flux through sides: $R\mathbf{k}$ is vertical, but $d\mathbf{S} = \mathbf{n} \, dS$ is horizontal, so the flux through the sides is 0.

Total flux: Adding the flux through the top, bottom, and sides gives

$$\iint_S R\mathbf{k} \cdot d\mathbf{S} = \iint_D (R(x, y, f_2(x, y)) - R(x, y, f_1(x, y)) \, dx \, dy.$$

This matches the right side calculation!

To prove the divergence theorem for a more general region, one could cut it into simple pieces.

29.2. **Divergence and gravitation.**

**Lemma 29.1.** *Let* $\mathbf{F}$ *be a 3D vector field pointing radially outward and whose magnitude is* $1/\rho^2$ *(defined everywhere except the origin). Then* $\operatorname{div} \mathbf{F} = 0$*.*

*Proof.* Since the radially outward unit vector is $\mathbf{r}/|\mathbf{r}| = \mathbf{r}/\rho$, this means that an explicit formula for $\mathbf{F}$ is

$$\mathbf{F} = \frac{1}{\rho^2}\frac{\mathbf{r}}{\rho} = \rho^{-3}\langle x, y, z\rangle = \langle \rho^{-3}x, \rho^{-3}y, \rho^{-3}z\rangle.$$

To calculate $\operatorname{div} \mathbf{F}$, we need to calculate partial derivatives of the three coordinate functions, while remembering that $\rho$ is really a function of $x, y, z$. We could just substitute $\rho = \sqrt{x^2+y^2+z^2}$ and calculate the partial derivatives explicitly. Alternatively, taking $\frac{\partial}{\partial x}$ of

$$\rho^2 = x^2 + y^2 + z^2$$

gives

$$2\rho\frac{\partial \rho}{\partial x} = 2x$$

so

$$\frac{\partial \rho}{\partial x} = \frac{x}{\rho}.$$

Now

$$\begin{aligned}\frac{\partial}{\partial x}(\rho^{-3}x) &= \rho^{-3} + (-3\rho^{-4})\frac{\partial \rho}{\partial x}x \\ &= \rho^{-3} - 3\rho^{-5}x^2.\end{aligned}$$

Summing this with the corresponding equations for $y$ and $z$ gives

$$\begin{aligned}\operatorname{div} \mathbf{F} &= 3\rho^{-3} - 3\rho^{-5}(x^2+y^2+z^2) \\ &= 3\rho^{-3} - 3\rho^{-5}\rho^2 \\ &= 0.\end{aligned}$$

□

Let $\mathbf{F} = \mathbf{F}(x, y, z)$ be the gravitational field of a point mass $M$ at $(0, 0, 0)$ (i.e., the force that it would exert on a unit mass at $(x, y, z)$). By Newton's inverse square law,

$$\mathbf{F} = -\frac{GM}{\rho^2}\frac{\mathbf{r}}{\rho}.$$

This is just a constant times the vector field in the lemma, so

$$\operatorname{div} \mathbf{F} = 0$$

at every point in $\mathbb{R}^3$ except $(0, 0, 0)$.

**Question 29.2.** Let $\mathbf{F}$ be the gravitational field of a point of mass $M$ at the origin. Let $S_a$ be the sphere of radius $a$ centered at the origin. Let $S_{2a}$ be the sphere of twice the radius. Let's compare the flux across $S_a$ with the flux across $S_{2a}$. Which of the following is correct?

(1) The flux across $S_{2a}$ is 4 times as much, because the integral is over a surface area that is 4 times bigger.
(2) The flux across $S_{2a}$ is $1/4$ as much, because the gravitational field is $1/4$ as strong.
(3) The fluxes are equal and nonzero.
(4) The fluxes are both 0, because the divergence theorem says

$$\oiint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \iiint_T \operatorname{div} \mathbf{F} \, dV = 0,$$

since $\operatorname{div} \mathbf{F} = 0$ everywhere.

Answer: The fluxes are equal and nonzero. (The two effects in the first two answers cancel each other out. The application of the divergence theorem in the last answer is wrong: $\mathbf{F}$ is not defined at $(0,0,0)$, so the right side of the divergence theorem does not even make sense.)

**Question 29.3.** What is the flux across the sphere $S_a$ of radius $a$ centered at $(0,0,0)$?

Answer: At every point, $\mathbf{F}$ and $\mathbf{n}$ are in *opposite* directions, so

$$\begin{aligned}
\text{flux} &= \oiint_{S_a} \mathbf{F} \cdot \mathbf{n} \, dS \\
&= \oiint_{S_a} -|\mathbf{F}| \, dS \\
&= \oiint_{S_a} -\frac{GM}{a^2} \, dS \\
&= -\frac{GM}{a^2} \operatorname{Area}(S_a) \\
&= -\frac{GM}{a^2} (4\pi a^2) \\
&= -4\pi GM.
\end{aligned}$$

It is independent of the radius $a$!

Even better, if $S$ is *any* closed surface enclosing the point mass, and $T$ is the 3D region between $S$ and a small sphere $S_a$ centered at the point mass (so $T$ has a bubble inside), then the extended divergence theorem shows that

$$\begin{aligned}
\oiint_S \mathbf{F} \cdot d\mathbf{S} - \oiint_{S_a} \mathbf{F} \cdot d\mathbf{S} &= \iiint_T \operatorname{div} \mathbf{F} \, dV \\
&= 0,
\end{aligned}$$

so

$$\begin{aligned}\oiint_S \mathbf{F} \cdot d\mathbf{S} &= \oiint_{S_a} \mathbf{F} \cdot d\mathbf{S} \\ &= -4\pi GM.\end{aligned}$$

The same claim is true if there are many point masses inside $S$, or even some planets inside $S$, because the force fields add. This proves Gauss's law:

$$\text{gravitational flux across } S \ = -4\pi GM,$$

where $M$ is the mass enclosed by $S$.

## 30. May 3

### 30.1. Gravitation, continued.

**Example 30.1.** What is the gravitational field inside a hollow spherical planet? Inside a centered sphere $S$ of radius $r$ inside the hollow part, symmetry implies that $\mathbf{F} = c\mathbf{n}$ for some $c$ depending only on $r$. Then the flux across $S$ is $4\pi r^2 c$, but Gauss's law says that it equals 0, so $c = 0$. Thus the gravitational field is $\mathbf{0}$ everywhere inside the hollow part.

**Question 30.2.** For a donut-shaped planet, if you are standing on the inner circle of the planet, is gravity pulling you towards the center of mass or is it pulling you towards the planet under your feet?

*Hint:* Imagine filling in the donut hole with a cylinder of very small height. The total flux through the cylinder is 0 by Gauss's law. On the other hand, is the outward flux through the top and bottom disks positive or negative? If you figure that out, that can help answer the question, because it must be cancelled by the flux through the lateral surface of the cylinder.

