**Massachusetts Institute of Technology**

DEPARTMENT OF MATHEMATICS
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# 1 The Levi-Civita Connection and its curvature

In this lecture we introduce the most important connection. This is the Levi-Civita connection in the tangent bundle of a Riemannian manifold.

## 1.1 The Einstein summation convention and the Ricci Calculus

When dealing with tensors on a manifold it is convient to use the following conventions. When we choose a local frame for the tangent bundle we write $e_1, \ldots e_n$ for this basis. We always index bases of the tangent bundle with indices down. We write then a typical tangent vector

$$X = \sum_{i=1}^{n} X^i e_i.$$

Einstein's convention says that when we see indices both up and down we assume that we are summing over them so he would write

$$X = X^i e_i$$

while a one form would be written as

$$\theta = a_i e^i$$

where $e^i$ is the dual co-frame field. For example when we have coordinates $x^1, x^2, \ldots, x^n$ then we get a basis for the tangent bundle

$$\partial/\partial x^1, \ldots, \partial/\partial x^n$$

More generally a typical tensor would be written as

$$T = T^i{}_{jk}{}^l e_i \otimes e^j \otimes e^k \otimes e_l$$

Note that in general unless the tensor has some extra symmetries the order of the indices matters. The lower indices indicate that under a change of frame $f_i = C_i{}^j e_j$ a lower index changes the same way and is called covariant while an upper index changes by the inverse matrix. For example the dual coframe field to the $f_i$, called $f^i$ is given by

$$f^i = D^i{}_j e^j$$

where $D^i{}_j$ is the inverse matrix to $C^j{}_i$ (so that $D^i{}_j C^j{}_k = \delta^i_k$.) The components of the tensor $T$ above in the $f_i$ basis are thus

$$T^i{}_{jk}{}^l = T^{i'}{}_{j'k'}{}^{l'} D^i{}_{i'} C^{j'}{}_j C^{k'}{}_k D^l{}_{l'}$$

Notice that of course summing over a repeated upper and lower index results in a quantity that is independent of any choices.

Given a vector bundle over our manifold which is not the tangent bundle or tensors on the tangent bundle we use a distinct set of indices to indicate tensors with values on that bundle. If $V \to M$ is a vector bundle of rank $k$ with a local frame $v_\alpha$, $1 \le \alpha \le r$ we would write

$$s = c^\alpha_i v_\alpha \otimes dx^i$$

for a typical section of the bundle $T^*M \otimes V$

Given a $\nabla$ connection in $V$ we write

$$\nabla s = s^\alpha{}_{;i} dx^i \otimes v^\alpha.$$

That is we think $\nabla s$ as a section of $T^*M \otimes V$ as opposed to the possibly more natural $V \otimes T^*M$. Or more concretely the semi-colon is also indicating that the indices following the semi-colon are to really be thought of comming first and the opposite order. Our convention here is designed to be more consistent with the mathematical literature. In the physics literature for example "Graviation" by Misner, Throne and Wheeler. So for a connection in the tangent bundle if we have a vector field with components $X^i$ we have write $X^i{}_{,j}$ for the components of its covariant derivative. The Christoffel symbols of a connection are the components of the covariant derivatives of the basis vectors:

$$\nabla_{e_i} v^\alpha = \Gamma_i{}^\alpha{}_\beta v^\beta.$$

Then we can write more explicitly

$$s^\alpha{}_{;i} = e_i s^\alpha + \Gamma_i{}^\alpha{}_\beta s^\beta$$

One often uses the short hand

$$e_i s^\alpha = s^\alpha{}_{,i}$$

so that

$$s^\alpha{}_{;i} = s^\alpha{}_{,i} + \Gamma_i{}^\alpha{}_\beta s^\beta$$

## 1.2 The low brow approach to the Levi-Civita connection

The Levi-Civita connection is characterized by two properties. First that it is metric compatible, i.e. that

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Y Z\rangle$$

or equivalently that parallel transport is an isometry. For a connnection in the tangent bundle there is another property to ask for called *torsion free.* The torsion of a connection in the tangent bundle is the tensor

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

and the torsion free condition means that this tensor is indentically zero. Since the coordinate vector field commute the expression for the components of torsion in terms of a coordinate frame is

$$T_i{}^k{}_j = \Gamma_i{}^k{}_j - \Gamma_j{}^k{}_i.$$

The Let $\nabla$ a torsion free connection metric compatible connection.

**Lemma 1.1.**

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}(g_{jl,i} + g_{il,j} - g_{ij,l})$$

*Proof.* Notice that there are $n^3$ distinct functions involved in defining a connection. Recall that the torsion free condition implies $n^2(n-1)/2$ relations amongst these functions:

$$\Gamma_{ij}^k = \Gamma_{ji}^k. \tag{1}$$

The metric compatibility implies $n^2(n+1)/2$ relations amongst these functions:

$$\begin{aligned} g_{ij,l} &= \langle \nabla_{\frac{\partial}{\partial x^l}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle + \langle \frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^l}} \frac{\partial}{\partial x^j} \rangle \\ &= g_{kj}\Gamma_{li}^k + g_{ik}\Gamma_{lj}^k. \end{aligned}$$

Hence using equation 1

$$\begin{aligned} g_{jl,i} + g_{il,j} - g_{ij,l} &= g_{kl}\Gamma_{ij}^k + g_{jk}\Gamma_{il}^k \\ &\quad + g_{kj}\Gamma_{ji}^k + g_{ik}\Gamma_{jl}^k \\ &\quad - g_{kj}\Gamma_{li}^k - g_{ik}\Gamma_{lj}^k \\ &= 2g_{kl}\Gamma_{ij}^k \end{aligned}$$

and the result follows. □

## 1.3 The method of moving frames

We will now redo the existence of the Levi-Civita connection from the point of view of the induced connection on the cotangent bundle. This method is more compuationally effective. A connection $\nabla$ in the tangent bundle induces a connection $\nabla^*$ in the cotangent bundle by the requiring

$$X\theta(Y) = \nabla_X^*\theta(Y) + \theta(\nabla_X Y).$$

After this section we'll drop the $*$ from the notation nice the meaning will be clear from context. To justify this definition think about the condition that the parallel transport for the two connections is equivalent.

Thus if $\Gamma_{ij}^k$ are the Christoffel symbols for $\nabla$ we have

$$\nabla^*_{\frac{\partial}{\partial x^i}} dx^k = -\Gamma_{ij}^k dx^j.$$

**Lemma 1.2.** *The condition that $\nabla$ is torsion free is equivalent to the condition that for any one-form $\alpha$ we have*

$$d\alpha(X,Y) = \nabla^*_X\alpha(Y) - \nabla^*_Y\alpha(X).$$

*Proof.* To see this notice

$$\begin{aligned}\nabla^*_X\alpha(Y) - &\nabla^*_Y\alpha(X) - d\alpha(X,Y) \\ &= X\alpha(Y) - \alpha(\nabla_X Y) - Y\alpha(X) + \alpha(\nabla_Y X) \\ &\quad - (X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])) \\ &= \alpha(-\nabla_X Y + \nabla_X Y + [X,Y]).\end{aligned}$$

□

A crucial step is to use an orthonormal frame $e_1,\ldots,e_n$ and dual coframe $e^1,e^2,\ldots,e^n$ then we have the Christoffel symbols for the connection

$$\nabla_{e_i}e_j = \Gamma^k_{ij}e_k$$

$$\nabla^*_{e_i}e^k = -\Gamma^k_{ij}e^j$$

Thus connection matrix for the dual connection is

$$\nabla e^k = -\Gamma^k_{ij}e^i \otimes e^j$$

The torsion free condition then implies

$$de^k = -\Gamma^k_{ij}e^i \wedge e^j$$

From the equation

$$e_i\langle e^j, e^k\rangle = 0$$

we derive that if $\nabla$ is metric compatible then

$$\Gamma^k_{ij} = -\Gamma^j_{ik}.$$

Let's prove, from this point of view, the basic uniqueness theorem the Levi-Civita connection.

**Lemma 1.3.** *There is a unique torsion free metric compatible connection.*

*Proof.* Suppose that $\nabla'$ is another metric compatible torsion free connection with connection matrix $\eta_j^k$ and dual connection matrix $-\eta_j^k$. Write the difference as

$$-\theta_j^k + \eta_j^k = f_{ij}^k e^i.$$

Then the torsion free condition implies

$$0 = f_{ij}^k e^i \wedge e^j$$

so that $f_{ij}^k = f_{ji}^k$. Metric compatible implies that $f_{ij}^k = -f_{ik}^j$ and so

$$f_{ij}^k = -f_{ik}^j = -f_{ki}^j = f_{kj}^i = f_{jk}^i = -f_{ji}^k$$

and so $f_{ij}^k = 0$.

To finish this discussion we need to see that we can solve the equation

$$de^k = -\theta_j^k \wedge e^j.$$

Where $\theta_j^k = \Gamma_{ij}^k e^i$ and $\Gamma_{ij}^k = -\Gamma_{ik}^j$ so there are $n \times n(n-1)/2$ unknowns. This is $n$ equations for two forms hence is $n \times n(n-1)/2$ equations. We have seen that the solution is unique and hence exists. $\square$

To make this a useful method for computing we need to understand more concretely how to compute with the answer. Suppose that

$$de^k = A_{ij}^k e^i \wedge e^j.$$

When we do calculations we are naturally get these $A_{ij}^k$ with $A_{ij}^k = -A_{ji}^k$. To find the Christoffel symbols we simply set

$$\Gamma_{ij}^k = -A_{ij}^k + A_{ik}^j + A_{jk}^i.$$

Note that with this definition

$$\Gamma_{ij}^k = -\Gamma_{ik}^j.$$

Since $A_{ik}^j + A_{jk}^i$ is symmetric in $i$ and $j$ we still have

$$de^k = \Gamma_{ij}^k e^i \wedge e^j.$$

For example in hyperbolic space

$$\mathbb{H}^{n+1}$$

with the metric

$$ds^2 = \frac{1}{x_0^2}((dx^0)^2 + (dx^1)^2) + \ldots + (dx^n)^2)$$

we have

$$e^i = \frac{dx^i}{x^0}$$

and hence $de^0 = 0$ while for $i > 0$ we have

$$de^i = -\frac{dx^0 \wedge dx^i}{(x^0)^2} = -e^0 e^i.$$

So for $i > 0$

$$A^i_{0i} = \frac{1}{2} = -A^i_{i0}.$$

Thus

$$\theta^i_0 = (A^i_{j0} - A^0_{ji} - A^j_{i0})e^j = -e^i$$

and $\theta^i_j = 0$ if $i, j > 0$.

## 1.4 Working on the frame bundle

The calculations above are straighforward but it is natural to look for a home for them. The bundle of orthonormal frames provides the right framework. After all in particular in the method of moving frames our choice was a local orthonormal frame field. A better way to view what is going on is to work with all frames simulatenously. The primordial object is the bunlde of frames (bases) of the tangent bundle $\pi : \mathrm{Fr}(M) \to M$.

