THE 5-LOCAL HOMOTOPY OF eo_4

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1. Introduction

Hopkins and Miller have shown that the Lubin-Tate spectrum E_n admits an E_∞ structure such that the group of E_∞ self-maps is the extended Morava stabilizer group $\mathbb{G}_n$ [4]. This allows the contruction of the "higher real K-theories" EO_n corresponding to E_n^{hG} for G a finite subgroup of $\mathbb{G}_n$. Hopkins and Miller also showed that the elements in higher Adams Novikov filtration in the homotopy of EO_{p-1} are easy to compute. The difficulty is understanding the zero line.

1.1. **Organization.** In §2, we review the Gorbounov-Hopkins-Mahowald Hopf algebroid and use it to compute the rational homotopy of eo_4. The middle sections of the paper compute the Adams-Novikov E_2 term for the homotopy of eo_4, loosely following Bauer's computation of the 3-local homotopy of tmf [1]. We introduce the Bockstein spectral sequences needed for computation in §3, and we carry out the prime independent computations. In §4, we restrict attention to the prime 5, competing the computations for eo_4. Finally, in §5, we compute the Adams differentials.

2. The Hopf Algebroid and Rational Computations

The Hopf algebroid used is the one developed by Gorbounov, Hopkins, and Mahowald in [2], based on deformations of the Artin-Scheier curve

$$y^{p-1} = x^p - x.$$

In [2], it is shown that the formal completion of the Jacobians of the family of curves

$$y^{p-1} = x^p + a_1 x^{p-1} + \cdots + a_p, \quad x \mapsto x + r, \ (x, y) \mapsto (\lambda^{p-1} x, \lambda^p y) \tag{1}$$

carries the Lubin-Tate universal deformation of the Honda formal group, together with an action of $\mathbb{Z}/p \ltimes \mathbb{Z}/(p-1)^2$, a maximal finite subgroup of $\mathbb{G}_{p-1}$. This implies that an appropriate sheaf of E_∞-ring spectra over the moduli stack of all such curves provides a good geometric model for the rigidification of deformations of a height n formal group given by the Hopkins-Miller theorem [4], just as with tmf and the moduli stack of elliptic curves.

The scaling action on the family given by Equation 1 allows us to split off a graded Adams summand, giving the actual spectrum in question. This splitting is algebraically realized by considering Equation 1 as a homogeneous graded equation, where $|x| = 2(p-1)$, $|y| = 2p$, $|r| = 2(p-1)$ and $|a_i| = 2i(p-1)$, and the λ action fixes the graded pieces. This also reflects the information that r is to correspond in the topology to $\mathcal{P}_5^1$. Just as with the computations of the homotopy of tmf, computations are done over the affine substack corresponding to the corepresenting Hopf algebroid:

$$(A, \Gamma) = \big(\mathbb{Z}_{(p)}[a_1, \ldots, a_p], A[r]\big).$$

At an arbitrary prime, we have the following formula for $\eta_R(a_i)$:

$$\eta_R(a_i) = \sum_{j=0}^{i} \binom{p-j}{i-j} a_j r^{i-j}.$$

At the prime 5, this gives

$$\begin{aligned}
\eta_R(a_1) &= a_1 + 5r \\
\eta_R(a_2) &= a_2 + 4a_1 r + 10r^2 \\
\eta_R(a_3) &= a_3 + 3a_2 r + 6a_1 r^2 + 10r^3 \\
\eta_R(a_4) &= a_4 + 2a_3 r + 3a_2 r^2 + 4a_1 r^3 + 5r^4 \\
\eta_R(a_5) &= a_5 + a_4 r + a_3 r^2 + a_2 r^3 + a_1 r^4 + r^5
\end{aligned}$$

Following Bauer's computation, we can use a faithfully flat base change and change of rings to get rid of a_p. This imposes the condition

$$\eta_R(a_p) = r^p + a_1 r^{p-1} + \cdots + a_{p-1} r = 0$$

in the comorphism ring. If we let $\bar{A}$ be $A/(a_p)$, then the quotient map

$$(A, \Gamma) \to (\bar{A}, \bar{A} \otimes_A \Gamma \otimes_A \bar{A})$$

induces an equivalence of the categories of comodules and therefore an equivalence in Ext.