### 30.2. Application of the divergence theorem to an electric field.

Let $\mathbf{E} = \mathbf{E}(x, y, z)$ be the electric field of a point charge $Q$ at $(0, 0, 0)$. Coulomb's law:

$$\mathbf{E} = \frac{Q/4\pi\epsilon_0}{\rho^2} \frac{\mathbf{r}}{\rho}$$

where $\epsilon_0$ is a constant.

The physics is different, but the math is the same as for gravitation, with the constant $Q/4\pi\epsilon_0$ in place of $-GM$. The gravitational flux was $-4\pi GM$, which is $4\pi$ times the constant appearing in the inverse square law for gravitation. Similarly, the electric flux is

$$4\pi(Q/4\pi\epsilon_0) = \frac{Q}{\epsilon_0}.$$

Summary:

| | field | flux |
|---|---|---|
| gravitational | $\mathbf{F} = -\frac{GM}{\rho^2}\frac{\mathbf{r}}{\rho}$ | $-4\pi GM$ |
| electric | $\mathbf{E} = \frac{Q/4\pi\epsilon_0}{\rho^2}\frac{\mathbf{r}}{\rho}$ | $Q/\epsilon_0$ |

So we get Gauss's law for an electric field (also called the Gauss-Coulomb law):

> The electric flux across a closed surface $S$ equals the charge enclosed divided by $\epsilon_0$,
>
> $$\oiint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}.$$

30.3. **3D Curl.** Recall: if $\mathbf{F} = \langle P, Q \rangle$ is a 2D vector field, then

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \\ &= Q_x - P_y. \end{aligned}$$

**Definition 30.3.** If $\mathbf{F} = \langle P, Q, R \rangle$ is a 3D vector field, then

$$\begin{aligned} \operatorname{curl} \mathbf{F} &:= \nabla \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}. \end{aligned}$$

*Warning:*

- If $\mathbf{F}$ is a 2D vector field, then $\operatorname{curl} \mathbf{F}$ is a *scalar-valued* function of $(x, y)$.
- If $\mathbf{F}$ is a 3D vector field, then $\operatorname{curl} \mathbf{F}$ is a *vector-valued* function of $(x, y, z)$.

30.3.1. *Physical interpretation.* If $\mathbf{F}$ is a velocity field, $\operatorname{curl} \mathbf{F}$ measures the rotational component of $\mathbf{F}$:

- direction of $\operatorname{curl} \mathbf{F}$ is along the axis of rotation (have the fingers of your right hand curl in the direction of rotation; then your thumb gives the direction)
- length of $\operatorname{curl} \mathbf{F}$ is 2(angular velocity).

**Example 30.4.** If $\mathbf{F} = \langle -\omega y, \omega x, 0 \rangle$ (rotation around $z$-axis with angular velocity $\omega$), then $\operatorname{curl} \mathbf{F} = 2\omega\mathbf{k}$.

30.4. **Line integrals in 3D.** Let $C$ be a piecewise smooth curve in $\mathbb{R}^3$.

Three kinds of line integrals:

- Line integral with respect to arc length: $\int_C f(x, y, z)\, ds$. (Remember: $s$ = distance traveled = arc length.)
- Line integral with respect to coordinate variables: $\int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$
- Line integral of a vector field: $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Each of these is approximated by a sum of products; e.g.,

$$\int_C f(x, y, z)\, ds \approx \sum_{i=1}^{n} f(P_i)\, \Delta s_i$$

To compute it: convert to an ordinary integral over a straight line segment by substituting a parametrization of $C$, say $\mathbf{r}(t)$ for $t \in [a, b]$.

30.5. **Fundamental theorem of calculus for line integrals.** FTC for line integrals: If $C$ starts at $A$ and ends at $B$, and $f(x, y, z)$ is a function that is continuously differentiable at each point of $C$, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

*Warning:* This can be used to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ only when $\mathbf{F}$ is a gradient!

**Definition 30.5.** A differential $P\, dx + Q\, dy + R\, dz$ is exact if it equals $df$ for some $f$.

Since $df := f_x\, dx + f_y\, dy + f_z\, dz$, *exact* means that there is an $f$ such that $P = f_x$, $Q = f_y$, $R = f_z$.

30.6. **Test for gradient.** Let $\mathbf{F} = \langle P, Q, R \rangle$ be a continuously differentiable vector field in a 3D region $T$.

Six conditions:

(1) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve $C$ inside $T$
(2) $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is path independent (for paths inside $T$)
(3) $\mathbf{F} = \nabla f$ (i.e., $P = f_x$, $Q = f_y$, and $R = f_z$) for some $f = f(x, y, z)$ on $T$
(4) $P\, dx + Q\, dy + R\, dz$ is exact (equal to $df$ for some $f$)
(5) $R_y = Q_z$, $P_z = R_x$, and $Q_x = P_y$ at every point of $T$
(6) $\operatorname{curl} \mathbf{F} = \mathbf{0}$ at every point of $T$.

Conditions (1)-(4) are equivalent, and if satisfied, $\mathbf{F}$ is called conservative.

Conditions (5)-(6) are equivalent.

Conditions (1)-(4) imply (5)-(6). E.g., (3) says $P = f_x$, $Q = f_y$, $R = f_z$, which imply $R_y = (f_z)_y = f_{zy} = f_{yz} = (f_y)_z = Q_z$, and so on.

If $T$ is simply connected, then all six conditions are equivalent.

30.7. **Computing a potential function.** Consider the vector field

$$\mathbf{F} = (3x^2y + 5yz)\,\mathbf{i} + (x^3 + 7z + 5xz)\,\mathbf{j} + (7y + 5xy + e^z)\,\mathbf{k}$$

on $\mathbb{R}^3$.

Problem 1: Is there a function $f$ such that $\nabla f = \mathbf{F}$?

Solution: It turns out that $\mathbb{R}^3$ is simply connected, so we can use any of the six conditions. Let's test (6):

$$\begin{aligned}
\operatorname{curl}\mathbf{F} &:= \nabla \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y + 5yz & x^3 + 7z + 5xz & 7y + 5xy + e^z \end{vmatrix} \\
&= \mathbf{i}((7+5x) - (7+5x)) - \mathbf{j}(5y - 5y) + \mathbf{k}((3x^2+5z) - (3x^2+5z)) \\
&= \mathbf{0}.
\end{aligned}$$

So the answer is yes.

Problem 2: Can you find such an $f$?

First solution (antiderivative method): We know

$$\begin{aligned}
f_x &= 3x^2y + 5yz \\
f_y &= x^3 + 7z + 5xz \\
f_z &= 7y + 5xy + e^z.
\end{aligned}$$

The $f_x$ equation implies

$$f = x^3y + 5xyz + g(y,z)$$

for some $g(y,z)$. Taking $\frac{\partial}{\partial y}$ gives

$$f_y = x^3 + 5xz + g_y$$

and comparing with the given $f_y$ equation shows that

$$\begin{aligned}
g_y &= 7z \\
g &= 7yz + h(z) \qquad \text{for some } h(z). \\
f &= x^3y + 5xyz + 7yz + h(z).
\end{aligned}$$

Taking $\frac{\partial}{\partial z}$ gives

$$f_z = 5xy + 7y + h_z$$

and comparing with the given $f_z$ equation shows that

$$h_z = e^z + c \qquad \text{for some constant } c.$$
$$f = x^3y + 5xyz + 7yz + e^z + c.$$

We check that this really has the right gradient.