Given a frame $f_1, \ldots, f_n$ and an matrix $a_{ij}$ we get a new orthonormal frame $f'_i = \sum_{j=1}^n a_{ij} f_j$ or a little more clearly we think of the orginal frame $\mathbf{f} = \{f_1, \ldots, f_n\}$ as giving an isomorphism

$$\mathbf{f} : \mathbb{R}^n \to T_x M.$$

Then an isomorphism of $a : \mathbb{R}^n \to \mathbb{R}^n$ we get a new frame

$$\mathbf{f} \circ a : \mathbb{R}^n \to T_x M.$$

This make $\mathrm{Fr}(M)$ into a principal $Gl(n)$ bundle. That is there is a map

$$\mathrm{Fr}(M) \times Gl(n) \to \mathrm{Fr}(M)$$

so that

$$(\mathbf{f}A)B = \mathbf{f}(AB).$$

Each orbit of the action is precisely one fiber of the projection and the action is effective ($\mathbf{f}A = \mathbf{f} \implies A = 1$). Informally it is a bundle of groups where the transition functions act by left multiplicition on the group leaving the right action to act on the resulting bundle.

A connection gives rise to an $Gl(n)$-invariant *horizontal* subbundle of the tangent bundle of $\mathrm{Fr}(M)$. By this we mean a subbundle $H \subset T\mathrm{Fr}(M)$ so that $\pi_*|_H : H \to \pi^* TM$ is an isomorphism and so that for all $a \in Gl(n)$ we have $(R_a)_* H = H$. The horizontal bundle associated to a connection is the lift of the tangent space given by the connection.

There is an exact sequence

$$0 \to VT\mathrm{Fr}(M) \to T\mathrm{Fr}(M) \to \pi^*(TM) \to 0.$$

Here $VT\mathrm{Fr}(M)$ denotes the "vertical tangent bundle" to $\mathrm{Fr}_{\mathrm{Or}}(M)$ i.e. the kernel of $d\pi$. The vertical tangent bundle is trivial and each fiber is isomorphic to the Lie algebra of $Gl(n)$ as follows. Let $A$ be a matrix and consider the path $\mathbf{e}\exp(tA)$. The derivative of the path at $t = 0$ defines a map

$$\iota_{\mathbf{e}} \colon \mathfrak{gl}(n) \to VT_{\mathbf{e}}\mathrm{Fr}(M).$$

We can thus rewrite this exact sequence as

$$0 \to P \times \mathfrak{gl} \to T\mathrm{Fr}(M) \to \pi^*(TM) \to 0.$$

A connection is then a splitting of this exact sequence

$$0 \to P \times \mathfrak{gl}_{\mathfrak{n}} \overset{\leftarrow}{\to} T\mathrm{Fr}(M) \to \pi^*(TM) \to 0.$$

Let us work this out explictly. Given a connection in a vector bundle if we choose local coordinates $x^i$ in the base and a local frame $\mathbf{e} = (e_\alpha)$. The frame $\mathbf{e}$ gives us a local trivialization so the principal bundle becomes

$$U \times Gl_n$$

We get Christoffel symbols

$$\nabla_{\frac{\partial}{\partial x^i}} e_\alpha = \Gamma^\beta_{i\alpha} e_\beta.$$

So given a curve $\gamma(t) = (x^1(t), x^2(t), \ldots, x^n(t))$ its parallel lift means finding $c^\alpha(t)$ so that

$$\frac{\partial c^\alpha}{\partial t} + \frac{\partial x^i}{\partial t}\Gamma^\alpha_{i\beta}c^\beta = 0.$$

In particular if $\gamma(t) = (0, \ldots, t \ldots, 0)$ and if the initial condition is one of the basis vector $e_\beta$

$$\frac{\partial c^\alpha}{\partial t}(0) + \Gamma^\alpha_{i\beta}(0, \ldots, 0) = 0.$$

From this we deduce that the horizontal space at the given orthonormal frame $\mathbf{e}$ is spanned by the $n$ vectors

$$(\frac{\partial}{\partial x^i}, -\Gamma^\beta_{i\alpha}). \tag{2}$$

Here we are writing the tangent space of $Gl_n$ at the identity as $M_n$ the space of $n \times n$-matrices.

There is a distinguished $\mathfrak{gl}_n$-valued one form on $Gl_n$ namely the one form which identifies the tangent space at $A$ with the tangent space at the identity. This one form, called the Mauer-Cartan form at a matrix $A$ can be written

$$\omega_{MC}|_A = A^{-1}dA$$

In other words the Mauer-Cartan form sends $\xi \in T_A Gl_n$ to $A^{-1}\xi \in T_I Gl_n$. The vectors (2) are in the kernel of the $\mathfrak{gl}_n$-valued one form

$$\theta_{\mathbf{e}} = \omega_{MC} + dx^i\Gamma^\beta_{i\alpha}$$

Notice that if we change frame so that $\tilde{\mathbf{e}} = \mathbf{e}g$ where $g : U \to Gl_n$. This means that

$$\tilde{e}_\alpha = g^\beta_\alpha e_\alpha$$

then the Chirstoffel symbols change by

$$\tilde{\Gamma}^\beta_{i\alpha} = (g^{-1})^\gamma_\alpha \frac{\partial g^\beta_\gamma}{\partial x^i}(g^{-1})^\beta_\gamma + \Gamma^\delta_{i\gamma}g^\delta_\alpha$$

while the Mauer-Cartan form changes by

$$\omega_{MC}|_{Bg} = g^{-1}(B^{-1}dB)g = g^{-1}\omega_{MC}|_B g + g^{-1}dg$$

If we change frame i.e. we consider the map $U \times Gl_n \to U \times Gl_n$ given by

$$\tilde{g} : (x, A) \mapsto (x, Ag(x))$$

and we pull back

Note that if $\tilde{\gamma}(t)$ is the horizontal lift at a frame $\mathbf{e}$ of $\gamma$ then $\tilde{\gamma}(t)A$ is the horizontal lift of $\gamma$ at $\mathbf{e}A$. This implies that right translation preserves the horizontal space of the connection and in particular that

$$\theta_{\mathbf{e}A} = R_A^* \theta_{\mathbf{e}}.$$

Note that that a section of the tangent bundle is given as a map

$$X : \mathrm{Fr}(M) \to \mathbb{R}^n$$

with the equivariance property

$$X(\mathbf{f}A) = A^{-1}X(\mathbf{f})$$

Given a connection we can

This bundle carries a tautological $\mathbb{R}^n$-valued one form

$$\theta : T\mathrm{Fr}(M) \to \mathbb{R}^n.$$

It is defined as follows. Given a tangent vector $v$ to $T\mathrm{Fr}(M)$ at a frame $\mathbf{f}$ we can project $v$ to $TM$ and expand it in terms of $\mathbf{f}$

$$\theta(v) = \mathbf{f}^{-1}(\pi_*(v)).$$

The tautological one form gives a geometric interpretation of torsion. Under the action of $Gl(n)$ this one-form transforms by

$$(R_a)_*\theta = a^{-1}\theta$$

Since $\theta$ has values in vector space

To this end we set $\pi\colon \mathrm{Fr}_{\mathrm{Or}}(M) \to M$ to be the bundle of orthonormal frames on $M$. This is principal $O(n)$ bundle. Given a frame $e_1, \ldots, e_n$ and an orthogonal matrix $a_{ij}$ we get a new orthonormal frame $f_i = \sum_{j=1}^n a_{ij}e_j$ or a little more clearly we think of the orginal frame $\mathbf{e} = \{e_1, \ldots, e_n\}$ as giving an isometry

$$\mathbf{e} : \mathbb{R}^n \to T_xM.$$

Then an isometry of $a : \mathbb{R}^n \to \mathbb{R}^n$ we get a new frame

$$\mathbf{e} \circ a : \mathbb{R}^n \to T_xM.$$

Given a connection its parallel transport gives rise to a lift through any point $\mathbf{e}$ of the total space of each path in a the base passing through $\pi(\mathbf{e})$. Taking derivatives we get a lifting of the $T_{\mathbf{e}}M$ to $T_{\mathbf{e}}\mathrm{Fr}_{\mathrm{Or}}(M)$ for each $\mathbf{e}$. Notice that the tangent bundle of $\mathrm{Fr}_{\mathrm{Or}}(M)$ is trivial. As have seen that Given a tangent vector $e_i$ in the base its lift to $\mathrm{Fr}_{\mathrm{Or}}(M)$ at $\mathbf{e}$ is $e$

Consider more generally a principal $G$ bundle $P \to M$ where $G$ is a Lie group. We will always think of $G$ as acting on $P$ on the right. The vertical tangent bundle of any principal bundle is alway trival and indeed each fiber can be identified with the Lie Algebra of the structure group. To see this let $\xi \in \mathfrak{g}$ be an element of the structure group which we always identify with left invariant vector fields on $G$.

## 1.5 A first pass at the curvature

The curvature of the connection is

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The coordinate expression of $R$ is

$$R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k} = R_{ijk}{}^l \frac{\partial}{\partial x^l}.$$

In the trivialization given by our orthonormal frame the curvature of the Levi-Civita connection has the form

$$\Omega^l_k = d\theta^l_k + \theta^l_m \wedge \theta^m_k.$$

This is related to the curvature tensor $R$ as follows. Defining as above

$$R_{ijkl} = \langle R(e_i, e_j)e_k, e_l \rangle.$$

then we have

**Lemma 1.4.**

$$[\Omega^l_k] = [\frac{1}{2} R_{ijkl} e^i \wedge e^j].$$

*Proof.* Since both side of the equation are independent of the frame it suffice to work in normal coordinates about $x$ and compute at $x$.

$$\begin{aligned}\Omega^l_k(e_i,e_j) &= d\theta^l_k(e_i,e_j)+\theta^l_m\wedge\theta^m_k(e_i,e_j)\\ &= e_i\theta^l_k(e_j)-e_j\theta^l_k(e_i)\\ &= e_i\Gamma^l_{jk}-e_j\Gamma^l_{ik}\end{aligned}$$

which is $R_{ijkl}$ at $x$. □

## 1.6 Symmetries of the curvature tensor

We defined the $(3,1)$-Riemann curvature tensor to be

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z.$$

where $\nabla$ is the Levi-Civita connection of the metric $g$ or $\langle,\rangle$. We also define the $(4,0)$ curvature tensor

$$R(X,Y,Z,W)=\langle R(X,Y)Z,W\rangle.$$

The curvature has the following symmetries

**Proposition 1.5.** *1.* $R(X,Y,Z,W)=-R(Y,X,Z,W)$

*2.* $R(X,Y,Z,W)=-R(X,Y,W,Z)$

*3.* $R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W)=0$

*4.* $R(X,Y,Z,W)=R(Z,W,X,Y)$

*Proof.* The first item is obvious. The second is also obvious from our point of view. The third follows from Jacobi identity. To see the fourth notice that

$$\begin{aligned}0 &= R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W)\\ &\quad +R(Y,X,W,Z)+R(W,Y,X,Z)+R(X,W,Y,Z)\\ &\quad -(R(Z,Y,W,X)+R(Y,W,Z,X)+R(W,Z,Y,X))\\ &\quad -(R(X,Z,W,Y)+R(Z,W,X,Y)+R(W,X,Z,Y))\\ &= 2(R(X,Y,Z,W)-R(Z,W,X,Y))\end{aligned}$$

□

There is another differential equation obeyed by the curvature called the second Bianchi identity.

