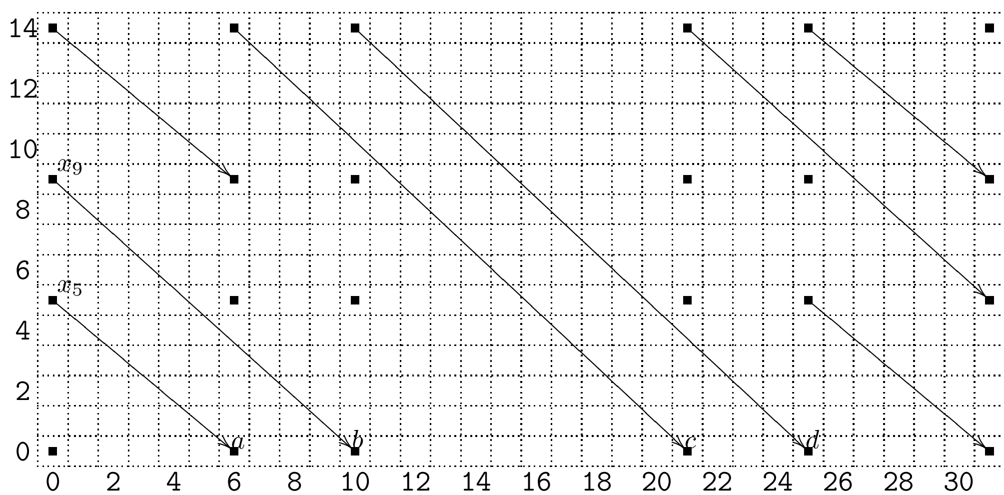
THE RATIONAL COHOMOLOGY OF $Sp(5)/SU(5)$

M.H.

As an algebra, the rational cohomology of $Sp(5)/SU(5)$ is indistinguishable from that of $S^6 \times S^{25} \# S^{10} \times S^{21}$. However, these spaces are not rationally equivalent, as the following computation of Massey products shows. Rationally, the exterior classes x_3 and x_7 in $H^*(SU(5);\mathbb{Q})$ give rise to the classes x_3 and x_7 in $H^*(Sp(5);\mathbb{Q})$. For this reason, we know they are permanent cycles in the Serre spectral sequence, and we can ignore them in our computations. Here is the E_2 term for the Serre spectral sequence, considering only the classes coming from x_5 and x_9, together with all of the differentials:



The long, d_{15} differentials give an example of the Massey product lemma:

$$c = d_{15}(ax_5x_9) =< a, d_6(x_5), d_{10}(x_9) >=< a, a, b >,$$

and

$$d = d_{15}(bx_5x_9) =< b, d_6(x_5), d_{10}(x_9) >=< a, b, b > .$$

This behavior actually shows us that these Massey products must be non-trivial, allowing us to distinguish rationally between the connect-sum of products of spheres and this homogeneous space.

Integrally, the story is much trickier. The action of the Steenrod algebra forces the class x_3 in the cohomology of $Sp(5)$ to map to $2x_3$ in the cohomology of $SU(5)$. A similar result holds for x_7. This makes the entire algebra more subtle, as there is a large amount of two torsion. On the other hand, there is only 2-torsion, so at other places, the computation is the same as the rational one.