THE 3-LOCAL *tmf* HOMOLOGY OF $B\Sigma_3$

MICHAEL A. HILL

Abstract. In this paper, we introduce a Hopf algebra, developed by the author and André Henriques, which is usable in the computation of the *tmf* homology of a space. As an application, we compute the *tmf* homology of $B\Sigma_3$ in a manner analogous to Mahowald's computation of the *ko* homology $\mathbb{R}P^\infty$ in [8].

1. Introduction

In this paper we compute the 3-local *tmf* homology and *tmf* Tate cohomology of the symmetric group Σ_3. This computation is motivated as follows. Mahowald's computation of $ko_*(\mathbb{R}P^\infty)$ has proved useful in a variety of contexts. In particular, Mahowald used $ko_*(\mathbb{R}P^n)$ and $ko_*(\mathbb{R}P^\infty/\mathbb{R}P^k)$ to get information about v_1 metastable homotopy theory in the EHP sequence [10]. Mahowald has also used $ko_*(\mathbb{R}P^\infty)$ to detect elements in his η_j family [9]. At the prime 3, the role of the spectrum *ko* is most naturally played by the spectrum *tmf*. To generalize these results of Mahowald's, the initial piece of data needed is the *tmf* homology of $B\Sigma_3$. Both of the aforementioned results should be generalizable starting from this point.

A theorem of Arone and Mahowald shows that v_n periodic information is captured by the first p^n stages of the Goodwillie tower [1]. This recasts Mahowald's result from [10] into a more readily generalizable form. To get v_2 periodic information at the prime 3, the initial data needed comes in part from QS^0 and $Q(B\Sigma_3{}_k^\infty)$, where $B\Sigma_3{}_k^\infty$ is a particular Thom spectrum of $B\Sigma_3$. Just as Mahowald uses knowledge of the *ko* homology of stunted projective spaces to reduce the questions involved to ones of J homology, we hope that a similar analysis, using Behrens' $Q(2)$, spectrum will allow an analysis of the v_2 primary Goodwillie tower at 3 [3].

Minami shows that the odd primary η_j family will be detectable in the Hurewicz image of the *tmf* homology of the n-skeleton of $B\Sigma_3$ for appropriate choices of n [12]. While determining the full Hurewicz image is a trickier task, understanding the groups and simple *tmf* operations on them could help determine if the conjectural η_j elements actually survive at the prime 3.

1.1. **Organization of Paper.** In §2, we introduce the main computational Hopf algebra $\mathcal{A}$, Ext over which is the Adams E_2 term for computing *tmf* homology. In §3, we review Mahowald's computation of the *ko* homology of $\mathbb{R}P^\infty$, presenting it in a manner which can be most readily generalized. In §4, we carry out one of the computational steps analogous to Mahowald's, computing the *tmf* homology of the cofiber of the transfer map, and in §5, we complete the computation of $tmf_*(B\Sigma_3)$. Rounding out the computations, in §6, we compute the *tmf* homology of the finite skeleta of R_3, giving additional results about that of the finite skeleta of $B\Sigma_3$.

The last two sections present conjectures as to further results. A computation of the homotopy of the Σ_3 Tate spectrum for *tmf* is presented together with a non-splitting conjecture in §7. Conjectural generalizations to primes bigger than 3 are presented in §8, together with the implications to the eo_{p-1} homology of $B\Sigma_p$.

1.2. **Conventions and Notation.** We restrict attention to the prime 3 and assume that all spaces and spectra are 3-completed except in §3. For ease of readability, let H be $H\mathbb{Z}/3$. If X is a space or spectrum, let $X^{[n]}$ denote its n-skeleton.

Finally, we need some *tmf* specific notation. Let I denote the ideal of the Adams E_2 term for tmf_* generated by v_0, c_4 and c_6. Let $\bar{I}$ denote the ideal of tmf_* generated by 3, c_4, c_6, and their Δ and Δ^2 translates. The ideal I converges to the ideal $\bar{I}$, and I is the annihilator ideal of the elements α and β. For brevity, the reader is asked to always assume the relation $I(\alpha,\beta)$ in all Adams E_2 terms, unless explicitly stated otherwise. Moreover, we assume that the relations $c_4^3 - c_6^2 = 27\Delta$ always holds and will not be explicitly stated.

2. Fundamental Hopf Algebra

Our basic tool of computation will be a variant of the Adams spectral sequences based on ordinary cohomology. Since H is a module over *tmf*, we can build a cosimplicial resolution of $tmf \wedge B\Sigma_3$ by H-modules in the category of *tmf*-module spectra. This greatly simplifies our computations, as the role of the dual Steenrod algebra is played by the Hopf algebra

$$\mathcal{A} := \pi_*(H \wedge_{tmf} H).$$

Theorem 2.1 (Henriques-Hill). *As a Hopf algebra,*

$$\mathcal{A} = \mathcal{A}(1)_* \otimes E(a_2),$$

where $|a_2| = 9$, and $\mathcal{A}(1)_* = \mathbb{F}_3[\xi_1]/\xi_1^3 \otimes E(\tau_0,\tau_1)$ is dual to the subalgebra of the Steenrod algebra generated by β and $\mathcal{P}^1$. The elements in $\mathcal{A}(1)_*$ have their usual coproducts, and

$$\Delta(a_2) = 1 \otimes a_2 + \xi_1 \otimes \tau_1 - \xi_1^2 \otimes \tau_0 + a_2 \otimes 1.$$

Proof. That this is a Hopf algebra follows from a slight recasting of Adams' original analysis of the Adams spectral sequence, using the fact that $\mathcal{A}$ is flat over H_* [2]. We begin with an observation of Hopkins and Mahowald, as formulated by Behrens [5]. If we let

$$C = S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8,$$

then smashing with *tmf* gives

$$tmf \wedge C = tmf_0(2) = BP\langle 2\rangle \vee \Sigma^8 BP\langle 2\rangle.$$

The middle equality demonstrates that this is actually an E_∞-ring spectrum. If we smash $tmf \wedge C$ with $V(1)$, then we again get a ring spectrum, since the obstruction to $V(1)$ being a ring spectrum lies in positive Adams-Novikov filtration [5], and the homotopy of $tmf \wedge C$ is concentrated in filtration zero. The Atiyah-Hirzebruch spectral sequence allows us to compute the ring structure on homotopy, and we see that the natural map

$$S^8 \to tmf \wedge C \wedge V(1) = k(2) \vee \Sigma^8 k(2)$$

behaves like a square root of v_2 [4]. In other words,

$$\pi_*\big(tmf \wedge C \wedge V(1)\big) = \mathbb{F}_3[\sqrt{v_2}].$$

The cofiber of $\sqrt{v_2}$ is H, and we have therefore realized H as a quotient of an extended *tmf* module by itself via a *tmf* module map.