Lecture actually ended here, due to lack of time.

Second solution (FTC in reverse): If $\nabla f = \mathbf{F}$, then

$$\begin{aligned} f(a,b,c) - f(0,0,0) &= \int_C \nabla f \cdot d\mathbf{r} \\ &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_{(0,0,0)}^{(a,b,c)} (3x^2y + 5yz)\,dx + (x^3 + 7z + 5xz)\,dy + (7y + 5xy + e^z)\,dz. \end{aligned}$$

Choose the path $C_1 + C_2 + C_3$ where $C_1$ goes from $(0,0,0)$ to $(a,0,0)$, $C_2$ goes from $(a,0,0)$ to $(a,b,0)$, and $C_3$ goes from $(a,b,0)$ to $(a,b,c)$. Add up the three integrals. For $C_1$, use the parametrization $\mathbf{r}(t) := \langle t,0,0\rangle$ for $t \in [0,a]$; then $x = t$, $y = 0$, $z = 0$, $dx = dt$, $dy = 0$, $dz = 0$, so

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^a 0\,dt = 0.$$

Similarly, for $C_2$ use $\langle a,t,0\rangle$ for $t \in [0,b]$; then $x = a$, $y = t$, $z = 0$, $dx = 0$, $dy = dt$, $dz = 0$, so

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^b a^3\,dt = a^3b.$$

Finally, for $C_3$, use the parametrization $\langle a,b,t\rangle$ for $t \in [0,c]$, so $x = a$, $y = b$, $z = t$, $dx = 0$, $dy = 0$, $dz = dt$, so

$$\begin{aligned} \int_{C_3} \mathbf{F} \cdot d\mathbf{r} &= \int_0^c (7b + 5ab + e^t)\,dt \\ &= 7bc + 5abc + e^c - 1. \end{aligned}$$

Summing yields

$$\int_C \mathbf{F} \cdot d\mathbf{r} = a^3b + 7bc + 5abc + e^c - 1.$$

Thus

$$f(x,y,z) = x^3y + 7yz + 5xyz + e^z - 1$$

is one possibility. (The others are obtained by adding any constant.)

Third solution: Just guess one possible potential function $f$, and check that it has the right gradient; then the complete set of solutions is the set of functions of the form $f + c$, where $c$ is any number.

MIDTERM #4 COVERS EVERYTHING UP TO HERE

## 31. May 7

### 31.1. Stokes' theorem.

| | $\oint_C \mathbf{F} \cdot d\mathbf{r}$ | flux |
|---|---|---|
| 2D | Green's theorem | Green's theorem for flux |
| 3D | *Stokes' theorem* | Divergence theorem |

Remember Green's theorem?

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} \, dA.$$

Stokes' theorem is the 3D analogue.

Setup: Let $C$ be a closed curve, and let $S$ be any surface bounded by $C$. Roughly speaking, $S$ "fills in" $C$. (Here $S$ is usually not a closed surface; it does not bound a 3-dimensional region.) For example, $C$ could be the Equator on the Earth, in which case $S$ could be the Northern Hemisphere. Let $\mathbf{F}$ be a 3D vector field that is continuously differentiable everywhere on $S$.

Stokes' theorem[1]:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

Here $\operatorname{curl} \mathbf{F}$ means $\nabla \times \mathbf{F}$, and $d\mathbf{S}$ means $\mathbf{n} \, dS$.

One subtlety: both sides depend on a choice of orientation, and these must be chosen *compatibly* in order for the theorem to hold. Namely, the left side depends on the choice of direction along $C$, and the right side depends on the choice of unit normal vector $\mathbf{n}$ that is implicit in $d\mathbf{S} = \mathbf{n} \, dS$.

What does it mean for these to be compatible? Two ways to say it:

- If you walk along $C$ in the chosen direction, with $S$ to your left, then $\mathbf{n}$ is pointing up.
- If your right hand thumb is pointing in the direction of $\mathbf{n}$, then your fingers point in the chosen direction along $C$.

[1] Actually first proved by Lord Kelvin, who mailed it to Stokes, who included it as a question in a physics competition for students.

31.1.1. *Stokes' theorem for a flat surface.* Special case: Suppose that $S$ is a piece of the $xy$-plane bounded by a closed curve $C$ oriented counterclockwise, and suppose that

$$\mathbf{F}(x,y,z) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j} + 0\,\mathbf{k}.$$

Which way should $\mathbf{n}$ be pointing? Up.

In this special case, what are the two sides of Stokes' theorem?

Left side:

$$\begin{aligned} \oint_C \mathbf{F}\cdot d\mathbf{r} &= \oint_C \langle P, Q, 0\rangle \cdot \langle dx, dy, dz\rangle \\ &= \oint_C P\,dx + Q\,dy. \end{aligned}$$

Right side:

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} \\ &= (Q_x - P_y)\mathbf{k} \\ d\mathbf{S} &= \mathbf{n}\,dS \\ &= \mathbf{k}\,dS, \\ \iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} &= \iint_S (Q_x - P_y)\,dS. \end{aligned}$$

So Stokes' theorem for a flat surface says

$$\oint_C P\,dx + Q\,dy = \iint_S (Q_x - P_y)\,dS.$$

This is just Green's theorem!

31.1.2. *Intuition for why Stokes' theorem is true.*

- If $C$ and $S$ are in the $xy$-plane, then Stokes follows from Green.
- If $C$ and $S$ are in an arbitrary plane, the same holds, because of the geometric invariance of work and curl under rotations of space. (They have a meaning that is independent of the coordinate system you are working in.)
- If $S$ consists of many flat polygons hinged together (like a disco ball), then adding up Stokes' theorem for each polygon gives Stokes' theorem for $S$ (the line integrals on shared edges cancel).
- Any piecewise smooth surface $S$ can be approximated by a collection of tiny polygons.

31.1.3. *Example.*

**Problem 31.1.** Let $S$ be the boundary of the cylinder $x^2+y^2 \le 9$, $0 \le z \le 2$ excluding the base in the $xy$-plane. Let $\mathbf{F} = \langle y, -x, y^3 \rangle$. Compute the outward flux of $\operatorname{curl} \mathbf{F}$ across $S$ in as many ways as you can.