**Lemma 1.6.**

$$d\Omega^l_k + \theta^l_m \wedge \Omega^m_k - \Omega^m_k \wedge \theta^l_m = 0$$

*Proof.* Taking $d$ of the formula $\Omega^l_k = d\theta^l_k + \theta^l_m \wedge \theta^m_k$ yields

$$\begin{aligned}
d\Omega^l_k &= d\theta^l_m \wedge \theta^m_k - \theta^l_m \wedge d\theta^m_k \\
&= (\Omega^l_m - \theta^l_n \wedge \theta^n_m) \wedge \theta^m_k - \theta^l_m \wedge (\Omega^m_k - \theta^m_n \wedge \theta^n_k) \\
&= \Omega^l_m \wedge \theta^m_k - \theta^l_m \wedge \Omega^m_k
\end{aligned}$$

□

In terms of the curvature tensor $R$ the second Bianchi identity takes the form

$$(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = 0$$

## 1.7 Sectional curvature

The sectional curvature is the function on the Grassmanian of two planes given by

$$K(\Pi) = R(e,f,e,f)$$

where $(e,f)$ is an orthonormal frame for $\Pi$. More generally if $X,Y$ span $\Pi$ then

$$K(\Pi) = \frac{R(X,Y,X,Y)}{|X|^2|Y|^2 - \langle X,Y\rangle^2}$$

The sectional curvature determines the full curvature.

$$\frac{\partial^2}{\partial s \partial t}(R(X+tZ,Y+sW,X+tZ,Y+sW) - R(X+tW,Y+sZ,X+tW,Y+sZ)) = R(X,Y,Z,W)$$

In particular if $K$ is a constant on the fibers of the Grassman bundle we have

$$R($$

## 1.8 Lie groups

A Lie group is a differentiable manifold and a group for which the group laws are differentiable. If $G$ is a Lie group then for any $g \in G$ we have diffeomorphisms

$$L_g, R_g : G \to G$$

given by left and right multiplication respectively. Thus

$$(L_g)_* T_e G \to T_g G$$

is an isomorphism and we can trivialize the tangent bundle of $G$. The formal definition of the Lie algebra of a Lie group is to use this isomorphism to identify the tangent space at the identity with the left-invariant vector fields.

**Definition 1.7.** A one-parameter subgroup is a map

$$\gamma : \mathbb{R} \to G$$

so that $\gamma(t+s) = \gamma(t)\gamma(s)$.

Every $\xi \in \mathfrak{g}$ is tangent to a one pararmeter subgroup. To see this let $F_t : U \subset \mathbb{R} \times G \to G$ be the flow for the vector field $\xi$ where $U$ is define so that for each $g \in G$ $U \cap \mathbb{R} \times \{g\}$ is the maximal interval of definition of the flow. First of all notice that since $\xi$ is left invariant we have $L_g \circ F$ is also the flow for $\xi$ from which it follows easily that $U = \mathbb{R} \times G$ and that

$$L_g \circ F(t, h) = F(t, gh) \tag{3}$$

The flow has the semi-group property,

$$F(t+s, g) = F(t, F(s, g)) \tag{4}$$

so if $g = F(t, e)$ then

$$\gamma(t+s) = F(t+s, e) = F(s, F(t, e)) = F(s, ge) = gF(s, e) = \gamma(t)\gamma(s).$$

**Proposition 1.8.** *Every compact Lie groups admits a bi-invariant metric.*

**Proposition 1.9.** *Let $\langle , \rangle$ be a bi-invariant metric. Then for all left-invariant vector fields$X, Y, Z$ we have*

1. $\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle$.

2. $\nabla_X Y = \frac{1}{2}[X, Y]$.
3. $R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$
4. $R(X, Y, Z, W) = -\frac{1}{4}\langle [X, Y], [Z, W]\rangle$.

*Proof.* For the third item we compute:

$$\begin{aligned} R(X, Y)Z &= \frac{1}{4}[X, [Y, Z]] - \frac{1}{4}[Y, [X, Z]] - \frac{1}{2}[[X, Y], Z] \\ &= -\frac{1}{4}[[X, Y], Z] \end{aligned}$$

The fourth formula follows immediately from this. □

## 1.9 The curvature of a submanifold of a Riemannian manifold

Let $N \subset M$ be a submanifold. If $M$ has a Riemannian metric then there is an induced Riemannian metric on $N$ Let $\nu$ denote the normal bundle of $N$ in $M$¿ Then we can define the so called second fundamental form

$$\mathrm{II} : TN \times TN \to \nu.$$

by

$$\mathrm{II}(X, Y) = \Pi_\nu(\nabla_X Y).$$

**Lemma 1.10.** II *is tensorial and*

$$\mathrm{II}(X, Y) = \mathrm{II}(Y, X).$$

*Proof.* The first claim follows from the second so consider the

$$\begin{aligned} \mathrm{II}(X, Y) &= \Pi_\nu(\nabla_X Y) \\ &= \Pi_\nu(\nabla_Y X + [X, Y]) \\ &= \Pi_\nu(\nabla_Y X) \\ &= \mathrm{II}(Y, X). \end{aligned}$$

. □

For example if $M = \mathbb{R}^{n+1}$ and $N = S^n$ then

$$\mathrm{II}(X, Y) = \langle X, Y\rangle.$$

Also notice that there is the following nice formula

**Lemma 1.11.** *If $\nu$ is a normal vector field to $N$ and $X$ and $Y$ are tangent then:*

$$\langle \nabla_X \nu, Y\rangle = -\langle \nu, \mathrm{II}(X, Y)\rangle$$

The second fundamental form determines the curvature of $N$ is terms of the curvature of $M$.

**Proposition 1.12.** *(The Gauss equations) For vector fields $X, Y, Z$ tangent $N$ we have:*

$$R^N(X, Y, W, Z) = R^M(X, Y, Z, W) - \langle \mathrm{II}(X, Z), \mathrm{II}(Y, W)\rangle + \langle \mathrm{II}(X, W), \mathrm{II}(Y, Z)\rangle$$

*Proof.* Suppose without loss of generality that $X, Y, Z, W$ are vectors field on $M$ which are tangent along $N$ and that $X$ and $Y$ commute. Then we have

$$\begin{aligned}
\langle \nabla^M_X \nabla^M_Y Z, W\rangle &= \langle \nabla^M_X (\nabla^N_Y Z + \mathrm{II}(Y, Z), W\rangle \\
&= \langle \nabla^N_X \nabla^N_Y Z + \mathrm{II}(X, \nabla^N_Y Z) + \nabla^M_X \mathrm{II}(Y, Z), W\rangle \\
&= \langle \nabla^N_X \nabla^N_Y Z W\rangle - \langle \mathrm{II}(Y, Z), \mathrm{II}(X, W)\rangle.
\end{aligned}$$

and hence

$$R^N(X, Y, W, Z) = R^M(X, Y, Z, W) - \langle \mathrm{II}(X, Z), \mathrm{II}(Y, W)\rangle + \langle \mathrm{II}(X, W), \mathrm{II}(Y, Z)\rangle$$

□

Thus the curvature of $S^n$ is

$$R(X, Y, Z, W) = -\langle X, Z\rangle\langle Y, W\rangle + \langle X, W\rangle\langle Y, Z\rangle.$$

## 1.10 Semi-Riemannian manifolds.

So far nothing we have done really required the inner product on the tangent space to be definite merely non-degenerate. As an example lets compute the curvature of hyperbolic space.

Let $\mathbb{H}^n$ be the component of the hyperboloid

$$c = -x_0^2 + x_1^2 + \ldots x_n^2 = -1$$

containing $(1, 0, 0, \ldots, 0)$. Consider the $\mathbb{R}^{n+1}$ with the Lorentz inner product.

$$m = -dx_0^2 + dx_1^2 + \ldots dx_n^2.$$

Then

$$\frac{1}{2}dc = -dx_0 + dx_1 + \ldots + dx_n$$

so the normal vector using $m$ to $\mathbb{H}^n$ is still

$$\hat{n} = (x_0, x_1, \ldots, x_n).$$

and so

$$\mathrm{II}(X, Y) = -m(X, Y)\hat{n}$$

and so

$$m(X, Z)m(Y, W) - m(X, W)m(Y, Z)$$

exactly the opposite of the sphere.

## 1.11 The decomposition of space of curvature tensors into irreducible representations of the orthogonal group

Let $\mathcal{R}$ denote the sub representation of $\mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$ consisting of

## 1.12 The behavior of the curvature under conformal change of metric.

Metrics $g$ and $\tilde{g}$ are said to be conformal if there is a function $\sigma$ so that

$$\tilde{g} = e^{2\sigma}g$$

We will investgate how the various curvatures change conformal change of metric. We will write $\langle , \rangle$ for the $g$ inner product and hence $e^{2\sigma}\langle , \rangle$ denotes the $\tilde{g}$-inner product.

**Lemma 1.13.** *The Levi-Civita connections are related as follows:*

$$\tilde{\nabla}_X Y = \nabla_X Y + (X\sigma)Y + (Y\sigma)X - \langle X, Y\rangle \nabla\sigma \tag{5}$$

*Proof.* Recall that the Levi-Civita connection $\nabla$ is determined be the equation

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}(X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle - \langle [X, Y], Z\rangle)$$

and so $\tilde{\nabla}$ is determined by

$$\begin{aligned} e^{2\sigma}\langle \tilde{\nabla}_X Y, Z\rangle &= \frac{1}{2}(Xe^{2\sigma}\langle Y, Z\rangle + Ye^{2\sigma}\langle X, Z\rangle - Ze^{2\sigma}\langle X, Y\rangle + \\ &\quad e^{2\sigma}(\langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle - \langle [X, Y], Z\rangle)) \\ &= e^{2\sigma}( \\ &\quad \frac{1}{2}(X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ &\quad \langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle - \langle [X, Y], Z\rangle) \\ &= e^{2\sigma}((X\sigma)\langle Y, Z\rangle + (Y\sigma)\langle X, Z\rangle - (Z\sigma)\langle X, Y\rangle) \\ &\quad e^{2\sigma}\langle \nabla_X Y, Z\rangle \end{aligned}$$

and so

$$\langle \tilde{\nabla}_X Y, Z\rangle = \langle \nabla_X Y, Z\rangle + (X\sigma)\langle Y, Z\rangle + (Y\sigma)\langle X, Z\rangle - \langle X, Y\rangle\langle \nabla\sigma, Z\rangle)$$

as required. $\square$

A long tedious calculation shows that

$$\begin{aligned} \tilde{R}(X, Y)Z = R(X, Y)Z &+ \nabla d\sigma(X, Z)Y - \nabla d\sigma(Y, Z)X - \langle Y, Z\rangle \nabla_X \nabla\sigma + \langle X, Z\rangle \nabla_Y \nabla\sigma \\ &+ (Y\sigma)(Z\sigma)Z - (X\sigma)(Z\sigma)Y - (Y\sigma)\langle X, Z\rangle \nabla\sigma + (X\sigma)\langle Y, Z\rangle \nabla\sigma \\ &- \langle Y, Z\rangle |\nabla\sigma|^2 X + \langle X, Z\rangle |\nabla\sigma|^2 Y \end{aligned}$$

which can be rewritten as

$$\tilde{R}^{4,0} = e^{2\sigma}\big(R^{4,0} + g \bowtie (\nabla d\sigma - d\sigma \otimes d\sigma + \frac{1}{2}|\nabla\sigma|^2 g)\big)$$

From which we glean

$$\tilde{W}^{4,0} = e^{2\sigma} W^{4,0}$$

or that the (3,1)-Weyl curvature is conformally invariant. Notice then that the Weyl curvature is an obstruction to a metric being locally conformally equivalent to the standard flat metric.