To finish the proof, we smash this cofiber sequence with H over *tmf*, giving the cofiber sequence

$$\Sigma^8 H \wedge_{tmf} \big(tmf \wedge C \wedge V(1)\big) \xrightarrow{\sqrt{v_2}} H \wedge_{tmf} \big(tmf \wedge C \wedge V(1)\big) \to H \wedge_{tmf} H.$$

We begin by analyzing the homotopy of the first two *tmf* modules in this resolution:

$$\pi_*\Big(H \wedge_{tmf} \big(tmf \wedge C \wedge V(1)\big)\Big) = H_*\big(C \wedge V(1); \mathbb{Z}/3\big).$$

The structure of this as a graded vector space is that of $\mathcal{A}(1)_*$. Since $\mathcal{A}$ is a commutative Hopf algebra, the classification of Hopf algebras over a finite field ensures both that $\sqrt{v_2}$ is zero in homotopy and that the structure of this as an algebra is as listed [11]. This is immediate from considering the degrees of the elements, since odd elements must be exterior classes and the element in degree 4 must be the generator of a truncated polynomial algebra.

Since the structure map from *tmf* to H is a map of E_∞ ring spectra, $\mathcal{A}$ is a module over H_*H. Moreover, this map is also a map of coalgebras over H_*. Since the unit map $S^0 \to tmf$ is a 6-equivalence, the natural map

$$H \wedge H \to H \wedge_{tmf} H$$

is a 6-equivalence. This implies that the induced map in homotopy is a Hopf algebra isomorphism in the same range, and this gives the coproducts on the elements τ_0, τ_1 and ξ.

To determine the coproduct on a_2, we endow $\mathcal{A}$ with a filtration such that a_2 is primitive in the associated graded. This filtration gives rise to a spectral sequence

$$\mathrm{Ext}_{Gr(\mathcal{A})}(\mathbb{F}_3, \mathbb{F}_3) \Rightarrow \mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_3, \mathbb{F}_3)$$

converging to the E_2 term of the Adams spectral sequence which computes $\pi_*(tmf)$. We shall use the known computation of $\pi_*(tmf)$ to deduce differentials in this algebraic spectral sequence, and this will determine the coproduct on a_2.

We first filter $\mathcal{A}$ by letting $\mathcal{A}(1)_*$ have filtration 0 and putting a_2 in filtration 1. The initial piece of data needed is the cohomology of $\mathcal{A}(1)_*$. An elementary computation shows that as an algebra

$$\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3, \mathbb{F}_3) = \mathbb{F}_3[v_0, v_1^3, \beta] \otimes E(\alpha_1, \alpha_2)/\big(v_0(\alpha_1, \alpha_2),\ \alpha_1\alpha_2 = v_0\beta\big).$$

This is pictorally represented in Figure 1.

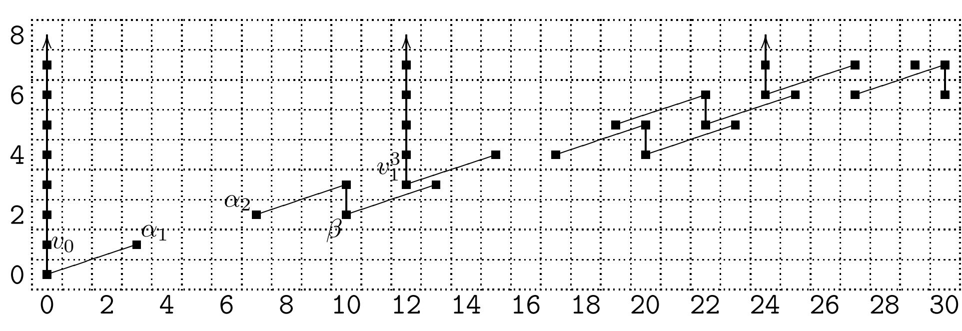

FIGURE 1. $\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3, \mathbb{F}_3)$

Since a_2 is primitive in the associated graded Hopf algebra, we know that

$$\mathrm{Ext}_{Gr(\mathcal{A})}(\mathbb{F}_3,\mathbb{F}_3)=\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3,\mathbb{F}_3)[\tilde{c}_4].$$

This Ext group is the E_1 page of a spectral sequence converging to the Adams E_2 term $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_3,\mathbb{F}_3)$. Since there is nothing in dimension 7 in tmf_*, we know that the element α_2 must be killed. The only possible way for to achieve this is for $d_1(\tilde{c}_4)=\alpha_2$. This E_1 page is given together with this necessary d_1 differential in Figure 2.

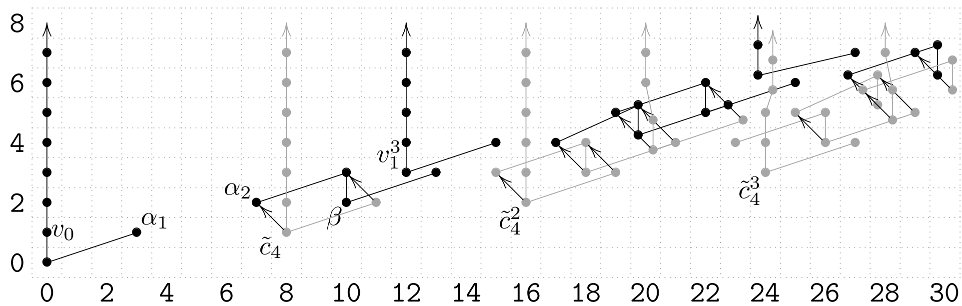

FIGURE 2. $\mathrm{Ext}_{Gr(\mathcal{A})}(\mathbb{F}_3,\mathbb{F}_3)$

At this point, we rename some of the remaining elements:

$$c_4=v_0\tilde{c}_4,\quad c_6=v_1^3,\quad \Delta=\tilde{c}_4^3.$$

For completeness, we note that a similar analysis gives the d_2 differentials:

$$d_2([\alpha_2\tilde{c}_4^2])=v_1^3\beta,\text{ and } d_2([v_0\tilde{c}_4^2])=v_1^3\alpha.$$

The E_2 page with the d_2 differentials is included as Figure 3.