*Solution 1: Use Stokes' theorem to convert it to a line integral on the circle $C$ given by $x^2+y^2=9$ in the $xy$-plane.*

$$\begin{aligned} \text{flux of } \operatorname{curl} \mathbf{F} \text{ across } S &= \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \\ &\overset{\text{Stokes'}}{=} \oint_C \mathbf{F} \cdot d\mathbf{r}. \end{aligned}$$

To compute the latter, choose a parametrization of $C$ (counterclockwise, so as to agree with the orientation of $S$). Let's use

$$\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 0 \rangle \qquad \text{for } 0 \le t \le 2\pi$$

so

$$\begin{aligned} d\mathbf{r} &= \langle -3\sin t, 3\cos t, 0 \rangle \, dt \\ \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle 3\sin t, -3\cos t, (3\sin t)^3 \rangle \cdot \langle -3\sin t, 3\cos t, 0 \rangle \, dt \\ &= \int_0^{2\pi} \left( -9\sin^2 t - 9\cos^2 t \right) \, dt \\ &= \int_0^{2\pi} -9 \, dt \\ &= -18\pi. \end{aligned}$$

*Solution 2: Compute the flux directly from the definition.* The surface $S$ consists of the top disk $S_1$ and the lateral surface $S_2$ of the cylinder. We have

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & y^3 \end{vmatrix} \\ &= 3y^2 \mathbf{i} - 2\mathbf{k}. \end{aligned}$$

On $S_1$, the unit normal is $\mathbf{n} = \mathbf{k}$, so

$$\begin{aligned} \operatorname{curl} \mathbf{F} \cdot \mathbf{n} &= -2 \\ \iint_{S_1} \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS &= -2 \operatorname{Area}(S_1) \\ &= -18\pi. \end{aligned}$$

For $S_2$, we use the parametrization in which $u = z$ is the height, and $v$ is the $\theta$ of cylindrical coordinates:

$$\mathbf{r}(u, v) := \langle 3\cos v, 3\sin v, u\rangle$$

for $u \in [0, 2]$ and $v \in [0, 2\pi]$. Then

$$\begin{aligned}
\mathbf{r}_u &= \langle 0, 0, 1\rangle \\
\mathbf{r}_v &= \langle -3\sin v, 3\cos v, 0\rangle \\
\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ -3\sin v & 3\cos v & 0 \end{vmatrix} \\
&= (-3\cos v)\,\mathbf{i} + (-3\sin v)\,\mathbf{j}.
\end{aligned}$$

This gives the *reverse* of the desired orientation for $\mathbf{n}$, so

$$\begin{aligned}
\iint_{S_1} \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS &= -\int_0^{2\pi}\int_0^2 \operatorname{curl}\mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,du\,dv \\
&= -\int_0^{2\pi}\int_0^2 \langle 3(3\sin v)^2, 0, -2\rangle\cdot\langle -3\cos v, -3\sin v, 0\rangle\,du\,dv \\
&= -\int_0^{2\pi}\int_0^2 -81\sin^2 v\cos v\,du\,dv \\
&= \int_0^{2\pi} 162\sin^2 v\cos v\,dv \\
&= 54\sin^3 v\Big|_0^{2\pi} \\
&= 0.
\end{aligned}$$

So the total flux of $\operatorname{curl}\mathbf{F}$ across $S$ is

$$-18\pi + 0 = -18\pi.$$

Lecture actually ended here.

*Solution 3: Use the divergence theorem to compute the flux of* $\operatorname{curl}\mathbf{F}$ *across the* entire *boundary of the cylinder, and then subtract the flux across the bottom disk.* We already calculated that

$$\operatorname{curl}\mathbf{F} = 3y^2\,\mathbf{i} - 2\,\mathbf{k}$$

so

$$\operatorname{div}\operatorname{curl}\mathbf{F} = \frac{\partial}{\partial x}3y^2 + \frac{\partial}{\partial y}0 + \frac{\partial}{\partial z}(-2) = 0 + 0 + 0 = 0.$$

By the divergence theorem, the outward flux of curl $\mathbf{F}$ across the entire boundary of the cylinder is

$$\begin{aligned}
\oiint_{\text{entire boundary}} (\operatorname{curl}\mathbf{F})\cdot d\mathbf{S} &= \iiint_{\text{solid cylinder}} \operatorname{div}\operatorname{curl}\mathbf{F}\,dV \\
&= \iiint_{\text{solid cylinder}} 0\,dV \\
&= 0.
\end{aligned}$$

On the other hand, the outward flux across the bottom disk of the cylinder is

$$\begin{aligned}
\iint_{\text{bottom disk}} (\operatorname{curl}\mathbf{F})\cdot d\mathbf{S} &= \iint_{\text{bottom disk}} (\operatorname{curl}\mathbf{F})\cdot(-\mathbf{k})\,dS \qquad \text{(since the outward unit normal is downward)} \\
&= \iint_{\text{bottom disk}} 2\,dS \\
&= 2\,\text{Area(bottom disk)} \\
&= 2(\pi\cdot 3^2) \\
&= 18\pi.
\end{aligned}$$

Subtracting gives the flux of curl $\mathbf{F}$ through $S$:

$$\iint_S (\operatorname{curl}\mathbf{F})\cdot d\mathbf{S} = 0 - 18\pi = -18\pi.$$

## 32. May 9

### 32.1. Review: extended divergence theorem.

**Problem 32.1.** Let $S_1$ be the unit sphere centered at the origin. Let $S_2$ be the sphere of radius 2 centered at the origin. Let $\mathbf{F}$ be an unknown continuously differentiable 3D vector field such that $\mathbf{F}(x,y,z) = \langle x,y,3\rangle$ for all $(x,y,z)\in S_1$, and $\mathbf{F}(x,y,z) = \langle x,y,z\rangle$ for all $(x,y,z)\in S_2$. What is the average value of div $\mathbf{F}$ on the region $T$ *between* the two spheres?

Solution: The average of any function $f$ on $T$ is defined as

$$\frac{\iiint_T f\,dV}{\text{Volume}(T)},$$

so the average value of div $\mathbf{F}$ is

$$\frac{\iiint_T \operatorname{div}\mathbf{F}\,dV}{\text{Volume}(T)}.$$

The denominator is just the difference of volumes of two balls:

$$\text{Volume}(T) = \frac{4}{3}\pi 2^3 - \frac{4}{3}\pi 1^3 = \frac{28\pi}{3}.$$

What about the numerator? We don't have a formula for $\mathbf{F}$ on points between the two spheres, so we cannot compute $\iiint_T \operatorname{div} \mathbf{F}\, dV$ directly. But the extended divergence theorem says that

$$\text{outward flux of } \mathbf{F} \text{ across } \underbrace{\text{the boundary of } T}_{\partial T} = \iiint\limits_T \operatorname{div} \mathbf{F}\, dV,$$

so we can try computing the left side instead. The boundary $\partial T$ consists of two pieces, $S_2$ with the outward unit normal, and $S_1$ with the unit normal pointing towards the origin (let's call that the inward unit normal, even though it is outward from the point of view of $T$). So the extended divergence theorem in reverse gives

$$\begin{aligned}\iiint\limits_T \operatorname{div} \mathbf{F}\, dV &= \text{outward flux of } \mathbf{F} \text{ across } \partial T \\ &= (\text{outward flux of } \mathbf{F} \text{ across } S_2) + (\text{inward flux of } \mathbf{F} \text{ across } S_1) \\ &= (\text{outward flux of } \mathbf{F} \text{ across } S_2) - (\text{outward flux of } \mathbf{F} \text{ across } S_1).\end{aligned}$$