We can derive the behavior of the Ricci curvature Ric and the scalar curvature s under conformal change of metric. Let $c_g$ denote the Ricci contraction for the metric $g$. If $e_1 \dots, e_n$ is a local orthonormal frame then

$$c_g(R)(e_j, e_k) = \sum_{i=1}^{n} R(e_j, e_i, e_k, e_i)$$

so that

$$\mathrm{Ric} = -c_g(R)$$

We will need the following.

**Lemma 1.14.** *For all $h \in Sym^2(T^*X)$ we have*

$$c_g(h \bowtie g) = \mathrm{tr}_g(h) + (n-2)h$$

*Proof.* A tedious calculation with indices. □

**Lemma 1.15.** *If $\tilde{g} = e^{2\sigma} g$ then we have*

$$\tilde{\mathrm{Ric}} = \mathrm{Ric} + (\Delta\sigma + (n-2)|\nabla\sigma|^2)g + (n-2)(\nabla d\sigma - d\sigma \odot d\sigma) \tag{6}$$

*and*

$$\tilde{s} = e^{-2\sigma}(s + 2(n-1)\Delta\sigma + (n-1)(n-2)|\nabla\sigma|^2) \tag{7}$$

## 1.13 Geodesics and curvature

Let $\gamma : (a, b) \to X$ be a smooth path. The energy of $\gamma$ is

$$E(\gamma) = \frac{1}{2}\int_a^b |\gamma_*(\frac{\partial}{\partial t})|^2 \mathrm{dt}$$

and the length is

$$L(\gamma) = \int_a^b |\gamma_*(\frac{\partial}{\partial t})| \mathrm{dt}.$$

A geodesic is a path which locally minimizes the length in the following sense. A variation of $\gamma$ is a function $F\colon (-\epsilon, \epsilon) \times (a, b) \to X$ so that $F(0, t) = \gamma(t)$. The infinitesimal variation of $\gamma$ corresponding to $F$ is the vector field along $\gamma$ $S = F_*(\frac{\partial}{\partial s})$. We denote by $T$ is the tangent vector field along $\gamma$, $T = \gamma_*(\frac{\partial}{\partial t})$. We have the following two important formulae. The first variational formula for the energy:

**Lemma 1.16.**

$$\frac{\partial}{\partial s}E(\gamma_s) = -\int_a^b \langle S, \nabla_T T\rangle \mathrm{dt} + \langle S, T\rangle|_a^b \tag{8}$$

*and the first variational formula for the length:*

$$\frac{\partial}{\partial s}L(\gamma_s) = -\int_a^b \langle S, \nabla_{T/|T|} T\rangle + \langle S, T/|T|\rangle|_a^b \tag{9}$$

*Proof.*

$$\begin{aligned}
\frac{\partial}{\partial s}E(\gamma_s) &= \int_a^b \langle T, \nabla_S T\rangle \mathrm{dt} \\
&= \int_a^b \langle T, \nabla_T S\rangle \mathrm{dt} \\
&= \int_a^b (\frac{\partial}{\partial t}\langle T, S\rangle - \langle \nabla_T T, S\rangle)\mathrm{dt} \\
&= \langle S, T\rangle|_a^b - \int_a^b \langle \nabla_T T, S\rangle \mathrm{dt}
\end{aligned}$$

The proof for the length is similar and left to the reader. □

As a consequence we have that $\gamma$ is a geodesic iff and only if

$$\nabla_T(T/|T|) = 0$$

in other words the unit tangent vector to $\gamma$ is parallel along $\gamma$. If we parameterize $\gamma$ proportional to arc length then $\nabla_T T = 0$. We also have the second variational formula.

**Lemma 1.17.** *Suppose that $\gamma$ is a geodesic. Then;*

$$\frac{\partial^2}{\partial s^2}E(\gamma_s) = \int_a^b (|\nabla_T S|^2 - R(S,T,S,T))\mathrm{dt} + \langle \nabla_S S, T\rangle|_a^b.$$

*Proof.*

$$\begin{aligned}
\frac{\partial^2}{\partial s^2}\frac{1}{2}\langle T,T\rangle &= \frac{\partial}{\partial s}\langle \nabla_S T, T\rangle \\
&= |\nabla_S T|^2 + \langle \nabla_S\nabla_S T, T\rangle \\
&= |\nabla_T S|^2 + \langle \nabla_S\nabla_T S, T\rangle \\
&= |\nabla_S T|^2 + \langle R(S,T)S, T\rangle + \langle \nabla_T\nabla_S S, T\rangle \\
&= |\nabla_S T|^2 + \langle R(S,T)S, T\rangle + \frac{\partial}{\partial t}\langle \nabla_S S, T\rangle - \langle \nabla_S S, \nabla_T T\rangle
\end{aligned}$$

The last term is zero if $\gamma$ is a geodesic so the result follows by integrating. □

The equation for a geodesic is in local coordinates say that if $\phi\circ\gamma(t) = (x^1(t),\ldots,x^n(t))$ so that $\phi_* T = \dot{x}^i\frac{\partial}{\partial x^i}$

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0$$

Standard existence and uniqueness theory for ODE's says that for each $x\in X$ and $V\in T_xX$ there is $\epsilon > 0$ and a geodesic

$$\gamma : (-\epsilon,\epsilon)\to X$$

so that $\gamma(0) = x$ and $\dot{\gamma}(0) = V$. Notice that the geodesic equation scales so that if $\gamma(t)$ is a geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = V$ then $\gamma(ct)$ is geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = cV$ so we can formulate a cleaner statement about the existence of geodesics.

Thus we see that for each $x\in M$ there is a neighborhood of $0\in T_xM$ and a map $\exp_x : U\to M$ define by setting

$$exp_x(v) = \gamma(1)$$

**Definition 1.18.** A point $y \in M$ is conjugate to $X$ if it is a critical value of the $exp_x$.

The Gauss lemma and the length minimizing property of geodesics. Consider a small ball in $T_x X$.

**Lemma 1.19.** *The image under the exponential map of spheres in the tangent space are orthogonal to the geodesics eminating from $x$.*

*Proof.* Let $y = \exp_x(v)$ and $w$ be a vector perpendicular to $v$. Let $T = d_v \exp_x(v)$ and $S = d_v \exp_x(w)$. We must show that

$$\langle T, S \rangle = 0.$$

We can extend $T$ and $S$ to vector fields along a variation by considering the the map $F(t,s) = \exp_x(t(v+sw))$. Then $T = F_*(\frac{\partial}{\partial t})$ and $S = F_*(\frac{\partial}{\partial s})$. Notice that $T$ has fixed lenght while $S$ does not.

$$\begin{aligned} \frac{\partial}{\partial t}\langle T, S \rangle|_{s=0} &= \langle \nabla_T T, S \rangle + \langle T, \nabla_T S \rangle \\ &= \langle T, \nabla_T S \rangle \\ &= \frac{1}{2}\frac{\partial}{\partial s}\langle T, T \rangle \\ &= 0 \end{aligned}$$

So the result follows as $S(0,0) = 0$. □

Let $\gamma : [0,b] \to M$ be a path eminating from $x$ and ending at $y$. Write $\gamma(t) = \exp_x(r(t)\omega(t))$. The $\tilde{\gamma}(t) = \exp_x(t/r(b)\omega(b))$ is also a path from $x$ to $y$ whose length is $r(b)$.

**Lemma 1.20.** *If $\gamma$ is a length minimizing path then $\dot{\omega} = 0$ and $r(t)$ is monotone.*

*Proof.* Write the tangent vector to $\gamma$ as

$$\dot{\gamma}(t) = \dot{r}(t)\frac{\partial}{\partial r} + r(t)\dot{\omega}(t).$$

So by the Gauss lemma

$$\|\dot{\gamma}(t)\| = \sqrt{\dot{r}^2 + r^2|\dot{\omega}|^2}$$

and hence

$$\ell(\gamma) \geq \int_0^b |\frac{dr}{dt}| dt \geq r(b)$$

with equality hold if and only if $\dot{\omega} = 0$ and $r(t)$ being monotone. $\square$

Having this in hand it is straighforward to show that the topology of $M$ as a manifold agrees with the topology of $M$ induced by the metric

$$\mathrm{d}(x, y) = \inf\{\ell(\gamma) | \gamma \text{ is a smooth path joining } x \text{ to } y.\}$$

We call a Riemannian manifold geodesically complete at $x$ if the exponential map is surjective at $x$ and geodesically complete if it is geodesically complete at each $x$.

**Theorem 1.21.** *The Hopf-Rinow Theorem The following are equivalent.*

- $(M, d)$ *is a complete metric space*
- $(M, g)$ *is geodesically complete.*
- $(M, g)$ *is geodesically complete at some $x$ in $M$.*
- *The ball $B(x, r)$ are compact for all $x \in M$ and $r > 0$.*

## 1.14 Moving around Geodesics and Jacobi fields

Consider a variation $F$ of a geodesic $\gamma : [a, b] \to M$ through geodesics. Then with the previous notation we have

$$\nabla_S \nabla_T T = 0$$

but as we saw before

$$\nabla_S \nabla_T T = \nabla_T \nabla_T S - R(T, S)T = -H_\gamma S$$

Thus the infinitesimal variation $S$ satisfies a second linear ODE, called the Jacobi equation

$$H_\gamma S = 0$$

The domain of this operator, the tangent vector fields along $\gamma$ is naturally but slightly loosely called $T_\gamma \mathcal{P}(M)$, the tangent space to the path space of $M$. The operator $H_\gamma : C$ is called the *Jacobi operator*. A solution to Jacobi's equation is called a *Jacobi field.* The dimension of the space of Jacobi fields along $\gamma$ is $2n$.

There are two obvious solutions to this equation

$$T \,\text{and}\, tT.$$

corresponding to changing $\gamma(t)$ to $\gamma(at+b)$. To understand $H_\gamma$ better consider a parallel orthonormal frame field $e_1, e_2, \ldots, e_n$ along $\gamma$ with $e_1 = T/|T|$. Let $K_{ij} = \langle R(e_1, e_i)e_1, e_j\rangle$. Writing $S = s^1 e_1 + \ldots + s^n e_n$ we have

$$\ddot{s}^1 = 0$$

and for $i = 2, \ldots, n$ we have:

$$\langle H_\gamma s, e_i\rangle = -\ddot{s}^i + \sum_{j=2}^{n} K_{ij} s^j.$$

Let us notice the following.

**Lemma 1.22.** *All the sectional curvatures are negative (nonpositive) at a point if and only if $K_{ji}(x)$ is a positive definite matrix.*

*Proof.* If $\Pi$ is spaned by the orthonormal vectors $e_1, e_2$ then

$$K(e_1, e_2) =$$

$\square$

We can formalize a little of what was going in the the proof of the Gauss lemma in the following way. Let $\gamma$ be a geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$, and $\gamma(1) = y$ Let $Y(t)$ be the Jacobi field along gamma with $Y(0) = 0$ and $\dot{Y}(0) = w$ then

**Lemma 1.23.**

$$d_v \exp(w) = Y(1).$$

*Proof.* Consider the same variation as in the proof of the Gauss lemma

$$F(s,t) = \exp_x(t(v + sw))$$

Note that $d_v \exp(w) = F_*(\frac{\partial}{\partial s})|_{s=1,t=1}$ Then as $s$ varies $f$ moves through geodesics and so

$$Y(t) = F_*(\frac{\partial}{\partial s})|_{s=0}$$

is a Jacobi field and has required properties. $\square$

Thus we have seen that

**Corollary 1.24.** *$y$ is conjugate to $x$ along $\gamma$ iff and only if there is a Jacobi field along $\gamma$ which vanishes at both $x$ and $y$.*

Thus from the above discussion we see that if the sectional curvature of $M$ is non-negative then the exponential map has no critical points.