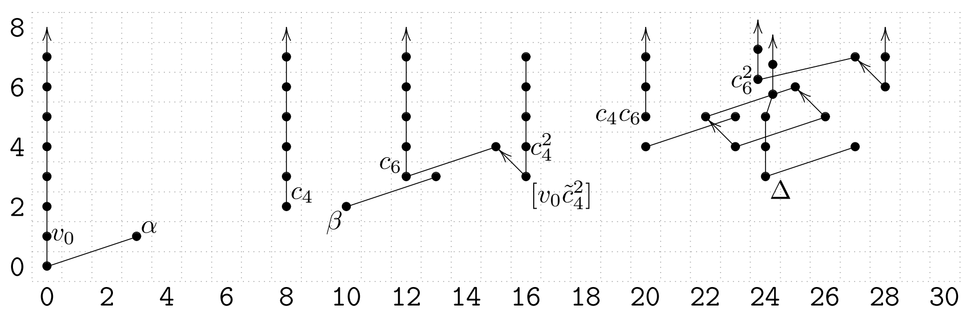

FIGURE 3. May E_2 page for $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_3,\mathbb{F}_3)$

For the d_1 to have the appropriate form, we must have

$$\psi(a_2)=1\otimes a_2+a_2\otimes 1\pm(\xi_1\otimes\tau_1-\xi_1^2\otimes\tau_0).$$

If the sign is negative, then we can simply replace a_2 by $-a_2$ to correct this. □

One can ask if there is a formal group intepretation to the Hopf algebra given in Theorem 2.1, similar to the interpretation of the Steenrod algebra as the automorphisms of the super additive formal group. This seems to be the case. If E is an elliptic spectrum, then the homotopy groups of $E \wedge_{tmf} E$ are the automorphisms of the formal group of E that extend to automorphisms of the associated elliptic curve. For the case $E = H$, we can proceed only by analogy, since the additive elliptic curve is not in the moduli stack used in the construction of TMF. However, if we consider the automorphisms of the additive formal group which extend to automorphisms of the additive elliptic curve, then we reconstruct the truncated polynomial part of Theorem 2.1. We conjecture that a full results can be recovered by considering super formal groups and super elliptic curves.

3. Review of $ko_*(\mathbb{R}P^\infty)$

In [8], Mahowald uses the homology of cofiber R_2 of the transfer map $B\Sigma_2 \to S^0$ to compute its *ko* homology and the *ko* homology of $\mathbb{R}P^\infty$. Since the method we will employ to handle $tmf_*(B\Sigma_3)$ is similar, we quickly review Mahowald's technique here. For this section only, all computations will be done at the prime 2.

3.1. **General Results and Definitions.** The homology of R_2 sits as an extension of the homology of $\Sigma\mathbb{R}P^\infty$ by the homology of S^0, and let e_i denote the generator of $H_i(R_2)$. The coaction of the dual Steenrod algebra on $H_*(R_2)$ is determined by the comodule structure on $H_*(\Sigma\mathbb{R}P^\infty)$ and the coaction formula

$$\psi(e_2) = \xi_1^2 \otimes e_0 + 1 \otimes e_2.$$

Let $A(1)$ be the spectrum whose cohomology is a free $\mathcal{A}(1)$-module of rank 1. Smashing $A(1)$ with *ko* gives a presentation of $H\mathbb{Z}$ as a *ko*-module spectrum. An analysis like that of the first section reestablishes the following classical result, normally proved using a change of rings argument.

Proposition 3.1. *There is a spectral sequence converging to the ko homology of a space X with E_2 term* $\mathrm{Ext}_{\mathcal{A}(1)_*}\big(\mathbb{F}_2, H_*(X)\big)$.

3.2. **The *ko* homology of R_2.** Mahowald's key observation was that there is a filtration of $H_*(R_2)$ such that the associated graded is a sum of comodules over $\mathcal{A}(1)_*$ whose Ext groups are easy to compute.

Proposition 3.2. *There is a filtration of $H_*(R_2)$ such that the associated graded is*

$$Gr = Gr\big(H_*(R_2)\big) = \bigoplus_{k=0}^{\infty} \Sigma^{4k} h\mathbb{Z},$$

where $h\mathbb{Z}$ is the $\mathcal{A}(1)_*$ comodule dual to $\mathcal{A}(1)//\mathcal{A}(0)$.

The proposition shows that if we compute Ext of Gr, then we see that it is torsion free, with a $\mathbb{Z}$ in dimensions congruent to $0 \mod 4$ (Figure 4).

Since this is concentrated in even degrees, both the algebraic extension spectral sequence and the Adams spectral sequence collapse. There are non-trivial extensions, though, as a ko_*-module.

Lemma 3.3. *As a module over ko_*,*

$$ko_*(R_2) = \mathbb{Z}_2\left[\frac{v_1^2}{4}\right].$$

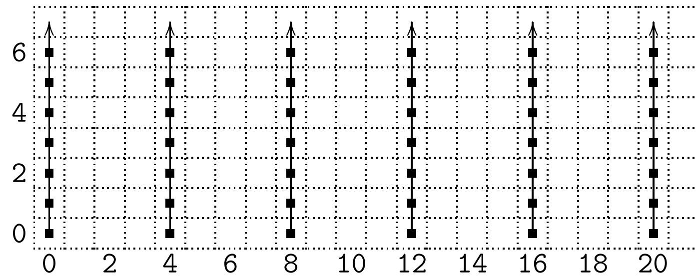

FIGURE 4. $\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_2, Gr)$

Proof. An elementary cobar computation for $\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_2, \mathbb{F}_2)$ shows that the generator of the $\mathbb{Z}$ in dimension 4 in ko_* is represented by

$$[2v_1^2] = \xi_1 \otimes \xi_2 \otimes \xi_2 + \dots.$$

If we look in the cobar complex computing $\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_2, h\mathbb{Z})$, then we see that there is a class x_7 such that

$$x_7 = \xi_1 \otimes \xi_1 \otimes e_5 + \dots, \quad \text{and} \quad d(x_7) = [2v_1^2],$$

where e_5 denotes the 5 dimensional class in $h\mathbb{Z}$. In $H_*(R_2)$, the coproduct on e_5 is

$$\psi(e_5) = \left(\xi_1^2 \otimes e_3 + \xi_2 \otimes e_2 + \xi_1^2\xi_2 \otimes e_0\right) + \xi_1 \otimes e_4 + 1 \otimes e_5.$$

This shows that $\xi_1 \otimes \xi_1 \otimes \xi_1 \otimes e_4$ is cohomologous to $[2v_1^2] \otimes e_0$. The homogeneity of the homology of R_2 then implies the result. □

Remark. *This lemma shows that Mahowald and Davis' result in* [6] *that $ko \wedge R_2$ splits as a wedge of copies of $H\mathbb{Z}$ is not true in the category of ko-module spectra.*

3.3. **Computing $ko_*(\mathbb{R}P^\infty)$.** Finishing the argument requires looking at the long exact sequence in ko homology for the cofiber sequence

$$S^0 \to R_2 \to \Sigma\mathbb{R}P^\infty.$$

The first map is the inclusion of the zero cell, and takes 1 to 1. From this, the result is easily determined (Figure 5).

