THE EHP SEQUENCE AND THE GOODWILLIE TOWER

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1. Introduction

DISCLAIMER: This document is preliminary. Unlike an earlier version dated March 2004, which contained a terrible number of misprints, this version is edited. Its theorems (read: "conjectures") have no proofs. In fact, the statements of the theorems are not even correct, but they are morally correct. The computations at the end need to be double checked, and there is ambiguity with $\sigma\eta$ and ϵ that I need to sort out for myself, but have not done so yet.

The EHP sequence allows for the computation of the unstable homotopy of all of the spheres simultaneously. The Goodwillie tower of the identity functor allows

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for the computation of the unstable homotopy groups of a given sphere in terms of the stable homotopy groups of a collection of spectra, known as the derivatives of the identity. We will investigate a relationship amongst these spectra which mirrors the EHP sequence in the stable category. From this relationship we will seek to understand how the EHP and Goodwillie perspectives on unstable homotopy interact.

More specifically, the EHP spectral sequence E_1-term is a list of all of the possible Hopf invariants of elements of the unstable homotopy groups of spheres. The differentials are given by Whitehead products. The E_∞ term is a collection of all of the Hopf invariants of actual homotopy elements.

In Section 4 We will show that the Goodwillie tower is the opposite. Namely, the E_1 term of the Goodwillie spectral sequence is a list of all possible iterated Whitehead products. In Section 5, we will recover the essentially known result that the differentials in the Goodwillie spectral sequences are given by Hopf invariants. The E_∞ term consists of all of the iterated Whitehead products which exist (i.e. have the correct spheres of origin to be defined) and are non-trivial.

Our core tool will be a Goodwillie reinterpretation of a decomposition of the stunted Whitehead spectra $L(k)_n$ in terms of $L(k-1)_m$. This will be described in Section 3.

Kuhn, Mahowald, and others have investigated the relationship of the EHPSS to the AHSS for $\mathbb{R}P^\infty$. Specifically, many differentials in the EHPSS may be deduced from attaching maps in $\mathbb{R}P^\infty$. In Section 6 we will demonstrate that there is a similar relationship between the differentials of the EHPSS and the attaching maps between the stunted Whitehead spectra that build up the derivatives.

In Section 7 we give some tables of computations in low dimensions at the prime 2 to give the reader some feeling for how the theory of this paper plays out in practice. The range we work in is well-known. In fact, it is entirely contained in Toda's range.

This paper is (for the time being) written at the prime 2 only. Thus everything in this paper is implicitly 2-local. Unless expressly stated otherwise, all ordinary homology and cohomology groups are taken with $\mathbb{F}_2$-coefficients. In order to avoid confusion, we shall denote the sphere spectrum by $\underline{S}$.

The author benefited enormously from conversations with Greg Arone, Brenda Johnson, Nick Kuhn, Mark Mahowald, and Haynes Miller.

2. Recollections about calculus

In this section we give a brief recollection of Goodwillie calculus. Let *Spaces* be the category of pointed spaces and let F be a functor from *Spaces* to *Spaces*. We wish to assume that F is a homotopy functor, in the sense that it sends weak equivalences to weak equivalences, and that $F(*) = *$. The functor F is said to be n-excisive if it takes strong homotopy cocartesian $n+1$-cubes to homotopy cartesian $n+1$-cubes. If a functor is n-excisive, then it is $n+1$ excisive.

Goodwillie [5] has developed a formal machinery that associates a functor F a sequence of functors $P_i(F)$ which are the best i-excisive approximations to F. Dwyer reformulates this process as an instance of localization [4]. The universal property of $P_i(F)$ implies the existence of natural transformations

$$P_{i+1}(F) \to P_i(F).$$

In nice situations, one recovers the values of F as a homotopy inverse limit

$$F(X) \simeq \varprojlim P_i(F)(X)$$

for X sufficiently highly connected. We shall always be working with functors that merely require X to be connected for this to hold.

Define the i^{th} Goodwillie layer of F to be the homotopy fiber

$$D_i(F) \to P_i(F) \to P_{i-1}(F).$$

One has

$$D_1(F)(X) = P_1(F)(X) = \varinjlim \Omega^n F(\Sigma^n X).$$

In [5], Goodwillie constructs the functors $D_i(F)$ so that they comes equipped with a canonical infinite delooping. Thus, there exists a functor

$$\mathbb{D}_i(F) : \mathit{Spaces} \to \mathit{Spectra}$$

such that $D_i(F) = \Omega^\infty \mathbb{D}_i(F)$.