From this we will deduce the

**Theorem 1.25.** *Cartan-Hadamard The universal cover of complete Riemannian manifold with non-positive sectional curvature is $\mathbb{R}^n$.*

We simply need the following lemma which is homework.

**Lemma 1.26.** *Let $M$ and $N$ be complete Riemannian manifolds and let $\phi : M \to N$ be a smooth map with $d_x\phi$ being an isometry for all $x \in M$. Then $\phi$ is a covering map.*

As the other end of the spectrum suppose we have

**Theorem 1.27.** *Myers Suppose that the Ricci curvature $Ric_{ij}$ satisfies*

$$\mathrm{Ric}_{ij} \geq \frac{n-1}{a^2} g_{ij}$$

*in the partial order of symmetric matrices. Then $M$ is compact of diameter less or equal to $\pi a$.*

*Proof.* Let $\gamma : [0,1] \to M$ a geodesic. So that $|\dot{\gamma}| = \ell(\gamma)$. On the $n$-sphere the Jacobi fields are

$$\sin(\pi t) w$$

Let $e_2, \dots, e_n$ be a parallel frame along $\gamma$ orthogonal $\dot{\gamma}$. Consider the variations

$$W_i(t) = \sin(\pi t) e_i.$$

$$\langle H_\gamma W_i, W_i \rangle = \int_0^1 (\sin(\pi t))^2 (\pi^2 + \ell^2 \langle R(e_1, e_i) e_1, e_i \rangle dt$$

so

$$\sum_{i=2}^{n} \langle H_\gamma W_i, W_i \rangle = \int_0^1 (\sin(\pi t))^2 (n-1)\pi^2 - \ell^2 \mathrm{Ric}_{11}) dt \leq 0$$

by the assumption. Thus there must be a conjugate point along $\gamma$ before $\gamma(1)$. Thus the image of the ball of radius $\ell$ in $T_x M$ covers the manifold and hence $M$ is compact.

□

**Corollary 1.28.** *Under there assumptions $\pi_1(M)$ is finite.*

*Proof.* The universal cover is compact. $\square$

Thus of particular interest is the behavior of $\gamma$ under variations fixing the endpoints, so we restrict $H_\gamma$ to the space of vector fields along $\gamma$ vanishing at the end points. Setting $\gamma(a) = x$ and $\gamma(y) = b$ we can think of the this space of vector fields as, $T_\gamma \mathcal{P}_{x,y}(M)$ the tangent space at $\gamma$ of the space of paths joining $x$ to $y$. Then $H_\gamma$ is the operator representing the quadratic form the Hessian of the energy functional on $\mathcal{P}_{x,y}(M)$

We make the following claims.

## 1.15 The basic constant curvature examples

Let compute everything for the manifolds

$$S^n(k) = \{x_0^2 + \ldots x_n^2 = \frac{1}{k^2}\}$$

$$\mathbb{R}^n$$

$$\mathbb{H}^n(-k) = \{-x_0^2 + x_1^2 + \ldots x_n^2 = \frac{-1}{k^2}\}.$$

So that we learn something new while doing this we'll digress to discuss

### 1.15.1 The curvature of a submanifold of a Riemannian manifold.

Let $N \subset M$ be a submanifold. If $M$ has a Riemannian metric then there is an induced Riemannian metric on $N$ Let $\nu$ denote the normal bundle of $N$ in $M$. Notice that the fiberwise exponential map give a diffeomorphism of a neighborhood of the zero section of $\nu$ with a neighborhood of $N$ in $M$. (Use the implicit function theorem. Then we define the *second fundamental form*

$$\mathrm{II} : TN \times TN \to \nu.$$

by

$$\mathrm{II}(X, Y) = \Pi_\nu(\nabla_X Y).$$

**Lemma 1.29.** II *is tensorial and*

$$\mathrm{II}(X, Y) = \mathrm{II}(Y, X).$$

*Proof.* The first claim follows from the second so consider the

$$\begin{aligned}\mathrm{II}(X,Y) &= \Pi_\nu(\nabla_X Y)\\ &= \Pi_\nu(\nabla_Y X + [X,Y])\\ &= \Pi_\nu(\nabla_Y X)\\ &= \mathrm{II}(Y,X).\end{aligned}$$

. ∎

An effective way to compute is give by the following.

**Lemma 1.30.** *If $\nu$ is a normal vector field to $N$ and $X$ and $Y$ are tangent then:*

$$\langle \nabla_X \nu, Y\rangle = -\langle \nu, \mathrm{II}(X,Y)\rangle$$

*Proof.* Extend $X, Y$ to vector fields defined in a neighborhood of the point under consideration.

$$0 = X\langle \nu, Y\rangle = \langle \nabla_X \nu, Y\rangle + \langle \nu, \mathrm{II}(X,Y)\rangle$$

∎

For example if $M = \mathbb{R}^{n+1}$ and $N = S^n(k)$ then

Since the normal is simply the position vector $v$ The covariant derivative is simply $\nabla_X v = X$ so we have

$$\mathrm{II}(X,Y) = -k\langle X, Y\rangle.$$

The second fundamental form is also used to define the Gaussian curvature of a submanifold. Let $\Pi$ be a two plane spaned by the orthonormal vectors $e_1, e_2$. Then we define

$$G(\Pi) = \langle \mathrm{II}(e_1,e_1), \mathrm{II}(e_2,e_2)\rangle - |\mathrm{II}(e_1,e_2)|^2.$$

For example for a hypersurface in $\mathbb{R}^3$ this measures the curvature of the ellpisoid or hyperbolid that fits the surface best at a point.

The second fundamental form determines the curvature of $N$ is terms of the curvature of $M$.

**Proposition 1.31.** *(The Gauss equations) For vector fields $X, Y, Z$ tangent $N$ we have:*

$$R^N(X,Y,W,Z) = R^M(X,Y,Z,W) - \langle \mathrm{II}(X,W), \mathrm{II}(Y,Z)\rangle + \langle \mathrm{II}(X,Z), \mathrm{II}(Y,W)\rangle$$

*and so*

$$K^N(\Pi) = K^M(\Pi) + G(\Pi)$$

**Corollary 1.32.** *The Gaussian curvature of a hypersurface in $\mathbb{R}^3$ agrees (upto sign) with its intrinsic curvature. This is called Gauss's "Theorema Egregium".*

*Proof.* Suppose without loss of generality that $X, Y, Z, W$ are vectors field on $M$ which are tangent along $N$ and that $X$ and $Y$ commute. Then we have

$$\begin{aligned}
\langle \nabla^M_X \nabla^M_Y Z, W\rangle &= \langle \nabla^M_X (\nabla^N_Y Z + \mathrm{II}(Y,Z), W\rangle \\
&= \langle \nabla^N_X \nabla^N_Y Z + \mathrm{II}(X, \nabla^N_Y Z) + \nabla^M_X \mathrm{II}(Y,Z), W\rangle \\
&= \langle \nabla^N_X \nabla^N_Y Z W\rangle - \langle \mathrm{II}(Y,Z), \mathrm{II}(X,W)\rangle.
\end{aligned}$$

and hence

$$R^N(X,Y,W,Z) = R^M(X,Y,Z,W) - \langle \mathrm{II}(X,Z), \mathrm{II}(Y,W)\rangle + \langle \mathrm{II}(X,W), \mathrm{II}(Y,Z)\rangle$$

□

Thus the curvature of $S^n(k)$ is

$$R(X,Y,Z,W) = -k^2\langle X,Z\rangle\langle Y,W\rangle + k^2\langle X,W\rangle\langle Y,Z\rangle.$$

More generally if $N$ is the level set of a function $f$ then we have

$$\mathrm{II}(X,Y) = -\nabla df(X,Y)\frac{\nabla f}{|\nabla f|}$$

**Definition 1.33.** A submanifold $N$ is called totally geodesic if all geodesics starting tangent to $M$ remain in $N$. A submanifold is called minimal if $\mathrm{trII} = 0$.

It is often useful to introduce Gaussian polar coordinates. By the Gauss lemma we have

$$\exp^*(g) = dr^2 + h_r.$$

where $h$ is some metric on $S^{n-1}$ which varies with $r$. As usual with polar coordinates we map $\mathbb{R}_+ \times S^{n-1}(1) \to \mathbb{R}^n$ by $(r, v) \mapsto rv$. The standard flat metric becomes

$$g = dr^2 + r^2 h_{S^{n-1}(1)}$$

For example if we consider $S^n(k)$ and introduce Gaussian coordinates about some point $u$. To compute what is going on fix and $v \perp u$ with $\|v\| = 1$ then

$$\cos(kt)u + \frac{1}{k}\sin(kt)v$$

is a unit speed geodesic starting at $u$ and hence

$$\gamma(t) = \cos(krt)u + \frac{1}{k}\sin(krt)v$$

is a geodesic which arrives at time 1 at $\exp(rv)$ a point distance $r$ from $u$. Now fix $w$ a unit vector orthogonal to both $u$ and $v$. Then

$$F(t, s) = \cos(krt)u + \frac{1}{k}\sin(krt)(\cos(s)v + \sin(s)w)$$

is a variation through geodesics and hence

$$Y(t) = \frac{1}{rk}\sin(krt)w$$

is a Jacobi field along $\gamma$ with $Y(0) = 0$ and $\dot{Y}(0) = w$. Finally

$$d_{rv}\exp(w) = Y(1) = \frac{1}{rk}\sin(kr)w$$

and so in geodesic polar coordinates the metric on $S^n$ is

$$dr^2 + \frac{\sin^2(kr)}{k^2} g_{S^{n-1}}$$

Note that this becomes 0 when $r = \pi k$ just as it should. The volume form is

$$\frac{\sin^{n-1}(kr)}{k^{n-1}} dr\, dvol_{S^{n-1}}$$

An important operator on a manifold is the Laplace (or Laplace-Beltrami) operator $\Delta$. It is defined by the equation

$$\int_X f\Delta f = \int_X |df|^2$$

Notice that if $f$ has compact support then

$$\int_X \Delta f = 0.$$

The Laplace operator on $S^n$ can be written in polar coords as

$$\Delta f = -\sin^{-(n-1)}(kr^{-(n-1)}\frac{\partial}{\partial r}\sin^{n-1}(kr)\frac{\partial f}{\partial r} + \frac{k^2}{\sin^2(kr)}\Delta_{S^{n-1}}.$$

In analysis on manifolds it is often important to write down the Green's function for the Laplace operator. This is operator that inverts the $\Delta$ to the extent possible. On a compact manifold we are search for and operator $G$ satifisying

$$\Delta G f = f - \frac{1}{\mathrm{Vol}(M)}\int f$$

We are happy if we can write down an integral kernel for this operator. This is a smooth function $\mathbf{g}\colon X \times X \setminus \mathrm{Diag} \to \mathbb{R}$ with

$$\mathbf{g}(x,y) = \mathbf{g}(y,x)$$

and satistying the distributional equation

$$\Delta_x \mathbf{g}(x,y) = \delta_x(y) - \frac{1}{\mathrm{Vol}(M)}$$

Assuming $\mathbf{g}(x,y) = f(r)$ a function of $r = \mathrm{dist}(x,y)$ get the differential equation

$$-\sin^{-(n-1)}(kr)\frac{\partial}{\partial r}\sin^{n-1}(kr)\frac{\partial f}{\partial r} = -\frac{1}{\mathrm{Vol}(S^n(k))}$$

subject to the initial condition

$$\lim_{r\mapsto 0}\sin^{n-1}(kr)f^{`}(r) = \frac{-1}{\mathrm{Vol}(S^{n-1}(k))}$$

so for example for $n = 2, 3$ we have

$$\mathbf{g}(x, y) = -\frac{1}{2\pi} \ln(\sin(\mathrm{dist}(x, y)/2))$$

and

$$\mathbf{g}(x, y) = \frac{1}{4\pi^2}(\mathrm{dist}(x, y) - \pi) \cot(\mathrm{dist}(x, y))$$

### 1.15.2 Semi-Riemannian manifolds.

So far nothing we have done really required the inner product on the tangent space to be definite merely non-degenerate. As an example lets compute the curvature of hyperbolic space.