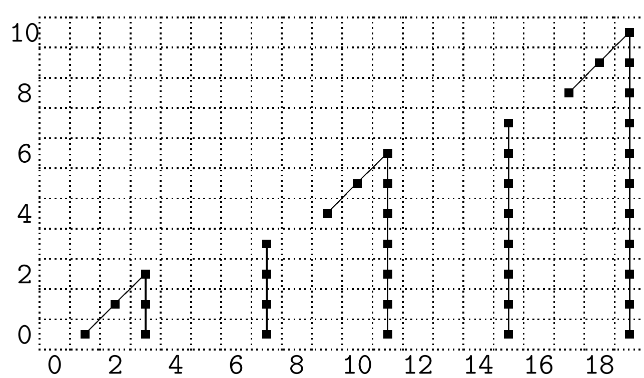

FIGURE 5. $ko_*(\mathbb{R}P^\infty)$

4. The *tmf* Homology of the Cofiber of the Transfer $B\Sigma_3 \to S^0$

Homologically, the situation at the prime 3 is analogous to the computation at 2. Let R_3 denote the cofiber of the transfer map $B\Sigma_3 \to S^0$. The homology of R_3 sits as an extension of the homology of $\Sigma B\Sigma_3$ by the homology of S^0, and again let e_i denote the generator of $H_i(R_3)$. The coaction of the dual Steenrod algebra on $H_*(R_3)$ is determined by the comodule structure on $H_*(\Sigma B\Sigma_3)$ and the coaction formula

$$\psi(e_4) = -\xi_1 \otimes e_0 + 1 \otimes e_4.$$

The *tmf* analogue $h\mathbb{Z}$ is again the comodule dual to the quotient module of $\mathcal{A}(1)$ by $\mathcal{A}(0)$, and the coproduct is the one induced by this structure.

Lemma 4.1. *$H_*(R_3)$ admits a filtration for which the associated graded is*

$$Gr\big(H_*(R_3)\big) = \bigoplus_{k=0}^{\infty} \Sigma^{12k} h\mathbb{Z}\,.$$

Proof. In fact, this lemma is quite easy to show. The $-k^{\text{th}}$ stage of the filtration is given by taking the subcomodule generated by the classes in dimensions $12n+1$ for all $n > k$. An elementary computation in the cohomology of the symmetric group shows that the associated graded is exactly what is claimed. □

Lemma 4.2.

$$\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_3, h\mathbb{Z}) = \mathbb{F}_3\left[v_0, \frac{c_4}{3}\right].$$

Proof. To prove this lemma we apply a long sequence of spectral sequences. First filter $\mathcal{A}$ as before by letting $\mathcal{A}(1)_*$ have filtration 0 and a_2 have filtration 1. This filtration extends to a filtration of $h\mathbb{Z}$ in an obvious way, and we have a spectral sequence

$$\mathrm{Ext}_{Gr(\mathcal{A})}(\mathbb{F}_3, h\mathbb{Z}) \Rightarrow \mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_3, h\mathbb{Z}).$$

As a Hopf algebra, $Gr(\mathcal{A})$ is very simple: the algebra structure stays the same, and now a_2 is primitive. Now we can use the two short exact sequences of Hopf algebras

$$\mathcal{A}(1)_* \to \mathcal{A} \to E(a_2) \quad \text{and} \quad E(a_2) \to \mathcal{A} \to \mathcal{A}(1)_*$$

to get a spectral sequence that converges to this Ext group and starts with

$$\mathrm{Ext}_{E(a_2)}\left(\mathbb{F}_3, \mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3, h\mathbb{Z})\right).$$

A final change of rings argument shows that

$$\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3, h\mathbb{Z}) = \mathbb{F}_3[v_0],$$

and this forces the result in question, since the target of any differential on the polynomial generator is zero for degree reasons. Again, from the previous computation of the E_2 term for the Adams spectral sequence for tmf_* in the category of *tmf*-modules, we see that the polynomial generator coming from a_2 is $\frac{c_4}{3}$. □

Since this algebra is concentrated in even degrees and since each of the graded pieces starts an even number of steps apart, the spectral sequence starting with Ext of the associated graded for $H_*(R_3)$ collapses. We are left with the following terms (Figure 6).

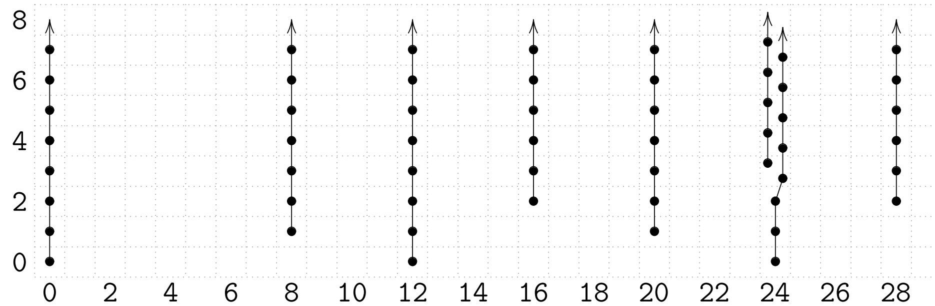

FIGURE 6. $\mathrm{Ext}_{\mathcal{A}}\left(\mathbb{F}_3, H_*(R_3)\right)$

Lemma 4.3.

$$\mathrm{Ext}_{\mathcal{A}}\left(\mathbb{F}_3, H_*(R_3)\right) = \bigoplus_{k=0}^{\infty} \Sigma^{12k}\mathbb{F}_3\left[v_0, \frac{c_4}{3}\right].$$

While there are no possible differentials in the Adams spectral sequence, there are non-trivial extensions in this, viewed as a module over tmf_*.

Theorem 4.4. *The Adams spectral sequence for the tmf homology of R_3 collapses, and as a tmf_*-module,*

$$tmf_*(R_3) = \mathbb{Z}_3\left[\frac{c_4}{3}, \frac{c_6}{27}\right].$$

Proof. We show this by returning to the cobar complex. Since the homology of R_3 has the very simple pattern of copies of $h\mathbb{Z}$ connected by a τ_0 comultiplication on the top class in each hitting the bottom class in the next, it will suffice to show that in the first copy, c_6 on the 0 cell is cohomologous to 27 on the 12 cell.

For simplicity, we will let i_n denote the class in dimension n in $h\mathbb{Z}$. The cobar complex for $\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3, h\mathbb{Z})$ shows that there is an element x_{16} such that

$$x_{16} = \tau_0 \otimes \tau_0 \otimes i_{13} + \dots \text{ and } d(x_{16}) = c_6 \otimes i_0.$$

This bounding cycle can be readily found by considering the Ext implications of the short exact sequence of comodules:

$$\mathbb{F}_3\{i_0, i_4, i_8\} \to h\mathbb{Z} \to \mathbb{F}_3\{i_5, i_9, i_{13}\}.$$

When we add in the next copy of $h\mathbb{Z}$, we change the coproduct on i_{13} to

$$\psi(i_{13}) = \left(\xi_1 \otimes i_9 + \xi_1^2 \otimes i_5 + \tau_1 \otimes i_8 + \xi_1\tau_1 \otimes i_4 + \xi_1^2\tau_1 \otimes i_0 + 1 \otimes i_{13}\right) + \tau_0 \otimes i_{12}.$$

This is the only change to the coproducts in our comodule, so when we consider again x_{16} and take its boundary, the only change is the addition of terms coming from this new term in the coproduct. However, the only instance of i_{13} in x_{16} is the one coming from $\tau_0 \otimes \tau_0 \otimes i_{13}$, so the real boundary is

$$d(x_{16}) = c_6 \otimes i_0 + \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes i_{12}.$$