Let's now compute each of the latter two fluxes separately. At each point of $S_2$, the value $\mathbf{F}(x, y, z) = \langle x, y, z\rangle$ is a radially outward vector of length 2, and $\mathbf{n}$ is a unit vector in the same direction, so we can use a shortcut:

$$\mathbf{F} \cdot \mathbf{n} = |\mathbf{F}||\mathbf{n}| \cos 0 = (2)(1)(1) = 2,$$

so

$$\begin{aligned}\text{outward flux of } \mathbf{F} \text{ across } S_2 &= \iint\limits_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS \\ &= \iint\limits_{S_2} 2\, dS \\ &= 2 \iint\limits_{S_2} dS \\ &= 2 \operatorname{Area}(S_2) \\ &= 2(4\pi 2^2) \\ &= 32\pi.\end{aligned}$$

For $S_1$, it is not true that $\mathbf{F} \cdot \mathbf{n}$ is constant, so the shortcut doesn't work. This means we have to parametrize $S_1$. We can use two of the spherical coordinates, namely $\phi$ and $\theta$ (since $\rho = 1$ on $S_1$). Then the parametrization is

$$\mathbf{r}(\phi, \theta) := \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi\rangle,$$

for $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi]$. To compute flux, we compute

$$\begin{aligned}
\mathbf{r}_\phi &= \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi \rangle \\
\mathbf{r}_\theta &= \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0 \rangle \\
\mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} \\
&= (\sin^2\phi\cos\theta)\mathbf{i} + (\sin^2\phi\sin\theta)\mathbf{j} + (\sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta)\mathbf{k} \\
&= \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi \rangle
\end{aligned}$$

(this is outward, so no need to change the sign for computing outward flux)

$$\begin{aligned}
d\mathbf{S} &= \mathbf{r}_\phi \times \mathbf{r}_\theta \, d\phi \, d\theta \\
&= \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi \rangle \, d\phi \, d\theta,
\end{aligned}$$

so

$$\begin{aligned}
\text{outward flux of } \mathbf{F} \text{ across } S_1 &= \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} \\
&= \iint_{S_1} \langle x, y, 3 \rangle \cdot \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi \rangle \, d\phi \, d\theta \\
&= \iint_{S_1} \langle \sin\phi\cos\theta, \sin\phi\sin\theta, 3 \rangle \cdot \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi \rangle \, d\phi \, d\theta \\
&= \iint_{S_1} (\sin^3\phi(\cos^2\theta + \sin^2\theta) + 3\sin\phi\cos\phi) \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} (\sin^3\phi + 3\sin\phi\cos\phi) \, d\phi \, d\theta.
\end{aligned}$$

Now

$$\int_0^{\pi} 3\sin\phi\cos\phi \, d\phi = 0$$

by symmetry, and

$$\begin{aligned}
\int_0^\pi \sin^3\phi\, d\phi &= \int_0^\pi (1-\cos^2\phi)\sin\phi\, d\phi \\
&= \int_1^{-1} (1-u^2)(-du) \qquad \text{(where } u=\cos\phi\text{)} \\
&= \int_{-1}^{1} (1-u^2)du \\
&= u - \frac{u^3}{3}\Big|_{-1}^{1} \\
&= (2/3) - (-2/3) \\
&= 4/3,
\end{aligned}$$

so we get

$$\text{outward flux of } \mathbf{F} \text{ across } S_1 = \int_0^{2\pi} \frac{4}{3}\, d\theta = \frac{8\pi}{3}.$$

Thus

$$\begin{aligned}
\iiint_T \operatorname{div}\mathbf{F}\, dV &= (\text{outward flux of } \mathbf{F} \text{ across } S_2) - (\text{outward flux of } \mathbf{F} \text{ across } S_1) \\
&= 32\pi - \frac{8\pi}{3} \\
&= \frac{88\pi}{3},
\end{aligned}$$

and finally, the average value of $\operatorname{div}\mathbf{F}$ on $T$ is

$$\frac{\iiint_T \operatorname{div}\mathbf{F}\, dV}{\text{Volume}(T)} = \frac{88\pi/3}{28\pi/3} = 22/7.$$

*Remark* 32.2. Actually, there is a sneaky trick that could have been used to simplify the computation of the flux through $S_1$. Namely, if $\mathbf{G}$ is the vector field defined by $\langle x, y, 3\rangle$ *on all of* $\mathbb{R}^3$, then $\mathbf{G}$ has the same values as $\mathbf{F}$ on $S_1$, so

$$\begin{aligned}
\text{outward flux of } \mathbf{F} \text{ across } S_1 &= \text{outward flux of } \mathbf{G} \text{ across } S_1 \\
&= \iiint_{\text{ball of radius 1}} \operatorname{div}\mathbf{G}\, dV \qquad \text{(by the divergence theorem)} \\
&= \iiint_{\text{ball of radius 1}} (1+1+0)\, dV \\
&= 2\left(\frac{4}{3}\pi 1^3\right) \\
&= \frac{8\pi}{3}.
\end{aligned}$$

32.2. **Review: a parade of differentials.** What is the difference between $ds$, $dS$, $d\mathbf{S}$, etc.?

- $d\mathbf{r}$ can be thought of as a tiny vector measuring change of position along a curve. Use: Line integrals of vector fields (e.g., work) have the shape $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$$d\mathbf{r} = \langle dx, dy, dz \rangle = \mathbf{T}\, ds.$$

- $ds$ can be thought of as the length of a little piece of a curve. Uses: Length of a curve, line integrals of *scalar* functions.

$$ds = |d\mathbf{r}| = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

- $dA$ represents a piece of area of a region in $\mathbb{R}^2$:

$$\begin{aligned} dA &= dx\, dy \qquad \text{(rectangular)} \\ &= r\, dr\, d\theta \qquad \text{(polar)} \\ &= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv \qquad \text{(change of variable).} \end{aligned}$$

  (The last formula is for a change of variable $x = x(u,v)$ and $y = y(u,v)$.)
- $\mathbf{n}\, ds$ in 2D is $\langle dy, -dx \rangle$, obtained by rotating $d\mathbf{r} = \mathbf{T}\, ds = \langle dx, dy \rangle$ clockwise $90^\circ$. Use: Flux of a 2D vector field $\mathbf{F}$ across a curve $C$ in $\mathbb{R}^2$ is $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$.
- $dS$ can be thought of as the area of a little piece of a surface. Uses: Surface area, surface integrals of *scalar* functions. If the surface is parametrized by $\mathbf{r}(u,v)$, then

$$dS = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$$

  (That's the area of a tiny parallelogram of sides $\mathbf{r}_u\, du$ and $\mathbf{r}_v\, dv$.) For a sphere of radius $\rho$ parametrized by $\phi, \theta$,

$$dS = \rho^2 \sin\phi\, d\phi\, d\theta.$$

- $d\mathbf{S}$ means $\mathbf{n}\, dS$. Use: Flux of a 3D vector field $\mathbf{F}$ across a surface $S$ is $\iint_S \mathbf{F} \cdot d\mathbf{S}$. In terms of $\mathbf{r}(u,v)$:

$$d\mathbf{S} = \mathbf{r}_u \times \mathbf{r}_v\, du\, dv.$$

  The length of $d\mathbf{S}$ is $dS$. The direction of $d\mathbf{S}$ is the unit normal