2.1. **Rational Information.**

The rational case is substantially easier to compute.

Lemma 1. *There are classes c_i of degree $2i(p-1)$ in A such that*

$$H^*(A \otimes \mathbb{Q}, \Gamma \otimes \mathbb{Q}) = H^0(A \otimes \mathbb{Q}, \Gamma \otimes \mathbb{Q}) = \mathbb{Q}[c_2, \ldots, c_p].$$

Proof. Since p is a unit, we can transform Equation 1 into one of the form

$$y^{p-1} = x^p + a_2 x^{p-2} + \cdots + a_p$$

by choosing r to be $-\frac{a_1}{p}$. Note that this fixes our choice of r. In the language of Hopf algebroids, we conclude that $(A \otimes \mathbb{Q}, \Gamma \otimes \mathbb{Q})$ is equivalent to the trivial Hopf algebroid $A = \Gamma = \mathbb{Q}[c_2', \ldots, c_p']$, where $c_i' = \eta_R(a_i)$ evaluated at our choice of r. The denominators of the elements c_i' are powers of p, so we can multiply by a sufficiently high power of p to get new generators that actually lie in A:

$$c_i = \sum_{j=0}^{i} \binom{p-j}{i-j} (-1)^j a_j a_1^{i-j} p^j.$$

□

At the prime 5, we have the elements c_i have the following form:

$$
\begin{aligned}
c_2 &= -2a_1^2 + 5a_2 \\
c_3 &= 4a_1^3 - 15a_1a_2 + 25a_3 \\
c_4 &= -3a_1^4 + 15a_1^2a_2 - 50a_1a_3 + 125a_4 \\
c_5 &= 4a_1^5 - 25a_1^3a_2 + 125a_1^2a_3 - 625a_1a_4 + 3125a_5
\end{aligned}
$$

2.2. **The Ring of Invariants.** We have generators c_i that rationally are polynomial generators. In our p-local context, this means their products can be written as some power of p times a sum of integral generators. To find the generators of $H^0(A,\Gamma)$, we have to add these and the obvious relations. The proof of the following theorem will be one of the goals for the rest of the paper:

Theorem 1. *As an algebra over* $\mathbb{Z}_{(5)}$,

$$H^0(A,\Gamma) = \mathbb{Z}_{(5)}[c_2, c_3, \Delta_i, \Delta'_{15}, \Delta'_{18}, \Delta]/\big(\mathit{rels}\big),$$

where i ranges from 4 to 22 and where the expressions of these elements in terms of the elements a_i and their relations are induced by the formulas from Appendix A, together with the natural inclusion of $H^0(A,\Gamma)$ into A.

This ring has a distinguished ideal:

$$\mathfrak{m} = (5, c_2, c_3, \Delta_i, \Delta'_j).$$

The ring $H^0(A,\Gamma)$ is the zero line of the Adams-Novikov E_2 term, and it is easier to compute the full E_2 term and then read off the zero line. The remainder of the paper does just that.

3. Preliminary, Prime Independent Remarks

We will compute the Adams-Novikov spectral sequence via a sequence of Bockstein spectral sequences. It is clear from the formulation of the right units that the chain of ideals

$$I_0 = (p) \subset I_1 = (p, a_1) \subset \cdots \subset I_{p-1} = (p, a_1 \ldots, a_{p-1})$$

is invariant. The quotients $(A/I_k, \Gamma/I_k)$ are therefore Hopf algebroids. Filtering by powers of these ideals give Bockstein spectral sequences of the form

$$H^*(A/I_k, \Gamma/I_k) \otimes \mathbb{Z}_{(p)}[a_{k-1}] \Rightarrow H^*(A/I_{k-1}, \Gamma/I_{k-1}).$$

This is a trigraded spectral sequence of algebras. If the degree of a homogeneous element x is written (s,t,u), where s is the cohomological degree, t is the internal dimension, and u is the Bockstein degree, then the degree of $d_r(x)$ is $(s+1, t, u+r)$.

The first three Bockstein spectral sequences are the same for all primes.