In other words, c_6 on the base class is (up to a sign) 27 times the class in dimension 12. □

5. The *tmf* Homology of $B\Sigma_3$

The most difficult of the computations now behind us, we can compute the *tmf* homology of $B\Sigma_3$ by simply considering the long exact sequence induced by applying tmf_* to the cofiber sequence

$$S^0 \to R_3 \to \Sigma B\Sigma_3.$$

The first map is the inclusion of the zero cell into R_3, and so this map in *tmf*-homology just takes 1 to 1. Since this is a map of tmf_*-modules, we see immediately that this map is injective on elements of Adams-Novikov filtration 0, with image

$$\mathbb{Z}_3\big[c_4, c_6, [3\Delta], [3\Delta^2], [c_4\Delta], [c_4\Delta^2], [c_6\Delta], [c_6\Delta^2], \Delta^3\big]/(27\Delta = c_4^3 - c_6^2) \subset \mathbb{Z}_3[\frac{c_4}{3}, \frac{c_6}{27}].$$

Additionally, since α and β act as zero on all of the classes in $tmf_*(R_3)$, the kernel of this first map is the submodule of tmf_* generated by α, β and their Δ translates. These together establish the following theorem about the *tmf* homology of $\Sigma B\Sigma_3$.

Theorem 5.1. *The tmf homology $\Sigma B\Sigma_3$ sits in a short exact sequence*

$$0 \to G_n \to tmf_n(\Sigma B\Sigma_3) \to \widehat{tmf}_{n-1} \to 0,$$

where $\widehat{tmf}_{n-1}$ is the subgroup of tmf_{n-1} of Adams Novikov filtration at least 1 and G_n, the cofiber of the map $tmf_n \to tmf_n(R_3)$, is given by

$$G_{24k+12j+8i} = \begin{cases} \mathbb{Z}/3 \oplus \bigoplus_{m=1}^{k} \mathbb{Z}/3^{6m} & k \equiv 1,2 \bmod 3, i+j=0 \\ \bigoplus_{m=0}^{k} \mathbb{Z}/3^{6m+3j+i} & k \equiv 0 \bmod 3 \\ \bigoplus_{m=0}^{k} \mathbb{Z}/3^{6m+3j+i} & k \equiv 1,2 \bmod 3, i+j>0 \\ 0 & \textit{otherwise} \end{cases},$$

where $j < 2$, and $i < 3$. The sequence is split as a sequence of groups. There is a hidden α extension originating on the copy of β^2 in $\widehat{tmf}_{20}$ and hitting the $\mathbb{Z}/3$ summand of G_{24}.

Proof. This short exact sequence is just a restatement of the earlier comments about the long exact sequence in *tmf* homology. It is split because the elements coming from G_n have Adams-Novikov filtration 0, and the convergence of the Adams-Novikov spectral sequence ensures a map of groups from $tmf_*(\Sigma B\Sigma_3)$ to G_n which is a left inverse to this inclusion.

The structure of the groups G_n is easy to show. A basis for $tmf_*(R_3)$ is given by the collection of monomials of the form $\Delta^k \tilde{c}_6^j \tilde{c}_4^i$, where $i < 3$, and $27\tilde{c}_6 = c_6$, $3\tilde{c}_4 = c_4$. This is simply because if we can solve the relation on Δ in $tmf_*(R_3)$. A basis for the Adams-Novikov filtration 0 subring of tmf_* is given by the monomials

$$\Delta^k c_6^j c_4^i \text{ for } k \equiv 0 \bmod 3 \text{ or } k \equiv 1,2 \bmod 3, i+j>0, \quad [3\Delta]\Delta^k, \text{ and } [3\Delta^2]\Delta^k.$$

Recalling that

$$\Delta^k c_6^j c_4^i = 3^{3j+i} \Delta^k \tilde{c}_6^j \tilde{c}_4^i$$

and collecting all terms of the same degree yields G_n.

The hidden extension can most readily been seen by considering the long exact sequence in Ext induced by the cofiber sequence. In this situation, Δ from the ground sphere kills Δ in the Adams E_2 term for $tmf_*(R_3)$, and $\alpha\beta^2$ on the ground sphere survives. $\square$

Remark. *The proof of this theorem also shows that the transfer induces a bijection between the elements of higher Adams-Novikov filtration elements of tmf_* and the elements of $tmf_*(B\Sigma_3)$ of Adams-Novikov filtration at least one (together with the $\mathbb{Z}/3$ coming from the 3-cell). This exactly repeats the situation at at the prime 2, where the transfer again mapped the higher Adams-Novikov elements in $ko_*(\mathbb{R}P^\infty)$ bijectively onto those in ko_*.*

6. The *tmf* Homology of the Finite Skeleta of R_3 and $B\Sigma_3$

For completeness, we include the tmf-homology of the finite skeleta of R_3 and $B\Sigma_3$. These computations serve as starting points for the program of Minami to detect the 3-primary η_j family [12].

6.1. **The Skeleta of R_3.** Let $n = 12k + i$, for $0 < i \leq 12$. We wish to compute the *tmf*-homology of $R_3^{[n]}$.

Lemma 6.1. *There is a filtration of $H_*(R_3^{[12k+i]})$ such that the associated graded is*

$$Gr\big(H_*(R_3^{[12k+i]})\big) = \left(\bigoplus_{n=0}^{k-1} \Sigma^{12n} h\mathbb{Z}\right) \oplus \Sigma^k M_i,$$

where M_i is the subcomodule of $h\mathbb{Z}$ generated by all classes of degree at most i for $i < 12$, and M_{12} is M_9 plus a primitive class in dimension 12.

Proof. The required filtration is just the restriction of the filtration used in the proof of Lemma 4.1 to the subcomodule $H_*(R_3^{[12k+i]})$. $\square$

The comodules M_i are the homology of $R_3^{[i]}$, and this splitting result and the follow theorem demonstrates that knowing their *tmf*-homology gives that of all finite skeleta. The proof of Theorem 4.4 shows the following

Theorem 6.2. *As a module over tmf_*,*

$$tmf_*(R_3^{[12k+i]}) = \mathbb{Z}_3\left[\frac{c_4}{3}\right]\{e_0, e_{12}, \ldots, e_{12(k-1)}\} \oplus \widetilde{M}_i e_{12k}/(c_6 e_{12j} - 27 e_{12(j+1)}),$$

where $\widetilde{M}_i$ is the tmf-homology of spectrum $R_3^{[i]}$.

The remainder of the section will be spent computing the modules $\widetilde{M}_i$. To save space, in what follows we use two indices: δ which ranges from 0 to 2 and ϵ which ranges from 0 to 1. When these appear, it means that all possible values of the index are actually present.

