5 Homework Solutions

18.335 - Fall 2004

5.1 Trefethen 20.1

$\Rightarrow$If A has an LU factorization, then all diagonal elements of U are not zero. Since $A = LU$ implies that $A_{1:k,1:k} = L_{1:k,1:k}U_{1:k,1:k}$ we get that $A_{1:k,1:k}$ is invertible.

$\Leftarrow$We prove by induction that $A_{1:k,1:k} = L_{1:k,1:k}U_{1:k,1:k}$ with

$$L_{1:k+1,1:k+1} = \begin{pmatrix} L_{1:k,1:k} & 0 \\ *** & 1 \end{pmatrix} \text{ and } U_{1:k+1,1:k+1} = \begin{pmatrix} U_{1:k,1:k} & * \\ 0 & u_{k+1} \end{pmatrix}$$

with all the elements on the diagonal of $U_{1:k,1:k}$ are non-zero for any k.

Step 1 For $k = 1$ we have $A_{1:1,1:1} = L_{1:1,1:1}U_{1:1,1:1}$ with $L_{1:1,1:1} = 1$, $U_{1:1,1:1} = A_{1:1,1:1} \neq 0$.

Step 2 If that is true for $k \leq m$ we prove it for $m + 1$. Simply choose:

$$A_{1:m+1,1:m+1} = \underbrace{\begin{pmatrix} L_{1:m,1:m} & 0 \\ X_m & 1 \end{pmatrix}}_{L_{1:m+1,1:m+1}} \underbrace{\begin{pmatrix} U_{1:m,1:m} & Y_m \\ 0 & u_{m+1} \end{pmatrix}}_{U_{1:m+1,1:m+1}}$$

with

$$\begin{aligned} X_m &= [a_{m+1,1}...a_{m+1,m}]U^{-1}_{1:m,1:m} \\ Y_m &= L^{-1}_{1:m,1:m} \begin{bmatrix} a_{1,m+1} \\ \vdots \\ a_{m,m+1} \end{bmatrix} \\ u_{m+1} &= -X_m Y_m \end{aligned}$$

Now we have $u_{m+1} \neq 0$ since $\det(A_{1:m+1,1:m+1}) = \det(U_{1:m,1:m})\, u_{m+1} \neq 0$. Now since $A = A_{1:n,1:n} = L_{1:n,1:n}U_{1:n,1:n}$ and $L_{1:n,1:n}$ is unit lower diagonal, $U_{1:n,1:n}$ is upper diagonal and we complete the proof.

5.2 Trefethen 21.6

Write

$$A = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

Proceed with the first step of Gaussian elimination:

$$\begin{pmatrix} a_{11} & A_{12} \\ 0 & A_{22} - \dfrac{A_{21}}{a_{11}} A_{12} \end{pmatrix}$$

Now for $A_{22} - \frac{A_{21}}{a_{11}} A_{12}$ we show that it has the property of strictly diagonally dominant matrices.

$$\sum_{j \neq k} \left| \left(A_{22} - \frac{A_{21}}{a_{11}} A_{12} \right)_{jk} \right| \leq \sum_{j \neq k} \left| (A_{22})_{jk} \right| + \sum_{j \neq k} \left| \frac{1}{a_{11}} (A_{21})_j (A_{12})_k \right|$$

A is strictly diagonally dominant, so we may write

$$\sum_{j \neq k} \left| (A_{22})_{jk} \right| < |(A_{22})_{kk}| - |(A_{12})_k| \text{ and } \sum_{j \neq k} \left| (A_{21})_j \right| < |a_{11}| - |(A_{21})_k|$$

so that in the end we get:

$$\begin{aligned} \sum_{j \neq k} \left| \left(A_{22} - \frac{A_{21}}{a_{11}} A_{12} \right)_{jk} \right| &< |(A_{22})_{kk}| - |(A_{12})_k| + \frac{|(A_{12})_k|}{|a_{11}|} \left(|a_{11}| - |(A_{21})_k| \right) \\ &< |(A_{22})_{kk}| - \frac{|(A_{12})_k| \, |(A_{21})_k|}{|a_{11}|} \leq \left| (A_{22})_{kk} - \frac{(A_{21})_k (A_{12})_k}{a_{11}} \right| \\ &\leq \left| \left(A_{22} - \frac{A_{21} A_{12}}{a_{11}} \right)_{kk} \right| \end{aligned}$$

Hence by induction if the property is true for any matrix of dimension $\leq m-1$ then it is true for any matrix A of dim $A = n$. This means that the submatrices that are created by successive steps of Gaussian elimination are also strictly diagonally dominant and hence we have no need for row swappings.