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

- $dV$ represents a piece of volume of a region in $\mathbb{R}^3$:

$$\begin{aligned} dV &= dx\, dy\, dz \qquad \text{(rectangular)} \\ &= dz\, r\, dr\, d\theta \qquad \text{(cylindrical)} \\ &= \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \qquad \text{(spherical)} \\ &= \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw \qquad \text{(change of variable).} \end{aligned}$$

- $dm$ can be thought of as a tiny bit of mass. It is $\delta\, ds$, or $\delta\, dS$, or $\delta\, dV$, depending on whether the object is of dimension 1, 2, or 3. (E.g., a wire, a warped metal plate, or a planet.) The density $\delta$ is mass per unit length, or mass per unit area, or mass per unit volume.

## 33. May 14

### 33.1. Fact to be used in understanding Maxwell's equations.

**Lemma 33.1.** *If* $\mathbf{F}$ *and* $\mathbf{G}$ *are continuous 3D vector fields such that*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{G} \cdot d\mathbf{S}$$

*for every surface* $S$*, then* $\mathbf{F} = \mathbf{G}$ *everywhere.*

*Proof.* Suppose not. Then we can choose a point $P$ where $\mathbf{F} \neq \mathbf{G}$. Let $S$ be a tiny disk perpendicular to $\mathbf{F} - \mathbf{G}$ at $P$. If $S$ is small enough, then

$$\iint_S (\mathbf{F} - \mathbf{G}) \cdot d\mathbf{S} \neq 0.$$

Thus

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \neq \iint_S \mathbf{G} \cdot d\mathbf{S},$$

a contradiction. □

### 33.2. Maxwell's equations.

Maxwell's equations relate electric and magnetic fields. They can be expressed in differential form (involving the derivative-like operators div and curl), or in integrated form (involving line integrals and surface integrals).

Mathematics proves none of them. Instead, the role of mathematics is to show that *the differential form is mathematically equivalent to the integrated form.*

Introduce the following quantities (in SI units):

$$\begin{aligned}
\mathbf{E} &= \text{electric field (newtons/coulomb = volt/m)} \\
\mathbf{B} &= \text{magnetic field (teslas)} \\
\rho &= \text{charge density (coulomb/m}^3\text{)} \\
\mathbf{J} &= \text{current density (amp/m}^2\text{)} \\
t &= \text{time (s)} \\
\mu_0, \epsilon_0 &= \text{constants.}
\end{aligned}$$

Here are Maxwell's equations in differential form (in SI units):

$$\begin{aligned}
\operatorname{div}\mathbf{E} &= \frac{\rho}{\epsilon_0} \\
\operatorname{div}\mathbf{B} &= 0 \\
\operatorname{curl}\mathbf{E} &= -\frac{d\mathbf{B}}{dt} \\
\operatorname{curl}\mathbf{B} &= \mu_0\epsilon_0\frac{d\mathbf{E}}{dt} + \mu_0\mathbf{J}.
\end{aligned}$$

And here are Maxwell's equations in integrated form (again in SI units):

$$\begin{aligned}
\oiint_S \mathbf{E}\cdot d\mathbf{S} &= \frac{Q}{\epsilon_0} && \text{(Gauss-Coulomb law)} \\
\oiint_S \mathbf{B}\cdot d\mathbf{S} &= 0 && \text{(Gauss's law for magnetism)} \\
\oint_C \mathbf{E}\cdot d\mathbf{r} &= -\frac{d}{dt}\iint_S \mathbf{B}\cdot d\mathbf{S} && \text{(Faraday's law)} \\
\oint_C \mathbf{B}\cdot d\mathbf{r} &= \mu_0\epsilon_0\frac{d}{dt}\iint_S \mathbf{E}\cdot d\mathbf{S} + \mu_0 I_S && \text{(Ampère's law).}
\end{aligned}$$

Here in the first two equations, $S$ is a closed surface (and $Q$ is the charge enclosed by $S$). In the last two equations, $C$ is the boundary curve of a surface $S$, and $I_S$ is the current flowing through $S$ (i.e., the flux of $\mathbf{J}$ across $S$).

Let's prove the equivalence of the two forms of the third Maxwell equation (Faraday's law). The key to the proof will be Stokes' theorem.

First suppose that

$$\operatorname{curl}\mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

at every point. Take the surface integral of both sides over $S$:

$$\iint_S \operatorname{curl}\mathbf{E}\cdot d\mathbf{S} = \iint_S -\frac{d\mathbf{B}}{dt}\cdot d\mathbf{S}.$$

Apply Stokes' theorem to the left side, and on the right side convert the integral of a derivative to the derivative of an integral (usually it is OK to do this, just as the sum of derivatives is the derivative of the sum):

$$\oint_C \mathbf{E}\cdot d\mathbf{r} = -\frac{d}{dt}\iint_S \mathbf{B}\cdot d\mathbf{S}.$$

Conversely, suppose that

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S}.$$

holds for every surface $S$ with boundary $C$. Reversing the steps above leads to

$$\iint_S \operatorname{curl} \mathbf{E} \cdot d\mathbf{S} = \iint_S -\frac{d\mathbf{B}}{dt} \cdot d\mathbf{S}.$$

By the fact from the beginning of lecture, this implies that

$$\operatorname{curl} \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

everywhere.

33.3. **Extended Stokes' theorem.** Sometimes the boundary of a surface $S$ in $\mathbb{R}^3$ may consist of *several* closed curves $C_1, \ldots, C_n$. In this situation, the extended Stokes' theorem says

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \cdots + \oint_{C_n} \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

It can also be written

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

where $\partial S$ is an abbreviation for the boundary of $S$.

It's even OK if the surface crosses itself.

**Example 33.2.** If $S$ is a pair of pants, oriented by the outward unit normal, then $\partial S$ consists of three curves: the waistline, and the two cuffs at the bottom. The waistline is oriented clockwise when viewed from the top.

33.4. **Simply connected regions in 3D.** The definition of "simply connected" for 3D regions is the same as for 2D regions:

**Definition 33.3.** A region $T$ in $\mathbb{R}^3$ is simply connected if every closed curve $C$ in $T$ is shrinkable in $T$ (i.e., can be continuously shrunk to a point without ever getting outside of $T$ during the shrinking).