The theory is especially interesting when applied to the identity functor Id. For brevity of notation, we shall refer to $D_i(Id)(X)$ as $D_i(X)$, and $P_i(Id)(X)$ as $P_i(X)$. In this case, for a connected space X, we get an inverse system of spaces

$$\begin{array}{ccccccc}
P_1(X) & \longleftarrow & P_2(X) & \longleftarrow & P_3(X) & \longleftarrow & \cdots \\
\| & & \uparrow & & \uparrow & & \\
D_1(X) & & D_2(X) & & D_3(X) & &
\end{array}$$

where the fibers $D_i(X)$ are the zeroth spaces of spectra. We shall refer to the spectral sequence associated to applying homotopy groups to this tower as the *Goodwillie spectral sequence*, and will use the abbreviation GSS. The GSS takes the form

$$E_1(X) = \pi_*(\mathbb{D}_*(X)) \Rightarrow \pi_*(X).$$

Thus it computes the unstable homotopy groups of the space X from the stable homotopy groups of the spectra $\mathbb{D}_i(X)$.

The spectra $\mathbb{D}_i(S^n)$ are well understood, thanks to work of Brenda Johnson, Greg Arone, Marja Kankaanrinta, Mark Mahowald, and Bill Dwyer [6], [2], [3], [1]. In fact, these spectra first appeared in the context of a conjecture posed by George Whitehead and solved by Nick Kuhn, and in stable splittings of classifying spaces investigated by Steve Mitchell, Stewart Priddy, and others (see for instance, [7], [9], [12]).

A sequence of integers $(l_1, \ldots, l_k)$ is said to be *completely unadmissible* (CU) if for all i, $l_i > 2l_{i+1}$. In [3] the following is proven

Theorem 2.1 (Arone-Mahowald)**.** If i is not of the form 2^k for any k, then $\mathbb{D}_i(S^n)$ is contractible. If i is of the form 2^k, we have an equality

$$H^*(\Sigma^k \mathbb{D}_{2^k}(S^n)) = \mathbb{F}_2\{Q^{l_1} \cdots Q^{l_k} \iota_n \ : \ (l_1, \ldots, l_k) \text{ is CU}, l_k \geq n\}$$

as modules over the Steenrod algebra.

The cohomology module in Theorem 2.1 should be regarded as a submodule of the cohomology of the extended power spectrum $H^*(\mathcal{D}_{\mathbb{Z}/2^{\wr k}}(S^n))$ associated to the k-fold iterated wreath product of $\mathbb{Z}/2$. The Q^j are Dyer-Lashoff operations, and the Steenrod action is determined by the Nishida relations. Thus at the prime 2 one need only consider the values of the (2^k)th layers of the identity on spheres.

The Goodwillie tower for S^1 has been well studied. In this case, the homotopy groups of S^1 are known, and Kuhn's theorem may be rephrased to say that the GSS for S^1 collapses at E_2. More precisely, define spectra

$$L(k)_n = e_{st}(B\mathbb{Z}/2^k)^{n\xi_k}$$

where ξ_k is the bundle induced from the reduced regular representation of $\mathbb{Z}/2^k$, and e_{st} is a lift of the Steinberg idempotent in $\mathbb{Z}[GL_k(\mathbb{Z}/2)]$. Here, we get an induced summand of the homotopy orbit spectra $(B\mathbb{Z}/2^k)^{\xi_k}$ since $GL_k(\mathbb{Z}/2)$ is the normalizer of the subgroup $\mathbb{Z}/2^k$ in Σ_{2^k}.

The spectrum $L(k)_1$ is commonly referred to as $L(k)$, and the spectrum $L(k)_0$ is often called $M(k)$. There are splittings (Mitchell-Priddy?)

$$M(k) \simeq L(k) \vee L(k-1).$$

One of the main results of [1, Corollary 9.6] is given below

Theorem 2.2 (Arone-Dwyer)**.** There are equivalences

$$L(k)_n \simeq \Sigma^{k-n}\mathbb{D}_{2^k}(S^n)$$

In the case $n = 1$, we have [7], [1]

Theorem 2.3 (Kuhn-Arone-Dwyer)**.** There are equivalences

$$\Sigma^{-k} Sp_{2^k}(\underline{S}^0)/Sp_{2^{k-1}}(\underline{S}^0) \simeq L(k) \simeq \Sigma^{k-1} D_{2^k}(S^1)$$

where $Sp_n(\underline{S}^0)$ is the n^{th} symmetric power of the sphere spectrum.