3.1. **Computation of $H^*(A/I_{p-1}, \Gamma/I_{p-1})$.** The Hopf algebroid $(A/I_{p-1}, \Gamma/I_{p-1})$ is the Hopf algebra $(\mathbb{F}_p, \mathbb{F}_p[r]/r^p)$. The cohomology of this is $\mathbb{F}_p[b] \otimes E(a)$, where $|a| = \big(1, 2(p-1)\big)$, $|b| = \big(2, 2p(p-1)\big)$, and in the cohomology of the bar complex,

$$a = [r], \quad b = \langle \underbrace{a, \ldots, a}_{p} \rangle = \Big[\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{j} r^{p-i} | r^i\Big].$$

3.2. Computation of $H^*(A/I_{p-2}, \Gamma/I_{p-2})$. .

We run the Bockstein spectral sequence for adding in a_{p-1}. The E_1 term is a polynomial algebra on elements a, b, and a_{p-1} of tridegrees

$$|a| = \big(1, 2(p-1), 0\big),\ |b| = \big(2, 2p(p-1), 0\big),\ |a_{p-1}| = \big(0, 2(p-1)^2, 1\big).$$

For dimension reasons, all of these are permanent cycles, so the spectral sequence collapses.

3.3. Computation of $H^*(A/I_{p-3}, \Gamma/I_{p-3})$.

The E_1 term of this Bockstein spectral sequence is a polynomial algebra on the elements from the previous part, together with a_{p-2}. The tridegrees of the elements a and b are not changed, while the rest are:

$$|a_{p-1}| = \big(0, 2(p-1)^2, 0\big),\ |a_{p-2}| = \big(0, 2(p-1)(p-2), 1\big).$$

It is also clear that a, b, and a_{p-2} are all permanent cycles which do not bound. The formulation of the right unit shows that

$$d_1(a_{p-1}) = 2aa_{p-2}.$$

This leaves us the following algebra for the E_2 page:

$$\mathbb{F}_p[b, a_{p-1}^p, a_{p-2}] \otimes E(a)/aa_{p-2}\{1, x_1, \ldots, x_{(p-1)}\}/(ax_k, a_{p-2}x_k),$$

where the x_k has tridegree $(1, 2(1+k(p-1))(p-1), 0)$ and is represented by aa_{p-1}^k.

Proposition. All of the x_k with the exception of x_{p-1} are non-bounding permanent cycles. We also have $d_{p-1}(x_{p-1}) = a_{p-2}^{p-1}b$.

Proof. For dimension reasons, the only possible non-trivial differentials on x_k are of the form $x_k \mapsto ba_{p-2}^n$. We therefore have the following dimension computation on the internal degree:

$$2(p-1)(1+k(p-1)) = 2(p-1)(p+n(p-2)) \Rightarrow (k-1)(p-1) = n(p-2).$$

This has a unique solution in our range: $k = p-1$, $n = p-1$.

For the prime 5, we can also show easily the second part via direct computation in the bar complex:

$$a_4^4r + 4a_4^3a_3r^2 + 3a_4^2a_3^2r^3 + 3a_4a_3^3r^4 \mapsto a_3^4\big(r^4|r + 2r^3|r^2 + 2r^2|r^3 + r|r^4\big) = a_3^4b.$$

For all primes, this result follows from the Massey product lemma of [3]:

$$d_{p-1}(x_{p-1}) = \langle a, \underbrace{d_1(a_{p-1}), \ldots, d_1(a_{p-1})}_{p-1}\rangle = \langle \underbrace{a, \ldots, a}_{p}\rangle a_{p-2}^{p-1} = ba_{p-2}^{p-1}.$$

□

This gives the following E_3 term, which, for dimension reasons, is also the E_∞ term:

$$\mathbb{F}_p[b, a_{p-2}, a_{p-1}^p] \otimes E(a)/(aa_{p-2}, a_{p-2}^{p-1}b)\{1, x_1, \ldots, x_{(p-2)}\}/(ax_k, a_{p-2}k).$$

There are also the following Massey product relations:

$$\langle x_k, a, a_{p-2}\rangle = x_{k+1} = \langle a_{p-2}^{k+1}, \underbrace{a, \ldots, a}_{k+2}\rangle.$$

The element a_{p-1}^p is a distinguished permanent cycle that we will call Δ.

