Graded problems, Part A
See attached photocopies.

Graded problems, Part B

1. (a) Figure 1 shows a graph of \( G(x) = \sin \left( \frac{1}{x} \right) \). What must be observed is that as \( x \) approaches zero, \( G(x) \) oscillates back and forth between \(-1\) and \(1\) infinitely many times. To see that this must be so, set \( z = 1/x \) and consider \( \sin z \). As \( x \to 0^+ \), \( z \to \infty \), which allows for infinitely many oscillations of \( \sin z \). The same thing happens when \( x \to 0^- \). We see that \( G(x) \) attains every value between \(-1\) and \(1\) infinitely many times as \( x \to 0 \), but it cannot be said to “approach” any one value in particular. Thus \( \lim_{x \to 0} G(x) \) does not exist.

![Figure 1: The function \( G(x) = \sin(1/x) \)](image)

(b) Choose \( f(x) = -|x| \) and \( h(x) = |x| \). Then since \( |\sin(1/x)| \leq 1 \) for all \( x \), we have \( f(x) \leq x \sin(1/x) \leq h(x) \) for all \( x \) (see Figure 2), and therefore
\[
\lim_{x \to 0} F(x) = 0.
\]

This is also the value of \( F(0) \) by definition, thus \( F \) is continuous at 0.

(c) If \( F(x) \) were differentiable at 0, its derivative there would be
\[
F'(0) = \lim_{h \to 0} \frac{F(0 + h) - F(0)}{h} = \lim_{h \to 0} \frac{h \sin(1/h) - 0}{h} = \lim_{h \to 0} \sin(1/h).
\]

But we saw in part (a) that this limit does not exist, thus \( F \) is not differentiable at 0.

(d) Using again the fact that \( |\sin(1/x)| \leq 1 \), we have
\[
-x^2 \leq x^2 \sin(1/x) \leq x^2,
\]
and since \(-x^2\) and \(x^2\) both equal zero at \( x = 0 \), \( \lim_{x \to 0} x^2 \sin(1/x) = 0 \). By the definition of the derivative,
\[
H'(0) = \lim_{h \to 0} \frac{H(0 + h) - H(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin(1/h) - 0}{h} = \lim_{h \to 0} h \sin(1/h) = 0,
\]
Figure 2: $F(x) = x \sin(1/x)$ along with two “squeezing functions,” $f(x) = -|x|$ and $h(x) = |x|$.

by the result of part (b).

2. (a) $\lim_{x \to 0} \frac{\sin ax}{x} = \lim_{x \to 0} \left( a \frac{\sin ax}{ax} \right) = a \lim_{ax \to 0} \frac{\sin ax}{ax} = a \cdot 1 = a$, since as $x \to 0$, so does $ax$.

(b) The polygon is made up of $n$ identical isosceles triangles, each with two sides of length $r$. For any one of these triangles, the angle between the two identical sides is $2\pi/n$ radians. Divide this triangle symmetrically into two right triangles, both with height $h$ and base $a$. Then

$$\frac{h}{r} = \cos \left( \frac{\pi}{n} \right) \quad \text{and} \quad \frac{a}{r} = \sin \left( \frac{\pi}{n} \right).$$

The area of the full isosceles triangle is then

$$\frac{1}{2} 2ah = ah = r^2 \cos(\pi/n) \sin(\pi/n) = \frac{r^2}{2} \sin(2\pi/n),$$

where we’ve used the double angle formula $\sin(2\theta) = 2 \sin(\theta) \cos(\theta)$. Multiplying by $n$, the polygon has area

$$A_n = \frac{nr^2}{2} \sin \left( \frac{2\pi}{n} \right).$$

(c) Using the result of part (a),

$$\lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{nr^2}{2} \sin \left( \frac{2\pi}{n} \right) = \frac{r^2}{2} \lim_{1/n \to 0} \frac{\sin \left( \frac{2\pi}{n} \right)}{1/n} = \frac{r^2}{2} 2\pi = \pi r^2.$$