**Question 33.4.** How many of the following are simply connected?

(1) $\mathbb{R}^3$
(2) a solid ball
(3) a solid torus
(4) $\mathbb{R}^3$ with a solid ball removed
(5) $\mathbb{R}^3$ with a point removed
(6) $\mathbb{R}^3$ with an infinite cylinder removed

Answer: Four of them, namely, (1), (2), (4), (5). In example (4), a curve going around the ball can be slipped around the ball.

Fact: If $T$ is simply connected and $C$ is a closed curve in $T$, then $C$ can be "filled in" with a (possibly self-crossing) surface $S$ contained in $T$. (The idea is to let $S$ be all the points that the curve passes through as it shrinks.)

Recall two of the conditions on a 3D vector field $\mathbf{F}$ on $T$:

(6) $\operatorname{curl} \mathbf{F} = \mathbf{0}$ at every point of $T$

(1) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve $C$ inside $T$.

*Proof that (6) implies (1) when $T$ is simply connected.* We are assuming that $\operatorname{curl} \mathbf{F} = \mathbf{0}$ at every point of $T$. Let $C$ be any closed curve inside $T$. Since $T$ is simply connected, $C$ is the boundary of some surface $S$ inside $T$. Then Stokes' theorem says

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \\ &= \iint_S \mathbf{0} \cdot d\mathbf{S} \\ &= 0.\end{aligned}$$

□

33.5. **Generalized Stokes' theorem.** The fundamental theorem of calculus, the fundamental theorem of calculus for line integrals, Green's theorem, Green's theorem for flux, the divergence theorem, and Stokes' theorem all fit the template

$$\text{“}\int_{\partial R} \mathbf{F} = \int_R d\mathbf{F}\text{”}$$

where $\partial R$ is the boundary of $R$ and $d\mathbf{F}$ is some kind of derivative/differential of $\mathbf{F}$.

*FTC:* The boundary of an interval $[a, b]$ is a set of two points $\{a, b\}$.

$$f(b) - f(a) = \int_a^b f'(x)\, dx.$$

*FTC for line integrals:* The boundary of an oriented curve $C$ from $A$ to $B$, is $\{A, B\}$.

$$f(B) - f(A) = \int_C \nabla f \cdot d\mathbf{r}.$$

*Green's theorem:* The boundary of a region $R$ in $\mathbb{R}^2$ is a closed curve $C$ (or many curves). Let $\mathbf{F}$ be a 2D vector field.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F}\, dA.$$

*Green's theorem for flux:* Same.

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA.$$

*Divergence theorem:* The boundary of a 3D region $T$ is a closed surface $S$. Let $\mathbf{F}$ be a 3D vector field.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_T \operatorname{div} \mathbf{F}\, dV.$$

*Stokes' theorem:* The boundary of a surface $S$ in $\mathbb{R}^3$ is a closed curve $C$ (or many curves). Let $\mathbf{F}$ be a 3D vector field.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

In fact, these are all special cases of a generalized Stokes' theorem that applies to differential forms on $n$-dimensional manifolds for any $n$!

## 34. May 16

### 34.1. Table of differentials and their uses.

| $n$-dim integral | Differential $n$-forms | Uses |
|---|---|---|
| $\int$ | $ds$, $d\mathbf{r}$, $\mathbf{n}\, ds$ | arc length, work, flux across curve |
| $\iint$ | $dS$, $d\mathbf{S}$ | surface area, flux across surface |
| $\iiint$ | $dV$ | volume |
| any | $dm$ | mass, average, centroid, moment of inertia |

Make sure that the dimension of the integral matches what you are trying to compute! For example, 2D flux is flux across a *curve*, so it should be a 1D integral (in fact, it is the integral of the normal component $\mathbf{F} \cdot \mathbf{n}$ of a vector field with respect to arc length along the curve, so it is $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$).

### 34.2. Practice with limits of integration.

**Problem 34.1.** Let $T$ be the 3D region defined by $x, y, z \geq 0$ and $x + y + z \leq 1$. (It's the tetrahedron in the first octant sliced off by the plane passing through $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.) Find limits of integration for an iterated integral over $T$ in *cylindrical coordinates*.

*Solution:* An iterated integral in cylindrical coordinates has the shape

$$\int_{\theta=?}^{?} \int_{r=f(\theta)}^{g(\theta)} \int_{z=h(r,\theta)}^{j(r,\theta)} \cdots\, dz\, r\, dr\, d\theta.$$

The range for $\theta$ is $[0, \pi/2]$. For a fixed value of $\theta$ in this range, the range for $r$ is from 0 to some function of $\theta$. That maximum value of $r$ is the length of the segment of the line $y = (\tan \theta)x$ inside the triangle in the $xy$-plane defined by $x, y \geq 0$ and $x + y \leq 1$. One endpoint of that segment is $(0,0)$; the other endpoint is a point $(r\cos\theta, r\sin\theta)$ satisfying $x+y=1$, so the maximum value of $r$ (for the fixed $\theta$) satisfies

$$r\cos\theta + r\sin\theta = 1$$
$$r = \frac{1}{\cos\theta+\sin\theta}.$$

Finally, once $\theta$ and $r$ in those ranges are fixed, $z$ ranges from 0 up to $1-x-y = 1-r\cos\theta-r\sin\theta$. Thus the integral has the shape

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{\frac{1}{\cos\theta+\sin\theta}}\int_{z=0}^{1-r\cos\theta-r\sin\theta} \cdots\, dz\, r\, dr\, d\theta.$$

34.3. **Review: vectors and matrices.** Some of the things you need to know:

- Geometric interpretation of dot product: $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$
- Geometric interpretation of cross product $\mathbf{a}\times\mathbf{b}$:
  - length: $|\mathbf{a}||\mathbf{b}|\sin\theta$
    (area of parallelogram spanned by $\mathbf{a}, \mathbf{b}$)
  - direction: perpendicular to $\mathbf{a}$ and $\mathbf{b}$
- Scalar component: $\operatorname{comp}_{\mathbf{b}}\mathbf{a} = \mathbf{a}\cdot\frac{\mathbf{b}}{|\mathbf{b}|}$
- Equation of a plane: If the normal vector is $\langle a,b,c\rangle$, then the equation is $ax+by+cz = d$ for some number $d$.
- det
- Laplace expansion (in terms of minors)
- going from a matrix to a linear transformation
- going from a linear transformation to a matrix (example: what matrix has the combined effect of dilating by a factor of 2 and rotating by $\theta$?)
- inverse matrix:
  - When does $A^{-1}$ exist?
  - How do you compute $A^{-1}$? Answer: matrix of minors, change signs (checkerboard pattern), transpose, and multiply by $1/\det A$.
- Solutions to square systems:

| | $\det A = 0$ | $\det A \neq 0$ |
|---|---|---|
| homogeneous $A\mathbf{x} = \mathbf{0}$ | infinitely many | only $\mathbf{0}$ |
| general $A\mathbf{x} = \mathbf{b}$ | none or infinitely many | only $A^{-1}\mathbf{b}$ |

  Geometric interpretation of $3\times 3$ system as intersection of 3 planes
- Parametric equations of lines: $\mathbf{r}(t) = \mathbf{a} + t\mathbf{v}$.
- Parametric equations of curves (Example: Parametrize the circle of radius 2 and center $(3,5)$ so that the speed is 7. Answer: $\langle 3,5\rangle + \langle 2\cos(7t/2), 2\sin(7t/2)\rangle$. Test answer by plugging in values of $t$.)