There connecting homomorphisms of the triple

$$Sp_{2^k}(\underline{S}^0)/Sp_{2^{k-1}}(\underline{S}^0) \xrightarrow{\partial} \Sigma Sp_{2^{k-1}}(\underline{S}^0)/Sp_{2^{k-2}}(\underline{S}^0)$$

induce transfers on the derivatives (applying Theorem 2.3)

$$D_{2^k}(S^1) \to BD_{2^{k-1}}(S^1)$$

which gives a chain deformation retract of the complex

$$\pi_*(D_1(S^1)) \xrightarrow{d_1} \pi_{*-1}(D_2(S^1)) \xrightarrow{d_1} \pi_{*-2}(D_4(S^1)) \xrightarrow{d_1} \cdots$$

to the complex

$$\mathbb{Z} \to 0 \to 0 \to \cdots .$$

Thus Kuhn's theorem is the statement that the GSS for S^1 collapses at E_2.

3. Building the derivatives

The EHP sequence is based on the homotopy fiber sequence

$$S^n \xrightarrow{E} \Omega\Sigma S^n \xrightarrow{H} \Omega\Sigma Sq(S^n)$$

where E is the suspension, and H is the James-Hopf map. The functor Sq is the squaring functor

$$Sq : X \mapsto X \wedge X.$$

In [3], it is explained how from Goodwillie's formal theory of calculus, there is are fiber sequences on the derivatives of the functors evaluated on spheres

$$D_{2^k}(Id)(S^n) \xrightarrow{E_*} \Omega D_{2^k}(\Sigma)(S^n) \xrightarrow{H_*} \Omega D_{2^k}(\Sigma \circ Sq)(S^n).$$

General properties of calculus imply that

$$D_{2^k}(\Sigma)(X) \simeq D_{2^k}(Id)(\Sigma X)$$
$$D_{2^k}(\Sigma \circ Sq)(X) \simeq D_{2^{k-1}}(Id)(\Sigma X).$$

We therefore have the following proposition.

Proposition 3.1 (Arone–Mahowald)**.** The natural transformations $E : Id \to \Omega\Sigma$ and $H : \Omega\Sigma \to \Omega\Sigma Sq$ induce cofiber sequences

$$\mathbb{D}_{2^k}(S^n) \xrightarrow{E_*} \Sigma^{-1}\mathbb{D}_{2^k}(S^{n+1}) \xrightarrow{H_*} \Sigma^{-1}\mathbb{D}_{2^{k-1}}(S^{2n+1})$$

Remark 3.2. These cofiber sequences were shown to exist in the context of the spectra $L(k)_n$ independently by Nick Kuhn [8] and Shin-ichiro Takayasu [14]. In this language, the cofiber sequences take the form

$$\Sigma^n L(k-1)_{2n+1} \to L(k)_n \to L(k)_{n+1}.$$

Thus the spectra $L(k)$ have a decreasing filtration $\{L(k)_n\}$, whose filtration quotients are $L(k-1)_m$'s. The simplicity of the $L(k)_n$ notation will lead us to use it throughout this paper. In the $L(k)_n$ notation, the GSS for S^n takes the form

$$E_1^{k,l}(S^n) = \pi_l(L(k)_n) \Rightarrow \pi_{n+l-k}(S^n).$$

We shall now explain how to compute this E_1 term from the stable homotopy groups of spheres using a sequence of iterated Atiyah-Hirzebruch spectral sequences.

The cohomology of $L(k)_n$ will be given in the notation that was used in the statement of Theorem 2.1 except that we will drop the ι_n from the notation. Letting L represent the sequence $(l_1 \dots l_k)$, with length $|L| = k$ and degree $\|L\| = \Sigma l_i$, we may rewrite the conclusion of Theorem 2.1 as

$$H^*(L(k)_n) = \mathbb{F}_2\{Q^L \ : \ L \text{ is CU}, \ |L| = k, \ l_k \geq n\}.$$

The cofiber sequences of Proposition 3.1 (Remark 3.2) break up into short exact sequences

$$\begin{array}{ccccc} H^*(L(k)_{n+1}) & \longrightarrow & H^*(L(k)_n) & \longrightarrow & H^*(\Sigma^n L(k)_{2n+1}) \\ \| & & \| & & \| \\ \mathbb{F}_2\{Q^L \ : \ {\scriptstyle L \text{ is CU}, |L|=k \atop l_k \geq n+1}\} & \longrightarrow & \mathbb{F}_2\{Q^L \ : \ {\scriptstyle L \text{ is CU}, |L|=k \atop l_k \geq n}\} & \longrightarrow & \mathbb{F}_2\{Q^L Q^n \ : \ {\scriptstyle L \text{ is CU}, |L|=k-1 \atop l_k > 2n}\} \end{array}$$

The iterated filtrations of $L(k)_n$ in terms of $L(k-1)_{2j+1}$'s give a sequence of Atiyah-Hirzebruch type spectral sequences

$$E_1^{i,j}(L(k)_n) = \begin{cases} \pi_i(L(k-1)_{2j+1}) & j \geq n \\ 0 & j < n \end{cases} \Rightarrow \pi_{i+j}(L(k)_n).$$

