Answer as many questions as you can. Write your solutions in the test booklet, showing enough work to demonstrate how you arrived at the answers. You’re encouraged to use scrap paper, but only the answers in your test booklet will be graded. Calculators are not permitted.

Please circle or box your answers for emphasis; e.g. the derivative of $x^2$ is $2x$.

1. [10 pts] Sketch the curve $r = 2(1 + \cos \theta)$, where $0 \leq \theta \leq 2\pi$.

2. [10 pts] Compute the circumference of the circle $r = 2 \cos \theta$.

\[
r = 2 \cos \theta \quad \iff \quad \sqrt{x^2 + y^2} = 2 \frac{x}{\sqrt{x^2 + y^2}}
\]
\[
\iff \quad x^2 + y^2 = 2x \iff (x-1)^2 + y^2 = 1
\]

\[
\text{Circumference} = \int_0^{2\pi} \sqrt{r^2 + (r')^2} \, d\theta
\]
\[
= \int_0^{2\pi} \sqrt{4\cos^2 \theta + 4 \sin^2 \theta} \, d\theta = 2 \int_0^{2\pi} \, d\theta = 2\pi
\]
3. [20 pts]
Compute the area common to \( r = a \cos \theta \) and \( r = b \sin \theta \).

\[
\int_{0}^{\frac{\pi}{2}} \frac{1}{2} (a \cos \theta)^2 d\theta + \int_{\frac{\pi}{2}}^{\tan^{-1} \left( \frac{a}{b} \right)} \frac{1}{2} (b \sin \theta)^2 d\theta
\]

\[
= \frac{a^2}{2} \left[ \frac{1 + \cos 2\theta}{2} \right]_0^{\frac{\pi}{2}} + \frac{b^2}{2} \left[ \frac{1 - \cos 2\theta}{2} \right]_{\frac{\pi}{2}}^{\tan^{-1} \left( \frac{a}{b} \right)}
\]

\[
= \frac{a^2}{2} \left[ \frac{\pi}{4} \right] + \frac{b^2}{2} \left[ \frac{1}{4} - \frac{\pi}{4} \right]
\]

4. [20 pts]
Compute the volume of the sphere \( x^2 + y^2 + z^2 = a^2 \) using both the disk and the shell methods.

* see example 1 page 226 of Simmons,

* see example 1 page 232.
5. [15 pts] Compute the integrals \( \int \frac{1}{x^2+\frac{1}{4}} \) and \( \int \ln(x) \, dx \).

\[ \int \ln(\sqrt{x}) \, dx = \int \ln(x) \, \frac{1}{2} \, dx = \frac{1}{2} \int \ln(x) \, dx = \frac{1}{2} \, x \ln(x) - \frac{1}{2} x + C \]

Int by parts \( \Rightarrow \)

\[ \int \ln(x) \, dx = x \ln(x) - \int 1 \, dx = x \ln(x) - x + C \]

\[ \int \ln(\sqrt{x}) \, dx = \frac{1}{2} (x \ln(x) - x) + C \]

\[ \int \frac{dx}{x^2+\frac{1}{4}} = \int \frac{dx}{(x+\frac{1}{2})^2+(\sqrt{\frac{3}{4}})^2} \quad \text{let } u = x+\frac{1}{2} \]

\[ \Rightarrow \int \frac{du}{u^2+(\sqrt{\frac{3}{4}})^2} = \frac{1}{\sqrt{\frac{3}{4}}} \tan^{-1}\left(\frac{u}{\sqrt{\frac{3}{4}}}\right) + C \]

\[ = \sqrt{\frac{4}{3}} \tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right) + C \]

6. [10 pts] Does the integral \( \int_{10}^{\infty} \frac{x^3}{x^5-x} \) converge/diverge (explain your answer):

\[ \frac{x^3}{x^5-x} \leq \frac{x^3}{x^5 - \frac{1}{2}x^5} \quad x \gg 1 \]

\[ \Rightarrow \int_{10}^{\infty} \frac{x^3 \, dx}{x^5-x} \leq \int_{10}^{\infty} \frac{x^3 \, dx}{\frac{1}{2}x^5} = 2 \int_{10}^{\infty} \frac{dx}{x^2} = -\frac{2}{x}\bigg|_{10}^{\infty} = \frac{1}{5} < \infty \]