Let $\mathbb{H}^n(-k)$ be the component of the hyperboloid

$$c = -x_0^2 + x_1^2 + \ldots x_n^2 = \frac{-1}{k^2}$$

containing $(1, 0, 0, \ldots, 0)$. Consider the $\mathbb{R}^{n+1}$ with the Lorentz or Minkowski inner product.

$$m = -dx_0^2 + dx_1^2 + \ldots dx_n^2.$$

Then

$$\frac{1}{2}dc = -dx_0 + dx_1 + \ldots + dx_n$$

so the normal vector using $m$ to $\mathbb{H}^n$ is still

$$\hat{n} = (x_0, x_1, \ldots, x_n).$$

The Minkowski metric is flat so its covariant derivative is the ordinary derivative and we still have.

$$\mathrm{II}(X, Y) = -m(X, Y)\hat{n}$$

and so

$$R(X, Y, Z, W) = m(X, Z)m(Y, W) - m(X, W)m(Y, Z)$$

exactly the opposite of the sphere because $n$ has $m(n, n) = -1$.

We can go through exactly the same calculations as before. Fix $u$ with $u \cdot u = \frac{-1}{k^2}$. To compute what is going on fix and $v \cdot u = 0$ with $\|v\|^2 = 1$ then

$$\cosh(kt)u + \frac{1}{k}\sinh(kt)v$$

is a unit speed geodesic starting at $u$ and hence

$$\gamma(t) = \cosh(krt)u + \frac{1}{k}\sinh(krt)v$$

is a geodesic which arrives at time 1 at $\exp(rv)$ a point distance $r$ from $u$. Now fix $w$ a unit vector orthogonal to both $u$ and $v$. Then

$$F(t,s) = \cosh(krt)u + \frac{1}{k}\sinh(krt)(\cos(s)v + \sin(s)w)$$

is a variation through geodesics and hence

$$Y(t) = \frac{1}{rk}\sinh(krt)w$$

is a Jacobi field along $\gamma$ with $Y(0) = 0$ and $\dot{Y}(0) = w$. Finally

$$d_{rv}\exp(w) = Y(1) = \frac{1}{rk}\sinh(kr)w$$

and so in geodesic polar coordinates the metric on $S^n$ is

$$dr^2 + \frac{\sinh^2(kr)}{k^2} g_{S^{n-1}}$$

Note that this becomes 0 when $r = \pi k$ just as it should. The volume form is

$$\frac{\sin^{n-1}(kr)}{k^{n-1}} dr \wedge \omega_{S^{n-1}(1)}$$

Thus the Laplace operator on $\mathbb{H}^n$ can be written in polar coords as

$$\Delta f = -\sinh^{-(n-1)}(kr)^{-(n-1)}\frac{\partial}{\partial r}\sinh^{n-1}(kr)\frac{\partial f}{\partial r} + \frac{k^2}{\sinh^2(kr)}\Delta_{S^{n-1}}.$$

Now to find the kernel representing the Green's function we search for a function $f(r)$ satisfying

$$\sinh^{-(n-1)}(kr)^{-(n-1)}\frac{\partial}{\partial r}\sinh^{n-1}(kr)\frac{\partial f}{\partial r} = 0$$

with the initial condition

$$\lim_{r\to 0}\sinh^{n-1}(kr)\frac{\partial f}{\partial r} = -\frac{1}{\mathrm{Vol}(S^{n-1}(1))}$$

$$\mathbf{g}(x,y) = -\frac{1}{k^{n-1}\mathrm{Vol}S^{n-1}(1)}\int_{\mathrm{dist}(x,y)}^{\infty}\sinh^{-(n-1)}(ks)ds$$

## 1.16 Conformal deformations of metrics on two manifolds

We now consider the problem of deforming a metric on a two manifold to a constant curvature metric. From the previous calculations in the case $n = 2$ we must solve the equation

$$\tilde{s} = e^{-2\sigma}(s + 2\Delta\sigma)$$

or

$$2\Delta\sigma - \tilde{s}e^{2\sigma} = -s$$

The Gauss-Bonnet theorem implies that

$$\frac{1}{4\pi}\int_\Sigma s = \chi(\Sigma)$$

so if $\tilde{g}$ exists we must have

$$\tilde{s}\mathrm{Vol}(\Sigma, \tilde{g}) = 4\pi\chi(\Sigma).$$

We will use with out proof that the Laplacian

$$\Delta : C^{k+2,\alpha}(\Sigma) \to C^{k,\alpha}(\Sigma)$$

has the following properties:

1. $\Delta$ is a Fredholm operator of index zero.
2. The kernel of $\Delta$ is the constant functions.
3. The $L^2$ orthogonal complement of the range is also the constant functions.
4. The spectrum of $\Delta$ is a disrcete unbounded subset of $\mathbb{R}_{\geq 0}$.
5. The maximum principle holds.

### 1.16.1 The case of Euler characteristic zero

The simplest case to deal with is that of Euler characteristic zero, so $\Sigma$ is diffeomorphic to a torus. Then we need to solve

$$2\Delta\sigma = -s$$

where $\int_\Sigma s = 0$ and by the above properties of $\Delta$ this equation has a unique solution with $\int_\Sigma \sigma = 0$.

### 1.16.2 The case of negative Euler characteristic

Surprisingly the case of negative Euler characteristic is reasonably easy. Setting $u = 2\sigma$ and $-s = k$ we must solve

$$\Delta u + e^u = k$$

where $k$ is a function with

$$\int_\Sigma k > 0.$$

Call a function $u_+$ a (strict) supersolution if

$$\Delta u_+ + e^{u_+}(>) \geq k$$

and $u_-$ is called a (strict) subsolution if

$$\Delta u_+ + e^{u_+}(<) \leq k$$

This definition is justified by the following lemma

**Lemma 1.34.** *Suppose that $u$ and $v$ are $C^2$-functions with*

$$\Delta u + e^u > (\geq) \Delta v + e^v$$

*then*

$$u > (\geq) v$$

*Proof.* Suppose not. Then there is a point $x \in \Sigma$ where $u(x) \leq v(x)$ and $u - v$ is a minimum at $x$. Then

$$0 \geq \Delta(u - v)|_x > e^{v(x)} - e^{u(x)} \geq 0$$

a contradiction. $\square$

Thus if $u_+$ is supersolution, $u$ is a solution and $u_-$ is a subsolution then

$$u_+ \geq u \geq u_-$$

with strict inequalities if either is strict in particular we have uniqueness. If $u_1$ and $u_2$ solve the equation then $u_1 = u_2$.

**Lemma 1.35.** *Sub and super solutions always exist.*

*Proof.* Choosing $c \in \mathbb{R}$ so that $e^c > k$ we have that $u_+ = c$ is a supersolution. To find a subsolution argue as follows. Let $\bar{k}$ denote the average value of $k$. Notice that $\bar{k}$ is positive. We can find a unique function $v$ with

$$\Delta v + \bar{k} = k$$

Then choose $c$ so that $e^{v+c} < \bar{k}$. Then we claim that $u_- = v + c$ is a subsolution.

$$\Delta(v + c) + e^{v+c} = k - \bar{k} + e^{v+c} < k$$

as required. $\square$

Next we set up two iterations, one starting with a given supersolution and keep producing a strictly smaller supersolution another starting with a subsolution and producing a strictly bigger subsolution. To set up the iterations we need the following lemma.

**Lemma 1.36.** *Let $u_0$ be a supersolution and fix $M > e^{u_0}$. Suppose that $v$ and $w$ solve the equation:*

$$\Delta w + Mw = Mv - e^v + k.$$

*If $v$ is a super solution with $v < u_0$ then $w < v$ and $w$ is a supersolution while if $v$ is subsolution then $v < w$ and $w$ is a subsolution.*

*Proof.* Consider the case where $v$ is supersolution. If the first inequality is false then there is a point $x \in \Sigma$ where $w(x) \geq v(x)$ and $w(x) - v(x)$ is maximum so that

$$0 \geq \Delta(w - v)|_x = M(w - v)|_x - (\Delta v - e^v + k) \geq 0$$

a contradiction. For second inequality consider

$$\begin{aligned} \Delta w + e^w - k &= M(v - w) + e^w - e^v \\ &> e^v(v - w + e^{w-v} - 1) = e^v(e^{-(v-w)} - (1 - (v - w)) \\ &\geq 0. \end{aligned}$$

where we have used that $M > e^v$ and $v > w$ to go from the first line to the second and $e^x - (1 + x) \geq 0$ to go from the second to the third. The case where $v$ is subsolution is similar. $\square$

Now fix a supersolution $u_0$ and $M$ as in the lemma. Define a sequence $\overline{u}_i$ inductively for $i > 0$ by:

$$\Delta \overline{u}_{i+1} + M\overline{u}_{i+1} = M\overline{u}_i - e^{\overline{u}_i} + k.$$

By the previuos lemma we have that $\overline{u}_{i+1} < \overline{u}_i$ and all the $\overline{u}_i$ are supersolutions. Since there is a subsolution the sequence is bounded below pointwise and we get that the $\overline{u}_i$ converge pointwise to some function $\overline{u}$. We also have that $\overline{u}$ is lower semi continuous since its a pointwise limit of a sequence of strictly decreasing functions.

Similarly start with a subsolution $\underline{u}_0$ and define $\underline{u}_i$ inductively by

$$\Delta \underline{u}_{i+1} + M\underline{u}_{i+1} = M\underline{u}_i - e^{\underline{u}_i} + k.$$

Now we have $\underline{u}_{i+1} > \underline{u}_i$ and as above we get pointwise convergence to some function $\underline{u}$ which is uppersemicontinuous. We wish to improve this to uniform convergence to that at least we get a continuous limit. Consider the differences

$$\delta u_i = \overline{u}_i - \underline{u}_i.$$

Then we have

$$\Delta \delta u_{i+1} + M\delta u_{i+1} = M\delta u_i - e^{\overline{u}_i} + e^{\underline{u}_i}$$

Consider a point at which $\delta u_{i+1}$ achieves its max. Then we have

$$M\delta u_{i+1} \leq M\delta u_i - e^{\overline{u}_i} + e^{\underline{u}_i} \leq (M - e^{\underline{u}_0})(\delta u_i)$$

So we have

$$\sup_{x\in\Sigma} \delta u_{i+1}(x) < \frac{1}{M}(M - \inf_{x\in\Sigma}(e^{\underline{u}_0} \sup_{x\in\Sigma} \delta u_i$$

Thus $\delta u_i \mapsto 0$ so $\underline{u} = \overline{u} = u$ is continuous. Then limit also uniform by a similar arguement. We need to get better convergence. If $\Delta : C^2 \to C^0$ was Fredholm we'd be home free but its not!!!! However it is Fredholm from $L^2_2 \to L^2$ and the so we can assume that say $\overline{u}_i$ converges in $L^2_2$ to $u$ from the equations. But on a two-manifold $L^2_2$ embeds in $C^{0,\alpha}$ for all $\alpha$ hence we can assume that $u_i$ converges to $u$ in $C^{0,\alpha}$. Then Hölder elliptic regularity implies convergencein $C^{2,\alpha}$ from the equations. Repeating we get that $u_i$ converges to $u$ in $C^\infty$. $u$ then solves the equation:

$$\Delta u + Mu = Mu - e^u + k$$

or

$$\Delta u + e^u = k$$

as required.