Proposition 6.3. *The spectra $R_3^{[1]}$, $R_3^{[2]}$, and $R_3^{[3]}$ are simply S^0. This implies that*

$$\widetilde{M}_i = tmf_*, \quad 1 \leq i \leq 3.$$

Lemma 6.4. *The spectrum $R_3^{[4]}$ is the cofiber of α_1. The tmf-homology of this is the extension of the module generated by $[\Delta^\epsilon e_0]$ and $[\alpha e_4]$ and subject to the relations*

$$\alpha[\alpha e_4] = \beta e_0,\ \alpha[\Delta e_0] = \beta^2[\alpha e_4],\ \alpha e_0 = \beta^3[\Delta^\epsilon e_0] = I[\alpha e_4] = \beta^4[\alpha e_4]$$

by the module

$$\mathbb{Z}_3[c_4, c_6, \Delta]\{[3e_4], [c_4e_4], [c_6e_4].$$

The extension is determined by the two relations

$$c_4[3e_4] = 3[c_4e_4] \pm c_6e_0, \quad c_6[3e_4] = 3[c_6e_4] \pm c_4^2e_0.$$

Proof. The Adams E_2 term can be readily computed to be the extension of

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta, \beta]\{e_0\}$$

by

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta]\{[v_0e_4], [c_4e_4], [c_6e_4]\} \oplus \mathbb{F}_3[\Delta, \beta]\{[\alpha e_4]\},$$

subject to the relations

$$c_4[3e_4] = 3[c_4e_4] \pm c_6e_0, \quad c_6[3e_4] = 3[c_6e_4] \pm c_4^2e_0, \quad \alpha[\alpha e_4] = \beta.$$

This Adams spectral sequence is a spectral module over the Adams spectral sequence for the *tmf*-homology of the sphere, and the two differentials in the Adams spectral sequence for the sphere,

$$d_2(\Delta) = \alpha\beta^2, \quad d_3([\alpha\Delta^2]) = \beta^5,$$

imply that Δe_0 and Δ^2e_0 are d_2 cycles and that the following differentials hold:

$$d_2(\Delta[\alpha e_4]) = \beta^3e_0, \quad d_3(\alpha\Delta^2[\alpha e_4]) = \beta^5[\alpha e_4].$$

This last d_3 implies that in fact,

$$d_3(\Delta^2e_0) = \beta^4[\alpha e_4],$$

using the relation involving α multiplication on $[\alpha e_4]$. □

Lemma 6.5. *The spectra $R_3^{[5]}$, $R_3^{[6]}$, and $R_3^{[7]}$ are the cofiber of the extension of α over the mod 3 Moore spectrum. The tmf-homology of these spectra, $\widetilde{M}_i$ is the tmf_* module generated by*

$$[\tfrac{c_4}{3}\Delta^\delta e_0],\ [\tfrac{c_6}{3}\Delta^\delta e_0],\ [\Delta^\epsilon e_0],\ [\alpha e_4],\ [\beta e_5],$$

and subject to the relations

$$\alpha[\beta e_5] = \beta[\tfrac{c_4}{3}e_0],\ \alpha[\alpha e_4] = \beta e_0,\ \alpha[\Delta e_0] = \beta^2[\alpha e_4],$$
$$(\alpha, \beta^3)e_0 = I([\alpha e_4], [\beta e_5]) = \beta^4[\alpha e_4] = 0.$$

Proof. In the long exact sequence in Ext induced by the inclusion of the 4-skeleton into $R_3^{[5]}$, the inclusion of the 5-cell kills the element $[v_0e_4]$. The elements $[c_4e_4]$ and $[c_6e_4]$ survive, and the relations in the Ext term for the 4-skeleton ensure that in the Adams E_2 term for $\widetilde{M}_5$,

$$v_0[c_4e_4] = c_6e_0, \quad v_0[c_6e_4] = c_4^2e_0.$$

Moreover, since α and β multiplications on the class $[v_0e_4]$ are trivial, the classes $[\alpha e_5]$ and $[\beta e_5]$ survive to the Adams E_2 page. An elementary computation in the bar complex establishes that

$$v_0[\alpha e_5] = c_4e_0.$$

This shows that the Adams E_2 page, as a module over that for tmf_*, is

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta, \beta]\{e_0, [\tfrac{c_4}{v_0}e_0], [\tfrac{c_6}{v_0}e_0], [\alpha e_4], [\beta e_5]\}$$
$$\big/\big(\alpha[\alpha e_4] - \beta e_0, \beta[\tfrac{c_4}{v_0}e_0] - \alpha[\beta e_5], \alpha e_0, I([\beta e_5], [\alpha e_4])\big)$$

The differentials again follow from those in the Adams spectral sequence of tmf_*. □

At this point, the patterns of extensions and differentials repeats. This makes the final computations substantially easier.

Lemma 6.6. *The spectrum $R_3^{[8]}$ is the spectrum C from §2, where the middle cell is replaced by the mod 3 Moore spectrum. The module $\widetilde{M}_8$ sits in a short exact sequence*

$$0 \to tmf_*\{[\tfrac{c_4}{3}\Delta^\delta e_0], [\tfrac{c_6}{3}\Delta^\delta e_0], [\Delta^\delta e_0], [\beta e_5]\}/\big((\alpha,\beta)\big([\tfrac{c_4}{3}{}^\epsilon \Delta^\delta e_0], [\tfrac{c_6}{3}\Delta^\delta e_0]\big), I[\beta e_5]\big) \\ \to \widetilde{M}_8 \to \mathbb{Z}_3[c_4, c_6, \Delta]\{[3e_8], [c_4 e_8], [c_6 e_8]\} \to 0,$$

where the extension is determined by the two relations

$$c_4[3e_8] = 3[c_4 e_8] \pm c_4[\tfrac{c_4}{3}e_0], \quad c_6[3e_8] = 3[c_6 e_4] \pm c_4[\tfrac{c_6}{3}e_0].$$

Proof. The long exact sequence in Ext coming from the short exact sequence in homology induced by the inclusion of $R_3^{[5]}$ into $R_3^{[8]}$ is determined by the connecting homomophism which takes e_8 to $[\alpha e_4]$. The linearity of this map shows that the Adams E_2 term for $\widetilde{M}_8$ is an extension of

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta]\{e_0, [\tfrac{c_4}{3}e_0], [\tfrac{c_6}{3}e_0]\} \oplus \mathbb{F}_3[\Delta, \beta] \otimes E(\alpha)\{[\beta e_5]\}$$

by

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta]\{[v_0 e_8], [c_4 e_8], [c_6 e_8]\},$$

subject to the extensions

$$c_4[v_0 e_8] = v_0[c_4 e_8] \pm \tfrac{c_4^2}{3}e_0, \quad c_6[v_0 e_8] = v_0[c_6 e_8] \pm \tfrac{c_4 c_6}{3}e_0.$$

The differentials are again determined by those of tmf_*. The only classes which support non-trivial α multiplication are multiples of $[\beta e_5]$, and here, the differentials are the same as for $\widetilde{M}_5$:

$$d_2(\Delta^i[\beta e_5]) = i\alpha\beta^2\Delta^{i-1}[\beta e_5], \quad d_3([\alpha\Delta^2][\beta e_5]) = \beta^5[\beta e_5].$$

