

A MAY COMPUTATION OF $\text{Ext}_{\langle \mathcal{P}^1, \mathcal{P}^3 \rangle}(\mathbb{F}_3, \mathbb{F}_3)$

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1. INITIAL COMPUTATIONS

We start by filtering the dual of $\mathcal{A}(1)$, $\mathbb{F}_3[\xi_1, \xi_2]/(\xi_1^{p^2}, \xi_2^p)$. Let ξ_1 and ξ_1^p have filtration 1, and let ξ_2 have filtration 3. Assign filtrations to the rest of the elements by requiring that this be a filtration of algebras. The associated graded algebra is

$$\mathbb{F}_3[\xi_1, \xi_1^p, \xi_2]/(\xi_1^p, (\xi_1^p)^p, \xi_2^p),$$

and all of the generators are primitive. There is a spectral sequence starting with Ext over this algebra and converging to the Ext in question.

The spectral sequence starts very simply. The E_1 term is just Ext over the associated graded:

$$E_1 = \mathbb{F}_3[b_0, b_1, b_{20}](h_0, h_1, h_{20}),$$

where h_{ji} is represented in the bar complex as $[\xi_j^{p^i}]$ ($j = 1$ is suppressed) and b_{ji} is the 3-fold Massey product $\langle h_{ji}, h_{ji}, h_{ji} \rangle$.

The coproduct on ξ_2 in the dual to $\mathcal{A}(1)$ gives us our initial differential:

$$d_1(h_{20}) = h_0 h_1.$$

The elements

$$x_5 = h_0 h_{20}, \quad x_7 = h_1 h_{20}, \quad \text{and } h_0 h_1 h_{20}$$

all survive. These have more useful formulations as Massey products:

$$x_5 = \langle h_0, h_1, h_0 \rangle = \langle h_1, h_0, h_0 \rangle$$

$$x_7 = \langle h_1, h_0, h_1 \rangle = \langle h_0, h_1, h_1 \rangle.$$

From these it follows that

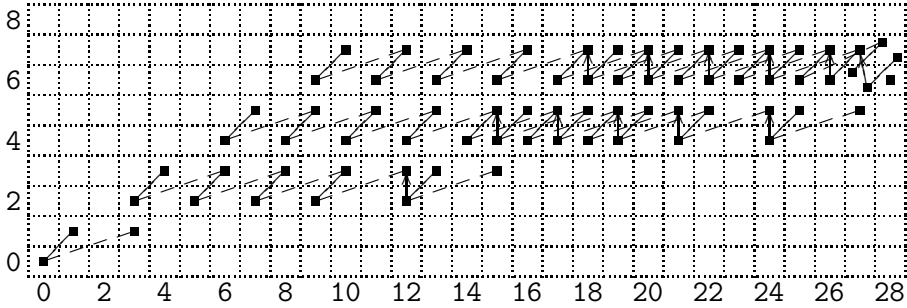
$$h_1 b_1 = h_0 x_5, \quad h_1 x_7 = h_0 b_2, \quad x_5^2 = b_1 x_7, \quad x_7^2 = b_2 x_5.$$

2. FIRST TRICKY DIFFERENTIAL

Recall that the Steenrod algebra acts on Ext in a way that is compatible with the differentials. We also know that $b_{20} = \beta \mathcal{P}^0(h_{20})$. This implies the following differential:

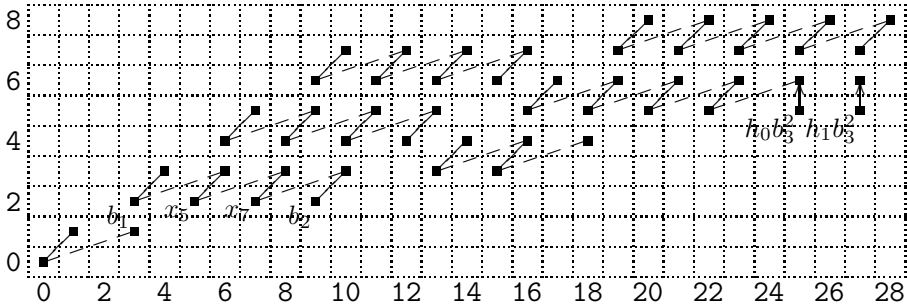
$$d_5(b_{20}) = \beta \mathcal{P}^0(d_1(h_{20})) = \beta \mathcal{P}^0(h_0) \mathcal{P}^0(h_1) + \mathcal{P}^0(h_0) \beta \mathcal{P}^0(h_1) = h_1 b_1.$$

This gives the following picture in which the y-axis is the cohomological degree and the x-axis is one quarter of the internal degree.



3. THE LAST DIFFERENTIAL

Adding in the multiplicative extensions given by Massey products, we get the following E_6 page:



A Massey product trick gives us the remaining differentials:

$$d_9(h_0b_3^2) = x_7b_2^2, \quad d_{10}(h_1b_3^2) = b_2^3.$$

For degree reasons, $E_{11} = E_\infty$, and since at each stage we resolved the extension questions using the Massey products at hand, we are done. A slightly more careful analysis shows that we also know $\text{Ext}_{\mathcal{A}(1)_*}$ as an algebra, again using our Massey product formulations of elements together with various shuffling lemmas. It should also be noted that similar considerations can be used to find Ext over larger quotients of the dual Steenrod algebra, and the same Massey product trick allows one to compute the differentials in the resulting spectral sequences.