### 1.16.3 The case of positive Euler characteristic.

The case of positive Euler characteristic is delicate via this method and we will not give the proof. Is does follow from the Riemann mapping theorem so you can look up a proof of that to round out this discussion. If $g$ is a metric on $S^2$ then $g$ induce a complex structure $j$ on $S^2$ and hence by the Riemann mapping theorem there is a holomorphic map $f : (S^2, j) \to (S^2, j_0)$ where $j_0$ is the standard complex stucture. $f_* g$ is then conformal to the standard metric.

## 1.17 Geodesics, Jacobi fields etc.

This is all boldly plagerised from Milnor since you can't say it better. Recall that a geodesic is curve $\gamma : (a, b) \to M$ whose tangent vector is parallel i.e.

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0.$$

or in coordinates

$$\ddot{x}^k + \Gamma_{ij} \dot{x}^i \dot{x}^j = 0.$$

Standard existence and uniqueness theory for ODE's tells us that

**Proposition 1.37.** *For all $x \in M$ there is a neighborhood $U$ of $x \in TM$ and $\epsilon > 0$ so that for all tangent vectors $X \in U$ there is a unique path $\gamma\colon (-\epsilon, \epsilon) \to M$ so that*

$$\dot{\gamma}(0) = X.$$

*Furthermore the induced map $F : (-\epsilon, \epsilon) \times U \to M$ is smooth.*

The geodesic equation has the additional property that if $\gamma\colon (a, b) \to M$ is a geodesic then so is $\gamma_c\colon (a/c, b/c) \to M$ given by

$$\gamma_c(t) = \gamma(ct).$$

Using this fact we can sharpen the statement above to read.

**Proposition 1.38.** *For all $x \in M$ there is a neighborhood $U$ of $x \in TM$ so that for all tangent vectors $X \in U$ there is a unique path $\gamma\colon (-2, 2) \to M$ so that*

$$\dot{\gamma}(0) = X.$$

*Furthermore the induced map $F : (-2, 2) \times U \to M$ is smooth.*

*Proof.* Let $U'$ and $\epsilon$ be as provided by Proposition 1.37. Take $c = 2/\epsilon$ and let $U$ be the set of vector $X \in TM$ so that $cX \in U'$. $\square$

**Definition 1.39.** If there is a geodesic $\gamma : [0,1] \to M$ with $\dot\gamma(0) = X \in T_xM$ then we define

$$\exp_x(X) = \gamma(1)$$

Notice that the geodesic $\gamma$ is defined by

$$\gamma(t) = \exp_x(tX)$$

Also notice that $\exp_x(0) = x$ and that the differential of the exponential at $0 \in T_xM$ is the identity.

$$D_{(x,0)} \exp(X) = X.$$

From Proposition 1.38 we see that there is neighborhood $U$ of $M$ in $TM$ so that $\exp : U \to M$ is well defined and is a smooth map whereever it is defined.

Next we wish to see that there is a unique shortest between close enough points. In fact we have the following:

**Lemma 1.40.** *For all $x \in M$ there is a neighborhood $W \subset M$ and an $\epsilon > 0$ so that*

1. *Any two points of $x, y$ of $W$ are joined by a unique geodesic of length $< \epsilon$.*

2. *The geodesic depends smoothly on its end points in the sense that if $\gamma(t) = \exp_x(tX)$ where $y = \exp_x(X)$ is required geodesic then $X$ depends smoothly on $x$ andy.*

3. *For $y \in W$ the map $\exp_y$ maps the $\epsilon$-ball in $T_yM$ onto an open set $U_y$.*

*Proof.* Consider the map $F\colon U \subset TM \to M \times M$ given by $F(X) = (\pi(X), \exp(X))$. The differential of $F$ at $(x,0)$ $D_{(x,0)}F : T_xM \times T_xM \to T_xM \times T_xM$ is

$$D_{(x,0)}F = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} \end{bmatrix}.$$

Thus for any $x \in M$ the implicit function theorem provides a neighborhoods $U$ of $O_x$ in $TM$ and $V$ of $(x, x)$ in $M \times M$ so that $F : U \to M \times M$ is a diffeomorphism onto $V$. We can choose $U$ so that a vector $X$ is in $U$ iff $\pi(X)$ is in a fixed neighborhood of $x$ and $\|X\| < \epsilon$ for some fixed $\epsilon$. Then choose $W$ so that $F(V) \supset W \times W$. □

Now we consider the relation between arc-length and geodesics.

**Theorem 1.41.** *Fixing $x \in M$ let $W$ and $\epsilon$ be as given in Lemma 1.40. For any pair $y, y' \in W$ if and $\gamma : [0, 1] \to M$ is geodesic of length $< \epsilon$ joining them and $\alpha :: [0, 1] \to M$ is any path then*

$$\int_0^1 \|\dot{\gamma}\| dt \leq \int_0^1 \|\dot{\alpha}\| dt.$$

*Equality can only hold if $\alpha([0, 1]) = \gamma([0, 1])$.*

*Proof.* We need two lemmas. Fix $y \in W$ and $U_y$ as in Lemma 1.40

**Lemma 1.42.** *The geodesics through $y$ are the orthogonal trajectories of the hypersurfaces $S_r = \{\exp_y(X) | \|X\| = r\}$*

*Proof.* We must show that the curves $t \mapsto \exp(rX(t))$ where $\|X(t)\| = 1$ and the curve $r \mapsto \exp(rX(0))$ are orthogonal. In other words if we consider the map

$$f(r, t) = \exp_y(rX(t))$$

Then

$$\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial r} \rangle|_{r=\epsilon, t=0} = 0. \tag{10}$$

But

$$\begin{aligned}
\frac{\partial}{\partial r} \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial r} \rangle &= \langle \nabla_{f_*(\frac{\partial}{\partial r})} \frac{\partial f}{\partial t}, \frac{\partial f}{\partial r} \rangle + \langle \frac{\partial f}{\partial t}, \nabla_{f_*(\frac{\partial}{\partial r})} \frac{\partial f}{\partial r} \rangle \\
&= \langle \nabla_{f_*(\frac{\partial}{\partial t})} \frac{\partial f}{\partial r}, \frac{\partial f}{\partial r} \rangle \\
&= \frac{1}{2} \frac{\partial}{\partial t} \langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial r} \rangle \\
&= 0.
\end{aligned}$$

Since $\|X(t)\| = 1$. Clearly $\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial r} \rangle|_{r=0, t=0} = 0$ so Equation **??** follows. □

Let $\alpha : [a,b] \to U_y - \backslash\{y\}$ be a piecewise smooth curve. We can write $\alpha(t) = \exp_y(r(t)U(t))$ for a piecewise smooth functions $r$ and $U$ where $\|U(t)\| = 1$.

**Lemma 1.43.** *Then the length*

$$\int_a^b \|\dot{\alpha}(t)\| dt \geq |r(a) - r(b)|$$

*with equality if and only if $U(t)$ is constant and $r(t)$ is monotone.*

*Proof.* Defining $f(r,t) = \exp_y(r(U))$ as above so that $\alpha(t) = f(r(t),t)$ we have

$$\dot{\alpha} = \frac{\partial f}{\partial r}\dot{r} + \frac{\partial f}{\partial t}$$

and hence

$$\|\dot{\alpha}\|^2 = |\dot{r}|^2 + \|\frac{\partial f}{\partial t}\|^2 \geq |\dot{r}|^2$$

with equality only if

$$\frac{\partial f}{\partial t} = 0$$

thus

$$\int_a^b \|\dot{\alpha}(t)\| dt \geq \int_a^b |\dot{r}| dt \geq |r(a) - r(b)|$$

with equality only if $r(t)$ is monotone and $U(t)$ is constant. $\square$

$\square$

## 1.18 Geodesics, completeness and the Hopf-Rinow theorem

A Riemannian manifold has a natural metric in the point set topology sense. Define

$$d(x,y) = \inf \ell(\gamma)$$

as $\gamma$ varies over broken $C^1$ paths joining $x$ to $y$. It is easy to check that this defines a metric. Lemma 1.40 implies easily the following:

**Proposition 1.44.** *The topology of $d$ is the same as the manifold topology.*

Also notice that Lemma 1.40 implies the following result.

**Proposition 1.45.** *If $\gamma : [a,b] \to M$ is a piecewise $C^1$ path that minimizes the length between its endpoints then up to reparameterization $\gamma$ is a geodesic.*

**Proposition 1.46.**

**Definition 1.47.** $(X, g)$ is geodesically complete if any geodesic $\gamma : [a,b] \to M$ can be extended to a geodesic $\tilde{\gamma} : \mathbb{R} \to M$.

## 1.19 Volume comparison theorems

## 1.20 Jacobi fields on the model spaces

Let $S^n(a)$ denote the sphere of constant sectional curvature $a$ i.e. the sphere of radius $1/a$. Suppose that $u$,$v$,$w$ are mutually orthogonal unit vectors then

$$\gamma(t) = \frac{1}{a}\cos(at)u + \frac{1}{a}\sin(at)v$$

is a unit speed geodesic and

$$F(t,s) = \frac{1}{a}\cos(at)u + \frac{1}{a}\sin(at)(\cos(s)v + \sin(s)w)$$

so that

$$\frac{\partial F}{\partial s}|_{s=0} = \frac{1}{a}\sin(at)w$$

is a Jacobi field along $\gamma$.

Similarly let Let $H^n(a)$ denote the sphere of constant sectional curvature $-a$ i.e. the sphere of radius $-x_0^2 + x_1^2 + x_2^2 + \dots x_n^2 = -\frac{1}{a^2}$. Suppose that $u$,$v$,$w$ are mutually orthogonal unit vectors then

$$\gamma(t) = \frac{1}{a}\cosh(at)u + \frac{1}{a}\sinh(at)v$$

is a unit speed geodesic and

$$F(t,s) = \frac{1}{a}\cosh(at)u + \frac{1}{a}\sinh(at)(\cos(s)v + \sin(s)w)$$

so that

$$\frac{\partial F}{\partial s}|_{s=0} = \frac{1}{a}\sinh(at)w$$

is a Jacobi field along $\gamma$.

## 1.21 Jacobi equation

With our conventions on the curvature a Jacobi field along the geodesic $\gamma$ is

$$-\ddot{Y}(t) + R(Y, \dot{\gamma})\dot{\gamma} = 0.$$

Notice that if $Y(t)$ is Jacobi field along $\gamma(t)$ then $Y(ct)$ is Jacobi field along $\gamma(ct)$.