□

Lemma 6.7. *The spectra $R_3^{[9]}$, $R_3^{[10]}$, and $R_3^{[11]}$ are the cofiber of the map from $\Sigma^4 C(\alpha)$ to C which is multiplication by 3 on the 4 and 8 cells. The module $\widetilde{M}_9$ can be expressed via the short exact sequence*

$$0 \to tmf_*\{[\alpha e_9]\} \to \widetilde{M}_9 \to \mathbb{Z}_3\left[\frac{c_4}{3}\right] e_0 \to 0,$$

where the only extension is given by

$$c_6 e_0 = 9[\alpha e_9].$$

Proof. The cofiber sequence coming from the inclusion of $R_3^{[8]}$ into $R_3^{[9]}$ induces a long exact sequence on Ext. The connecting homomorphism is

$$e_9 \mapsto [v_0 e_8] + [\tfrac{c_4}{3}e_0].$$

This is a map of modules over the Adams E_2 term for tmf_*, and just as before, the element $[\alpha e_9]$ is in the kernel of this map. This gives hidden extensions analogous to the ones for $\widetilde{M}_4$ and $\widetilde{M}_5$ in the Adams E_2 term for $\widetilde{M}_9$:

$$\alpha[\alpha e_9] = \beta e_5, \quad v_0[\alpha e_9] = [\tfrac{c_6}{3}e_0].$$

The c_4 and c_6 extensions coming from $[v_0e_8]$ give two more extensions:

$$v_0[c_4e_8] = c_4[\tfrac{c_4}{3}e_0], \quad v_0[c_6e_8] = c_4[\tfrac{c_6}{3}e_0].$$

This establishes that the Adams E_2 term is given by the extension of

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta]\{[\alpha e_9]\}$$

by

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta]\{e_0, [\tfrac{c_4}{v_0}e_0], [\tfrac{c_4^2}{v_0^2}e_0]\},$$

where $c_6e_0 = v_0^2[\alpha e_9]$. Just as before, the ordinary Adams differentials determine the differentials, recalling that $[\frac{c_6}{v_0^2}e_0] = [\alpha e_9]$:

$$d_2(\Delta^k[\tfrac{c_6}{v_0^2}e_0]) = k\alpha\beta^2\Delta^{k-1}[\tfrac{c_6}{v_0^2}e_0] = \beta^2[\beta e_5], d_3(\Delta^2[\beta e_5]) = \beta^5[\alpha e_9].$$

The Adams differentials here preserve the exact sequence, and this establishs the statement of the Lemma. □

Remark. *For completeness, we note that if we were to include a* 13*-cell, attaching it to the* 9*-cell via* α*, then the attaching map in long exact sequence in tmf homology would take the copy of* tmf_* *coming from the* 13*-cell isomorphically onto the factor* $tmf_*\{[\alpha e_9]\}$.

Proposition 6.8. *Since the twelve dimensional class is primitive in* M_{12}*, we conclude that as a* tmf_**-module,*

$$\widetilde{M}_{12} = \widetilde{M}_9 \oplus \Sigma^{12} tmf_*.$$

6.2. **The Skeleta of $B\Sigma_3$.** The analysis of the preceding section allows us to completely determine the structure of the groups $tmf_*(B\Sigma_3^{[n]})$. However, due to the complexity of the combinatorial problem, explicit demonstration of these groups in unenlightening. We instead present the following theorem concening bounds on the orders of these groups.

Theorem 6.9. *If* $n = 12k + i$*, then* 3^{3k+2} *annihilates the torsion subgroup of* $tmf_*(B\Sigma_3^{[n]})$*. Moreover, if* $i \geq 5$*, then there are elements of order exactly* 3^{3k+1}*, and if* $i \geq 9$*, then there are elements of order exactly* 3^{3k+2}*.*

Proof. This is immediate with the consideration that the large torsion subgroups are generated by high powers of $\frac{c_6}{27}$. If we consider only a finite skeleton of $B\Sigma_3$, then we include only finitely many powers of this element. The largest such element occurs in dimension $12k$. If i is at least 5, then we have the element $\frac{c_4}{3}$ on this element. If i is at least 9, then we have the element $\frac{c_4^2}{9}$ on this element. These provide the elements of exact order. □

7. The Σ_3 Tate Homology of *tmf*

The analysis used to compute the *tmf* homology of R_3 applies to compute the homotopy of

$$tmf^{t\Sigma_3} = (tmf \wedge B\Sigma_3)_{-\infty} = \varprojlim \big(tmf \wedge (B\Sigma_3)_{-n}\big).$$

7.1. **Computation of the Homotopy.** A mod 3 form of James periodicity shows that as $\mathcal{A}(1)_*$-comodules,

$$H_*\big((B\Sigma_3)_{-12k+3}\big) = \Sigma^{-12k} H_*\big((B\Sigma_3)_3\big).$$

The Adams spectral sequence argument in §5 shows that the map

$$\pi_*\big(tmf \wedge (B\Sigma_3)_{-12(k+1)+3}\big) \to \pi_*\big(tmf \wedge (B\Sigma_3)_{-12k+3}\big)$$

is surjective on the G_* summand and zero on the $\widehat{tmf}_*$ summand. This implies that there are no $\lim^1$ terms coming from the inverse system of homotopy groups. Moreover, this is a system of tmf_*-modules, and considering the action of c_4 and c_6 in each of the modules in the inverse system allows us to conclude

Theorem 7.1. *The homotopy of the Σ_3 Tate spectrum of tmf is an indecomposable tmf_* module, and*

$$\pi_*(tmf^{t\Sigma_3}) = \Sigma^{-1}\mathbb{Z}_3\left[\frac{c_4}{3}, \left(\frac{c_6}{27}\right)^{\pm 1}\right].$$

7.2. **A Conjectural Non-splitting Result.** We wish to establish a limit argument using the Adams spectral sequence to show that this spectrum does not split. When we consider the effect of homology on the James periodicity result, then it shows that there is a filtration of $H_*(B\Sigma_3)_{-\infty}$ such that the associated graded is

$$Gr\big(H_*(B\Sigma_3)_{-\infty}\big) = \bigoplus_{k=-\infty}^{\infty} \Sigma^{12k-1} h\mathbb{Z}.$$

This implies that the limit Adams spectral sequence for the homotopy of $tmf^{t\Sigma_3}$ collapses, reaffirming the previous result. However, analysis of the Adams E_2 term shows that the Adams filtrations seem to be wrong for a splitting result analogous to Mahowald and Davis' result. The classes in dimensions 3 mod 12 all have Adams filtration at least 2, whereas v_1 should lie in Adams filtration 1.