We can represent this E_∞ term as a picture for the prime 5 (Figure 1), with $t/8$ given by the horizontal axis and s given by the vertical one. This picture is repeated polynomially in Δ, represented by a box, and b, so we will only list the first part.

FIGURE 1. $H^*(A/I_{p-3}, \Gamma/I_{p-3})$

In the picture, a solid line of positive slope is multiplication by a, one of slope zero is multiplication by a_3, and the dotted lines are Massey products $\langle a_3, a, \cdot\rangle$. The case of the general prime is similar, except that the horizontal axis would be indexed as $t/2(p-1)$, and each row above the zeroth would have $p-1$ solid dots.

4. COMPLETION OF THE COMPUTATION AT THE PRIME 5

From this point on, we will restrict our attention to the prime 5. In this case, we can find explicit representatives of the elements x_1, x_2, and x_3.

$$x_1 = a_4r + a_3r^2,\ x_2 = a_4^2r + 2a_3a_4r^2 + 3a_3^2r^3,\ x_3 = a_4^3r + 3a_3a_4^2r^2 - a_3^2a_4r^3 - 3a_3^3r^4.$$

4.1. Computation of $H^*(A/I_1, \Gamma/I_1)$. .

The computation here starts largely as before. The elements a, b, a_2, x_1, and Δ are all permanent cycles, for dimension reasons. The element x_1 is now represented as $a_4r + a_3r^2 + a_2r^3$. However, beyond this the patterns of differentials becomes more complicated.

For clarity, we will rely on pictures of the E_r terms to describe the initial situations and tell us which elements could support a differential. In these Bockstein spectral sequences, we know from degree considerations that the d_r-differential of any element must be divisible by a_2^r (more generally, by the new element to the r^{th} power). If we make the convention that a solid horizontal line means multiplication by the new, Bockstein element and an open circle means a polynomial algebra on this element, then we see that the possible targets of a d_r differential are open circles preceded horizontally by r solid lines. If we additionally make the convention that circles with dots in them are the non-Bockstein multiplicative generators, then the differentials are totally determined by their values on these elements. These conventions will allow us to immediately see which elements could support a differential, reducing the number of things to check.

4.1.1. *The d_1 Differential.* We have a single differential coming immediately from the bar complex:

$$d_1(a_3) = 3a_2a.$$

If we extend this by multiplicativity, using the fact that $a_3a = 0$, we see that all elements of the form a_3^k are d_1-cycles. To see if there are any other differentials, we first look at the picture (Figure 2), in which dashed horizontal lines are a_3 multiplications.

FIGURE 2. E_1 page for $H^*(A/I_1, \Gamma/I_1)$

From this, we see the last possible d_1 differential:

Proposition. We have $d_1(x_3) = a_2a_3^2b$.

Proof. The element x_3 can be written as

$$x_3 = \langle a_3^3, a, a, a, a\rangle.$$

From this it follows from a simplification of May's work on Massey products, as presented in [5]that

$$d_1(x_3) = \langle d_1(a_3^3), a, a, a, a\rangle = \langle -a_2a_3^2a, a, a, a, a\rangle = -a_2a_3^2\langle a, a, a, a, a\rangle = -a_2a_3^2b.$$

□

4.1.2. *The d_2 Differential.* From the picture of the E_2 page (Figure 3), we see immediately that the only elements that can support a d_2 differential are a_3^3 and x_2.

Figure 3. The E_2 page for $H^*(A/I_1, \Gamma/I_1)$

Proposition. $d_2(a_3^3) = -a_2^2x_1$ and $d_2(x_2) = -a_2^2b$.

Proof. Again, we have Massey product proofs. The element x_2 is the Massey product

$$x_2 = \langle a_3^2, a, a, a\rangle.$$

This means, by the Massey product lemma, that

$$d_2(x_2) = \langle d_1(a_3), d_1(a_3), a, a, a\rangle = a_2^2b.$$

A slightly more general form of the Massey product lemma gives the other result:

$$d_2(a_3^3) = \langle a_3, d_1(a_3), d_1(a_3)\rangle = a_2^2\langle a_3, a, a\rangle = a_2^2x_1.$$

□

4.1.3. *The d_3 Differential and Beyond.* Given the sparsity of the spectral sequence above the filtration 0 line (Figure 4), it is clear that it now collapses.