## 1.22 The differential of the exponential map and the pullback of the volume form

Recall that:

$$D_v \exp_m(w) = Y(1)$$

where $Y(t)$ is Jacobi field along $\gamma$ where $\dot{\gamma}(0) = v$, $Y(0) = 0$ and $\dot{Y}(0) = w$. Notice that

$$D_{tv} \exp_m(w) = \frac{1}{t} Y(t)$$

The volume form for an oriented Riemannian manifold is

$$vol_g = e^1 \wedge e^2 \wedge \ldots \wedge e^n = \sqrt{\det(g_{ij})} dx^1 \wedge \ldots dx^n.$$

Thus fixing an orthonormal frame $e_2, \ldots, e_n$ for $\dot{\gamma}(0)^\perp$ and letting $Y_i(t)$ denote the Jacobi field with $Y_i(0) = 0$ and $\dot{Y}_i(0) = e_i$. Thus

$$\exp^*(vol_g)|_{tv} = \sqrt{(}\det(\langle ngle \frac{1}{t} Y_i(t), \frac{1}{t} Y_j(t) \rangle)) = \frac{1}{t^{n-1}} \sqrt{(\det(\langle ngle Y_i(t), Y_j(t) \rangle))} dx^1 \wedge \ldots dx^n. = sqr$$

Also let $e_i(t)$ be the parallel extention of $e_i$ and let $Y_i^r(t)$ the Jacobi field with

$$Y_i^r(0) = 0 \text{and} Y_i^r(r) = e_i(r).$$

## 1.23 Characteristic Classes

Recall that for a $U(1)$-bundle $\pi : P \to X$ a connection $A$ was an imaginary one form on $P$.

$$\pi^*(F_A) = dA$$

and is section of ad$P$ which is a trivial bundle in this case. Any two connections differ by an imaginary one form so

$$A' = A + \pi^* a$$

and

$$\pi^*(F_{A'}) = dA' = \pi^*(F_A) + d\pi^* a$$

Furthermore we see that

$$dF_A = 0.$$

Thus the cohomology class of $F_A$ is an invariant of the bundle and does not depend on the connection! Definition

$$c_1(P) = \frac{i}{2\pi} F_A$$

How to generalize this to arbitrary bundles. We need each of these step to generalize. The first problem is that ad$P$ is generally not trivial so we need a consider maps

$$\varphi : \mathfrak{g} \to \mathbb{R}$$

which are invariant under the adjoint action. For example if $G = U(n)$ then the symmetric function of the eigenvalues $\sigma_k(\xi)$ are invariant functions.

Formula 2 is still valid any two connections differ by the pull back of a one form with values in ad$P$.

$$A' = A + \pi^* a$$

The last equation does quite hold but what is true is

**Lemma 1.48.** *(The Bianchi Identity). The*

$$d_A F_A = 0$$

*or equivalently*

$$d\pi^*(F_A) - [A, \pi^*(F_A)] = 0$$

Proof. Lets be concrete about this and trivialize the bundle locally. Then

$$s^*(\pi^*(F_A)) = da + a \wedge a$$

where $s^*(A) = \theta_{\mathrm{MC}} + a$ so

$$\begin{aligned} dF_A &= da \wedge -a \wedge da \\ &= (F_A - a \wedge a) \wedge a - a \wedge (F_A - a \wedge a) \\ &= (F_A \wedge a - a \wedge F_A) \end{aligned}$$

# 2 Characteristic classes

## 2.1 The Chern Classes

Let $P \to B$ be a $U(n)$ bundle. Then define

$$c(P) = 1 + c_1(P) + c_2(P) + \ldots + = [\det(1 + \frac{i}{2\pi} F_A)]$$

So for example if $P$ is a $U(2)$ bundle and in a local trivalization over some open set $U$ we have

$$s^*(F_A) = \begin{bmatrix} i\alpha & \beta + i\gamma \\ -\beta + i\gamma & i\delta \end{bmatrix}$$

then

$$c_1(A) = -\frac{1}{2\pi}(\alpha + \delta)$$

and

$$c_2(A) = -\frac{1}{4\pi^2}(-\alpha \wedge \delta + \beta \wedge \beta + \gamma \wedge \gamma).$$

We can also define the Chern-classes of a vector bundle $E \to B$. Chooose a Hermitian metric $h$ in $E$ and let $P = P(E, h)$ be bundle of unitary frames of $E$ with respect to $h$.

*Exercise* 1. Show that if $h$ and $h'$ are two different hermitian metrics in $E$ then $P(E, h)$ and $P(E, h')$ are isomorphic.

Given this exercise we can define

$$c(E) = c(P).$$

Let $\langle mbda \to \mathbb{CP}^n$ be the tautological line bundle. The frame bundle of $\langle mbda$ for the natural Hermitian metric in $\langle mbda$ is

$$S^{2n+1} \subset \mathbb{C}^{n+1} \to \mathbb{CP}^n.$$

$U(1)$ acts on $S^{2n+1}$ by scalar mulptiplication. Writing

$$\mathbf{z} = \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_n \end{bmatrix}$$

the hermitian inner product is

$$h(\mathbf{z}, \mathbf{w}) = \mathbf{z}^t \bar{\mathbf{w}}$$

and the real inner product is the real part of this.

$$\langle ngle \mathbf{z}, \mathbf{w} \rangle = \Re(\mathbf{z}^t \bar{\mathbf{w}})$$

The infinitesimal generator of the action is

$$\hat{\xi} = i\mathbf{z}$$

The natural connection is the map

$$A_{\mathbf{z}} T_{\mathbf{z}} S^{2n+1} \to i\mathbb{R}$$

given by projection onto the generator of the action which is explicity by

$$A_{\mathbf{z}}(\mathbf{w}) = i\langle nglei\mathbf{z}, \mathbf{w} \rangle = -\Im(\mathbf{z}^t \bar{\mathbf{w}}).$$

Introduction the complex valued one forms

$$dz^\mu = dx^\mu + idy^\mu \quad \text{and} \quad d\bar{z}^\mu = dx^\mu - idy^\mu$$

we can write this more tidily as

$$A = -\Im(\mathbf{z}^t d\bar{\mathbf{z}})$$

Thus the curvature of $A$ is

$$F_A = dA = -\Im(d\mathbf{z}^t \wedge d\bar{\mathbf{z}}).$$

Here the curvature is written as a two-form on the total space of the bundle. To sort out which cohomology class $c_1(\langle mbda)$ repersents it suffices to evaluate it on $\mathbb{CP}^1 \subset \mathbb{CP}^n$ or more explicitly let $\phi : \mathbb{C} \to \mathbb{CP}^n$ be given by

$$\phi(z) = [z : 1 : 0 : 0 : \ldots 0].$$

$\phi$ lifts to cover a section of the bundle $\tilde{\phi} : \mathbb{C} \to S^{2n+1}$ given by

$$\tilde{\phi}(z) = (\frac{z}{\sqrt{1 + |z|^2}}, \frac{1}{\sqrt{1 + |z|^2)}}, 0, \ldots, 0).$$

Then we must compute

$$\frac{i}{2\pi}\int_{\mathbb{C}} \tilde{\phi}^*(F_A)$$

$$\begin{aligned}
\tilde{\phi}^*(F_A) &= -\Im(d\frac{z}{\sqrt{1+|z|^2}} \wedge d\frac{\bar{z}}{\sqrt{1+|z|^2}} + d\frac{1}{\sqrt{1+|z|^2}} \wedge d\frac{1}{\sqrt{1+|z|^2}} \\
&= -\Im(\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} \\
&= 2i\frac{dx \wedge dy}{(1+x^2+y^2)^2}
\end{aligned}$$

But from 18.01 we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \frac{dx \wedge dy}{(1+x^2+y^2)^2} &= 2\pi \int_0^\infty \frac{r\mathrm{d}r}{(1+r^2)^2} \\
&= \pi
\end{aligned}$$

And so all together we have

$$\langle nglec_1(\langle mbda), [\mathbb{CP}^1]\rangle = \frac{i}{2\pi}(2i)\pi = -1.$$

Whitney sum formula. If $E = E_1 \oplus E_2$ then we have

**Proposition 2.1.** $c(E) = c(E_1)c(E_2)$

If $E \to B$ is complex vector bundle then we can form the conjugate bundle $\bar{E}$. As real bundles $E$ and $\bar{E}$ are isomorphic but $z \in \mathbb{C}$ acts on $v \in \bar{E}$ by

$$z \bullet v = \bar{z}v.$$

## 2.2 The Pontryagin classes

Let $V$ be a real vector bundle. The Pontryagin classes are defined to be the Chern classes of the complexification with a sign twist that is the more common convention in the literature.

$$p_i(V) = (-1)^i c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C}) \in H^{2i}(B; \mathbb{R})$$

However to keep life from getting too complicated we will define the total Pontryagin class to be

$$p(V) = 1 - p_1(V) + p_2(V) - p_3(V) + \dots.$$

so that we can write

$$p(V) = c(V \otimes_{\mathbb{R}} \mathbb{C})$$

Properties. If $V = V_1 \oplus V_2$ then

$$p(V) =$$

For example if the curvature of a connection in a rank four bundle is

$$F_A = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}$$

Then the local expression for the total Pontryagin class is given by

$$1 - \frac{1}{4\pi^2}(a^2+b^2+c^2+d^2+e^2+f^2) + \frac{1}{16\pi^2}(a^2f^2+c^2d^2+b^2e^2-2abef+2acdf-2bcde).$$

## 2.3 The Euler class

The Pfaffian. There is an extra ad-invariant polynomial in the Lie algebra of $SO(2k)$. Notice that

$$\det \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} = a^2$$

and that

$$\det \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = (a^2f^2+c^2d^2+b^2e^2-2abef+2acdf-2bcde) = (af-be+cd)^2$$

In general the determinant of a real skew symmetric matrix is a perfect square. There is a polynomial in the entries of matrix called the *Pfaffian* which gives rise to

$$Pf : \mathfrak{so}(2k) \to \mathbb{R}$$

Here are three way to define the Pfaffian.

1. Any $2k \times 2k$ skew symmetric real matrix is conjugate by a matrix in $SO(2k)$ to a matrix of the form

$$\Lambda = \begin{bmatrix} 0 & -\lambda_1 & 0 & \dots & & 0 \\ \lambda_1 & 0 & 0 & \dots & & 0 \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & 0 & -\lambda_k \\ 0 & & & & \lambda_k & 0 \end{bmatrix}$$

(A Cartan subalgebra of the Lie algebra $\mathfrak{so}(2k)$.) Then we define

$$\text{Pfaff}(\Lambda) = \lambda_1 \lambda_2 \dots \lambda_k$$

2.

3. Associate to $M \in \mathfrak{so}(2k)$ the 2-form

$$\hat{M} = \sum_{i<j} M_{ij} e^i \wedge e^j.$$

Then $\hat{M}^k \in \lambda^{2k}(\mathbb{R}^{2k})$ and we define

$$\frac{1}{k!} \hat{M}^k = \text{Pfaff}(M) e^1 \wedge e^2 \dots e^{2k}$$

So for example $\hat{\Lambda} = \lambda_1 e^1 \wedge e^2 + \dots \lambda_k e^{2k-1} \wedge e^{2k}$.

# 3 Applications of Characteristic classes

You will have shown in your homework that

$$c(\mathbb{CP}) = (1+x)^{n+1}$$

where $x = -c_1(\langle mbda^*) \in H^2(\mathbb{CP}^n$.