- FTC for vector-valued functions:

$$\int_a^b \mathbf{r}'(t)\,dt = \mathbf{r}(b) - \mathbf{r}(a).$$

34.4. **Review: derivatives of multivariable functions.**

- Level curve of $f(x, y)$
- Meaning of partial derivative
- Approximation formula: $\Delta f \approx f_x \Delta x + f_y \Delta y$, or

$$f(x, y) \approx f(x_0, y_0) + \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0).$$

- Chain rule (draw dependency diagram)
- Geometric meaning of $\nabla f$
- Directional derivative $D_{\mathbf{u}} f = (\nabla f) \cdot \mathbf{u}$
- Differential of a function: $df := f_x\,dx + f_y\,dy$
- Non-independent variables: re-read the notes on these
- Tangent plane. . .
  - to a graph, e.g., $z = x^2 + y^2$ at the point above $(2, 3)$: The tangent plane is the equation that would hold if the approximation formula

$$\Delta z \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$$

    were exact:

$$z - f(a, b) = \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

    Get

$$z - 13 = 4(x - 2) + 6(y - 3).$$

  - to a level surface $f(x, y, z) = c$ at the point $(x_0, y_0, z_0)$: Use $\mathbf{n} = (\nabla f)(x_0, y_0, z_0)$ as normal vector. Example: $x^2 + y^2 + z^2 = 9$ at $(2, 2, 1)$. Get $\mathbf{n} = \langle 4, 4, 2 \rangle$. So the plane is

$$4x + 4y + 2z = d$$

    for some number $d$. It must pass through $(2, 2, 1)$, so it must be

$$4x + 4y + 2z = 18.$$

34.5. **Review: max/min.** Identify the function $f$ you are maximizing/minimizing. Identify the domain (region of possible inputs).

Example: Maximum value of $x^3 + y^4$ on the disk $x^2 + y^2 \leq 9$?

34.5.1. *Case 1: no constraint equations.* Here we are maximizing/minimizing $f(x)$ on an interval or $f(x,y)$ on a planar region or ..., not a bent curve or bent surface.

Need to check:

- critical points (points where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and ... simultaneously vanish)
- points where some partial derivative is undefined
- behavior at $\infty$ (what happens to $f$ as $(x,y) \to \infty$)
- boundary behavior (if there are constraint inequalities) — this may lead to another, lower-dimensional max/min problem.

Second derivative test is used to identify the *type* of a critical point $(a,b)$. (Usually it is not needed to find max/min.)

**Example 34.2.** Minimize $f := x + y + 1/xy$ on the first quadrant. Since $f \to \infty$ as $x \to 0$, $y \to 0$, or $(x,y) \to \infty$, there has to be a minimum somewhere inside, and it must be a critical point. Computing the partials shows that $f$ has just one critical point. So that's where the minimum is.

34.5.2. *Case 2: constraint equations.* Suppose that the possible inputs are constrained by $g = c$.

Solution 1: Parametrize $g = c$ to change it into a problem with unconstrained parameters. E.g., to maximize $x^3 + y^4$ subject to $x^2 + y^2 = 1$, substitute $x = \cos t$, $y = \sin t$ and maximize $\cos^3 t + \sin^4 t$ for $t \in [0, 2\pi]$.

Solution 2: Use Lagrange multipliers to maximize/minimize $f$ subject to $g = c$.

Need to check:

- points on $g = c$ where $\nabla f = \lambda \nabla g$ for some scalar $\lambda$
- points on $g = c$ where $\nabla g = 0$
- points on $g = c$ where $\nabla f$ or $\nabla g$ is undefined
- behavior at $\infty$ (what happens to $f$ as $(x,y) \to \infty$ along $g = c$?)
- boundary behavior (if there are constraint inequalities)

34.6. **Review: flux.**

- 2D flux across a curve: $\displaystyle\int_C \mathbf{F} \cdot \mathbf{n}\, ds$.

  To compute it in terms of a parametrization $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, use

  $$\mathbf{n}\, ds = \langle dy, -dx \rangle.$$

  and then substitute

  $$dx = x'(t)\, dt$$
  $$dy = y'(t)\, dt.$$

- 3D flux across a surface: $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$.

  To compute it in terms of a parametrization, use

$$\mathbf{n}\,dS = d\mathbf{S} = \mathbf{r}_u \times \mathbf{r}_v\,du\,dv.$$

34.7. **Review: applications of integrals.**

- mass is $\int_C dm$, or $\iint_S dm$, or $\iiint_T dm$, depending on the dimension of the object.
- (mass-weighted) average of a function $f$: For a 1D object (e.g., a wire),

$$\overline{f} := \frac{\int_C f\,dm}{\int_C dm}.$$

  The denominator is the total mass. If the density is constant, then

$$\overline{f} := \frac{\int_C f\,ds}{\text{length of } C}.$$

  For a 2D or 3D object, replace each $\int_C$ by $\iint_S$ or $\iiint_T$.
- centroid (also called center of gravity, or center of mass) is the point

$$(\overline{x}, \overline{y}, \overline{z})$$

  whose coordinates are the average values of the functions $x$, $y$, and $z$.
- moment of inertia of an object relative to an axis of rotation is

$$I := \int_C (\text{distance to axis})^2\,dm.$$

  For a 2D or 3D object, replace each $\int_C$ by $\iint_S$ or $\iiint_T$.
- work done by a force field $\mathbf{F}$ on an object moving along $C$ is $\int_C \mathbf{F}\cdot d\mathbf{r}$.
- gravitational force of a solid object $T$ acting on a point mass $m$ at the origin is

$$\mathbf{F} = \iiint_T \frac{Gm\,dM}{\rho^2}\,\frac{\mathbf{r}}{\rho}.$$

  Normally one would convert this to an iterated integral in spherical coordinates.
- gravitational field of a solid object $T$, measured at the origin, is

$$\mathbf{F} = \iiint_T \frac{G\,dM}{\rho^2}\,\frac{\mathbf{r}}{\rho}.$$

  (It's the gravitational force that would be exerted on a test mass with $m = 1$.)

34.8. **Review: vector calculus.**

- Change of variables
- Big theorems: FTC for line integrals, Green's, Green's for flux, divergence, Stokes'
- Conditions for $\mathbf{F}$ to be conservative, simply connected regions, finding a potential function
- Cylindrical and spherical coordinates, latitude/longitude
- Line integrals, surface integrals, surface area
- Gravitation, Gauss's law

THIS IS THE END!