5.3 Trefethen 22.1

Apply 1 step of Gaussian elimination to A :

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \xrightarrow[\text{of GE}]{\text{1 Step}} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & a_{22}^{(1)} & \cdots & a_{2m}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2}^{(1)} & \cdots & a_{mm}^{(1)} \end{pmatrix}$$

, where the entries $a_{ij}^{(1)} = a_{ij} - l_{ik} a_{kj}$. Since we used partial pivoting in our calculation, we must have $|l_{ik}| \leq 1$,

$$|\tilde{a}_{ij}| = |a_{ij} - l_{ik} a_{kj}| \leq |a_{ij}| + |l_{ik}| \, |a_{kj}| \leq |a_{ij}| + |a_{kj}| \leq 2 \max_{i,j} |a_{i,j}|$$

In order to form A we need $m-1$ such steps, so in the end we have:

$$|u_{ij}| \leq 2 \max_{i,j} \left| a_{i,j}^{(m-2)} \right| \leq 2 \max_{i,j} \left| a_{i,j}^{(m-3)} \right| \leq \ldots \leq 2 \max_{i,j} |a_{i,j}|$$

so that we obtain $|u_{ij}| \leq 2^{m-1} \max_{i,j} |a_{i,j}|$. Therefore

$$\rho = \frac{\max_{i,j} |u_{i,j}|}{\max_{i,j} |a_{i,j}|} \leq 2^{m-1}$$

5.4 Let A be symmetric and positive definite. Show that $|a_{ij}|^2 < a_{ii}$ a_{jj}.

Since A is symmetric and positive definite, it has all a_{ii} positive and for any vector x we have $x^T A x > 0$. Choose x such that $x_k = \delta_{jk} a_{jj} - \delta_{ik} a_{ij}$, where δ_{lm} is the Kronecker delta, meaning that all the entries of x are zero except the i-th and the j-th entries which equal to $-a_{ij}$ and a_{jj} respectively . Carrying out the calculation gives $x^T A x = a_{ii}\left(a_{ii}a_{jj} - a_{ij}^2\right) > 0$ thus completing the proof.

5.5 Let A and A^{-1} be given real n-by-n matrices. Let $B = A + xy^T$ be a rank-one perturbation of A. Find an $O(n^2)$ algorithm for computing B^{-1}. Hint: B^{-1} is a rank-one perturbation of A^{-1}.

Since B^{-1} is a rank-one perturbation of A^{-1} we may write $B^{-1} = A^{-1} + uv^T$. Then

$$\begin{aligned} BB^{-1} &= \left(A + xy^T\right)\left(A^{-1} + uv^T\right) \\ I &= I + Auv^T + xy^T A^{-1} + xy^T uv^T \\ 0 &= Auv^T + xy^T A^{-1} + xy^T uv^T \end{aligned}$$

Choosing $u = A^{-1}x$, allows us to write:

$$\begin{aligned} 0 &= xv^T + xy^T A^{-1} + xy^T uv^T \\ 0 &= v^T + y^T A^{-1} + y^T uv^T \\ 0 &= v^T(1 + y^T u) + y^T A^{-1} \\ v^T &= -\frac{y^T A^{-1}}{1 + y^T A^{-1} x} \end{aligned}$$

Hence B^{-1} is given by:

$$B^{-1} = A^{-1} - \frac{A^{-1} x y^T A^{-1}}{1 + y^T A^{-1} x}$$

It is easy to see that the inverse can be computed in $O(n^2)$ operations