Figure 4. $H^*(A/I_1, \Gamma/I_1)$

For computational reasons, we give here some of the full names for some of the elements listed above. The elements a and b have their usual bar representatives, while

$$\begin{aligned} x_1 &= a_4r + a_3r^2 + a_2r^3 \\ [a_3^2] &= a_3^2 + 2a_2a_4 \\ [a_3^5] &= a_3^5 + 2a_2^3a_3^3 + a_2^4a_3a_4 \\ \Delta &= a_4^5 - 2a_3^4a_4^2 - a_2a_3^2a_4^3 + 2a_2^2a_4^4 + a_2^3a_3^2a_4^2 + a_2^4a_4^3 \end{aligned}$$

With these elements, we can also compute the structure of H^* as a ring:

Proposition. We have the multiplicative extension $2a[a_3^2] = a_2x_1$, and the full algebra of $H^*(A/I_1, \Gamma/I_1)$ is

$$\mathbb{F}_5\big[a, b, x_1, a_2, [a_3^2], [a_3^5], \Delta\big]/\Big((a, x_1)^2,\ a(a_2, [a_3^2], [a_3^5]),\ a_2^2(b, x_1),$$
$$[a_3^2]^5 - [a_3^5]^2 = a_2^3[a_3^2]^4 + a_2^6[a_3^2]^3 + 2a_2^5\Delta,\ 2a[a_3]^2 - a_2x_1\Big)$$

Proof. The algebra structure will follow from the first part by algebra. The first part follows from noting that the difference of these two elements is the bar differential of a_3a_4. □

4.2. **Computing $H^*(A/I_0, \Gamma/I_0)$.** Because things are so spread out, this is actually easier to compute than the previous term. We start with the observation that, for dimension reasons, a, b, Δ, and x_1 are all permanent cycles. The bar representative of x_1 is $-r^5$.

4.2.1. *The d_1 Differential.* We first note the differential coming immediately from the bar complex:

$$d_1(a_2) = -a_1 a.$$

To continue, we use the picture of E_1 (Figure 5), marking this differential. We will use similar notation as before, but here solid lines with represent a_1 multiplications while dashed lines will represent a_2 multiplication. To further simplify the picture, we use a circled star to indicate a polynomial algebra on both a_1 and a_2.

FIGURE 5. E_1 page for $H^*(A/I_0, \Gamma/I_0)$

The picture shows us another differential.

Proposition. We have $d_1([a_3^2]) = 3a_1x_1$.

Proof. The element a_3^2 can be realized as $\langle a_3, a, a_2\rangle$ or $\langle a_2, a, a_2, a\rangle$. Taking d_1 on this as on Massey products, we get

$$d_1(a_3^2) = \langle a_3, a, a_1, a\rangle = a_1x_1,$$

or

$$d_1(a_3^2) = \langle a_2, a, d_1(a_2), a\rangle = \langle a_2, a, a_1a, a\rangle = a_1x_1.$$

□

4.2.2. *The d_2 Differential and the E_∞ Page.* At this point, our spectral sequence is again very sparse (Figure 6).

FIGURE 6. E_2 Page of $H^*(A/I_0, \Gamma/I_0)$

We again see that we can have but a single coherent differential.

Proposition. We have $d_2(a_2x_1) = a_1^2b$.

Proof. On this page, $x_1 = \langle a_2, a, a, a\rangle$, so $a_2x_1 = \langle a_2^2, a, a, a\rangle$. The Massey product lemma shows that

$$d_2(\langle a_2^2, a, a, a\rangle) = \langle d_1(a_2), d_1(a_2), a, a, a\rangle = a_1^2b.$$

□

It is clear that no further differentials are possible, so the spectral sequence collapses here.

4.3. **Finally, $H^*(A,\Gamma)$.** Everything we have done so far has led us to compute what happens when we add in the number 5. There is already an obvious differential given by $a_1 \mapsto 5r$. Additionally, x_1 has survived this long because it has represented r^5 which, mod 5, is a cycle since r is. Now the binomial theorem tells us exactly what it will hit:

$$r^5 \mapsto 5r^4|r + 10r^3|r^2 + 10r^2|r^3 + 5r|r^4.$$

In other words, $d_1(x_1) = 5b$. This gives us all of the differentials for dimension reasons, as we shall immediately see.