Conjecture 7.2. *There does not exist a splitting of the form*

$$tmf^{t\Sigma_3} = \bigvee_{k=-\infty}^{\infty} \Sigma^{12k-1} BP\langle 1\rangle.$$

8. The Conjectural Case for Higher Primes

A similar result is conjectured to hold for the p-local case with eo_{p-1}, where eo_{p-1} is an E_∞ ring spectrum which $K(p-1)$-localizes to EO_{p-1} and whose homotopy groups are determined by the Gorbounov-Hopkins-Mahowald Hopf algebroid without inverting Δ or completing [7].

8.1. **Gorbounov, Hopkins, and Mahowald's Geometric Model.**

(1) $$y^{p-1} = x^p + a_1 x^{p-1} + \cdots + a_p, \quad x \mapsto x + r.$$

8.2. **Hopes of** eo_{p-1}**.** We note that the Gorbounov-Hopkins-Mahowald curves come equipped with an involution ι which on points looks like $(x,y)\mapsto(x,-y)$.

Conjecture 8.1. *If C_p is the p-cell complex*

$$S^0\cup_{\alpha_1} e^{2(p-1)}\cup_{\alpha_1}\cdots\cup_{\alpha_1} e^{2(p-1)^2},$$

then

$$\mathrm{eo}_{p-1}(\iota):=\mathrm{eo}_{p-1}\wedge C_p$$

is an E_∞ ring spectrum which corresponds to the geometric situation of a Gorbounov-Hopkins-Mahowald curve together with a fixed point of the above involution.

A fixed point of the involution is equivalent to the data of a curve of type 1 together with a root of the right hand side. By using the morphism $x\mapsto x+r$, we can force this fixed point to be $(0,0)$. In other words, the Adams-Novikov E_2 term is the cohomology of the trivial Hopf algebroid

$$A=\Gamma=\mathbb{Z}[a_1,\dots,a_{p-1}].$$

This in particular implies that the Adams-Novikov spectral sequence collapses.

Conjecture 8.2. *As a Hopf algebra,*

$$\mathcal{A}:=\pi_*(H\mathbb{Z}/p\wedge_{\mathrm{eo}_{p-1}} H\mathbb{Z}/p)=\mathcal{A}(1)_*\otimes E(a_2,\dots,a_{p-1}),$$

where again $\mathcal{A}(1)_*$ is dual to the subalgebra generated by β and $\mathcal{P}^1$, and where $|a_i|=2i(p-1)+1$. The elements in $\mathcal{A}(1)$ again have their usual coproducts, while

$$\psi(a_j)=\sum_{k=0}^{j}\tfrac{1}{k!}\xi_1^k\otimes a_{j-k}+a_j\otimes 1,$$

where $a_1=\tau_1$ and $a_0=\tau_0$.

Indicative Sketch. The spectrum EO_{p-1} is the homotopy fixed points of E_{p-1} under an action of an extension of $\mathbb{Z}/p$ by $\mathbb{Z}/(p-1)^2$. Since E_{p-1} is a p complete spectrum, the prime to p part of the group serves only to carve out an "Adams summand" for EO_{p-1}. The p-cell spectrum C_p, when smashed with EO_{p-1}, undoes the $\mathbb{Z}/p$ homotopy fixed points, resulting in a torsion free spectrum that is the $K(p-1)$-localization of a wedge of copies of $BP\langle p-1\rangle$. This implies that just as in the case of $p=2$ or $p=3$, $\mathrm{eo}_{p-1}(\iota)$ should split as a wedge of copies of $BP\langle p-1\rangle$, reaffirming Conjecture 8.1.

We should actually have a stronger statement. The coproducts on the elements actually follow from the right units of the elements a_i of the Gorbounov-Hopkins-Mahowald Hopf algebroid restricted to the (non-full) substack representing $\mathrm{eo}_{p-1}(\iota)$. Better said, r corresponds to ξ_1, p corresponds to τ_0, and a_i corresponds to the element of the same name in $\mathcal{A}$. □

8.3. **The** eo_{p-1} **homology of** $B\Sigma_p$**.** Assuming Proposition 8.2, we can reprove most of the results true for the prime 3. If we again consider the cofiber R_p of the transfer map $B\Sigma_p\to S^0$, then there is an analogue to Lemma 4.1

Proposition 8.3. *There is a filtration of $H_*(R_p)$ such that the associated graded is*

$$Gr\big(H_*(R_p)\big)=\bigoplus_{k=0}^{\infty}\Sigma^{2p(p-1)k}h\mathbb{Z}.$$

The same argument that showed that $\mathrm{Ext}_{\mathcal{A}}$ of this was torsion free works at other primes, so we see that $\mathrm{Ext}_{\mathcal{A}_p}\left(\mathbb{F}_p, H_*(R_p)\right)$ is an evenly generated polynomial algebra with generators corresponding to p and certain fractional multiplies of rational generators of eo_{p-1*}. The extension problems can be similar solved.

It is easy to see that $\mathrm{eo}_{p-1*}(R_p)$ is again indecomposable as a module over eo_{p-1*}, since $R_p \otimes \mathbb{Q} = S^0_{\mathbb{Q}}$. We moreover conjecture that the eo_{p-1} image of the transfer map again contains all of the higher Adams-Novikov filtration elements, since these are generated by α and β, and these elements will again not be present in $\mathrm{eo}_{p-1*}(R_p)$.

References

1. Greg Arone and Mark Mahowald, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, Invent. Math. **135** (1999), no. 3, 743–788.
2. Andrew Baker and Andrej Lazarev, *On the Adams spectral sequence for R-modules*, http://hopf.math.purdue.edu//Baker-Lazarev/Rmod-ASS.pdf.
3. Mark Behrens, *A modular description of the K(2)-local sphere at the prime 3*.
4. ———, *Buildings, elliptic curves, and the $K(2)$-local sphere*, 2005.
5. Mark Behrens and Satya Pemmaraju, *On the existence of the self map v_2^9 on the Smith-Toda complex $V(1)$ at the prime 3*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, Contemp. Math., vol. 346, pp. 9–49.
6. Donald M. Davis and Mark Mahowald, *The spectrum $(P \wedge bo)_{-\infty}$*, Math. Proc. Cambridge Philos. Soc. **96** (1984), no. 1, 85–93.
7. V. Gorbounov and M. Mahowald, *Formal completion of the Jacobians of plane curves and higher real K-theories*, J. Pure Appl. Algebra **145** (2000), no. 3, 293–308.
8. M. Mahowald and R. James Milgram, *Operations which detect Sq^4 in connective K-theory and their applications*, Quart. J. Math. Oxford Ser. (2) **27** (1976), no. 108, 415–432.
9. Mark Mahowald, *A new infinite family in ${}_2\pi_*{}^s$*, Topology **16** (1977), no. 3, 249–256.
10. ———, *The image of J in the EHP sequence*, Ann. of Math. (2) **116** (1982), no. 1, 65–112.
11. John W. Milnor and John C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264. MR MR0174052 (30 #4259)
12. Norihiko Minami, *On the odd-primary Adams 2-line elements*, Topology Appl. **101** (2000), no. 3, 231–255.