The Bockstein spectral sequence here is best drawn by having a circle represent $\mathbb{Z}_{(5)}$ (Figure 7). The differentials again preserve the internal dimension and change the cohomological degree by 1. We label the algebra generators fed to us by the Bockstein spectral sequence by the name of their Bockstein leading term (i.e. the name of the element on the E_1 page of the Bockstein spectral sequence in which it first appears):

Figure 7. E_1 Page for $H^*(A,\Gamma)$

Since there are no more possible differentials, we conclude that $E_2 = E_\infty$. Additionally, we know from the above picture the names of the algebra generators of H^0, and we therefore see by these Bockstein spectral sequences that the algebra generators given in Appendix 1 are the generators of H^0.

Corollary. *As an algebra, $H^0(A,\Gamma)$ is as described in Theorem 1*

Proof. The Bockstein spectral sequences demonstrated that the classes given are the algebra generators. The relations are simple consequences of algebraic manipulations, so these are also immediate. □

Putting everything we have seen so far together allows us to show the following theorem.

Theorem 2. *As an algebra,*

$$H^*(A,\Gamma) = H^0[a,b]/\Big(a^2, \mathfrak{m}(a,b)\Big).$$

Proof. The only surprise relation is $\mathfrak{m}(a,b)$, and this follows from the earlier fact that terms dominated in $(a_1,a_2,a_3)(a,b)$ was zero by the time we reached this last page. □

5. ADAMS DIFFERENTIALS AND THE 5-LOCAL HOMOTOPY OF eo_4

In this section, we compute the Adams' differentials for the homotopy of eo_4. Since the unit $S^0 \to \mathrm{eo}_4$ takes the elements $\alpha, \beta_1 \in \pi_*(S^0)$ to the classes $a, b \in \pi_*(\mathrm{eo}_4)$, and since we have the Toda relation that $\alpha\beta_1^p = 0$, we must conclude:

Theorem 3. *We have $d_9(\Delta) = ab^4$.*

This additionally gives us some hidden multiplicative extensions. We can see these by considering the Massey product representatives of the "left-over" classes $[a\Delta]$, $[a\Delta^2]$, and $[a\Delta^3]$.

Proposition. We have

$$\begin{aligned}[a\Delta] &= \langle \iota, ab^4, a\rangle \\ [a\Delta^2] &= \langle \iota, ab^4, ab^4, a\rangle \\ [a\Delta^3] &= \langle \iota, ab^4, ab^4, ab^4, a\rangle.\end{aligned}$$

Additionally, we have a hidden multiplicative extension

$$a[a\Delta^3] = b^{13}.$$

Proof. The first relations are immediate from the form of d_8. The hidden extension follows by "shuffling" in the a and then "shuffling" out the b^4 terms. □

The hidden extension and these Massey product forms force on us a final differential:

Theorem 4. *We have* $d_{33}([a\Delta^4]) = 4b^{17}$.

Proof. The Adams form of the Massey product lemma gives

$$d_{33}(a\Delta^4) = \langle a, d_9(\Delta), d_9(\Delta), d_9(\Delta), d_9(\Delta)\rangle = b^{17}.$$

□

The spectral sequence collapses at this point, as there are not enough things in higher filtration to be the target of any further differentials.

Appendix A. Formulas Relating the classes Δ_i

References

1. Tilman Bauer, *Computation of the homotopy of the spectrum tmf*, arXiv:math.AT/0311328.
2. V. Gorbounov and M. Mahowald, *Formal completion of the Jacobians of plane curves and higher real K-theories*, J. Pure Appl. Algebra **145** (2000), no. 3, 293–308.
3. Michael A. Hill, *A computational lemma for differentials in spectral sequences*, http://math.mit.edu/ mikehill/Papers.
4. Michael J. Hopkins and Haynes Miller, *E_n has a unique E_∞ structure*, Yet unpublished.
5. Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press Inc., Orlando, FL, 1986.