Through k iterated AHSS's, we may begin from the stable stems and compute $\pi_*(L(k)_n)$.

$$\bigoplus_{\substack{L \\ L \text{ is CU}, |L|=k, l_k \geq n}} \pi_i(\underline{S}^{\|L\|}) \Rightarrow \cdots \Rightarrow \pi_i(L(k)_n)$$

We shall refer to this as the iterated AHSS for $L(k)_n$. We shall refer to elements in the first E_1 term by the index L which identifies its summand of origin. Given $\alpha \in \pi_i(\underline{S}^{\|L\|})$, we shall use the notation $\alpha[L]$ to refer to the corresponding element of the first E_1-term of the iterated AHSS.

4. The Goodwillie tower and Whitehead products

In this section, we would like to indicate the relationship of Whitehead products to the GSS. Suppose $\alpha \in \pi_{n+i}(S^n)$ is detected in the E_1 term of the GSS by $\beta \in \pi_{i+k}(L(k)_n)$, and suppose that β is detected in the iterated AHSS by the element

$$\gamma[j_1, \dots, j_k]$$

where γ is an element of $\pi_{i+k}(\underline{S}^{\|J\|})$. Loosely speaking, what we would like to prove is that α is given by the iterated Whitehead product

$$[\iota_{j_k}, [\iota_{j_{k-1}}, [\cdots [\iota_{j_1}, \widetilde{\gamma}] \cdots]$$

where $\widetilde{\gamma}$ is a suitable desuspension of γ. In fact, every term of the Whitehead product must be desuspended appropriately for the expression to make sense. The slogan here is:

> "Elements of the unstable homotopy groups of spheres in Goodwillie filtration 2^k break up into k-fold Whitehead products"

We would like to make the above statement precise and accurate. Let

$$P_n : \Omega^2 S^{2n+1} \to S^n$$

be the connecting homomorphism of the EHP sequence. The relation of P_n to Whitehead products is the following proposition (see [16, ?]).

Proposition 4.1. *If α is an element of $\pi_{i+n+1}(S^{2n+1})$, and there is an element $\widetilde{\alpha} \in \pi_i(S^n)$ so that $\alpha = E^{n+1}\widetilde{\alpha}$, then we have*

$$P_n(\alpha) = [\iota_n, \widetilde{\alpha}].$$

What we shall actually claim is that given certain GSS elements were permanent cycles, we have a 'formula'

$$\alpha = E^{-l_k} P_{j_k} E^{-l_{k-1}} P_{j_{k-1}} \cdots E^{-l_1} P_{j_1} \widetilde{\gamma}$$

where E is the suspension, l_m is the quantity $j_m - 2j_{m+1} - 1$ for $m < k$, and l_k is the quantity $j_k - n$.

Specifically, something like the following theorem should be true, but it is probably only morally true as stated.

Theorem 4.2 (**Relation between GSS representatives and Whitehead products).** *Suppose that $\alpha[J]$ is a permanent cycle in the iterated AHSS for $\pi_*(L(k)_n)$, where $J = (j_1, \dots, j_k)$ is a CU sequence with $j_k \geq n$. Let N_m be the quantity given by*

$$N_m = \begin{cases} 2j_{m+1} + 1, & \text{for } 0 \leq m < k \\ n, & \text{for } m = k \end{cases}$$

The elements

$$\alpha[j_1, \dots, j_m]$$

are permanent cycles in the iterated AHSS for $\pi_*(L(m)_{N_m})$. Suppose that each of these subsequence elements $\alpha[j_1, \dots, j_m]$ detect elements in $\pi_*(L(m)_{N_m})$ which are permanent cycles in the GSS for $\pi_*(S^{N_m})$. Then, for $0 \leq m \leq k$, there exist elements $\alpha_m \in \pi_*(L(m)_{N_m})$ and $\beta_m \in \pi_*(S^{N_m})$ so that

1. *the element α_m is detected by $\alpha[j_1, \dots, j_m]$ in the iterated AHSS for $\pi_*(L(m)_{N_m})$.*
2. *the element β_m is detected by α_m in the GSS for $\pi_*(S^{N_m})$.*

(3) modulo elements of Goodwillie filtration greater than 2^m, we have an equality
$$E^{j_m - N_m}\beta_m \equiv P_{j_m}\beta_{m-1}.$$

5. The Goodwillie differentials and Hopf invariants

In this section we will explain how to compute the differentials in the GSS. They are all Hopf invariants, as explained in [2] and elsewhere. In this section, we spell out this relation from a computational point of view.

We must begin by discussing various generalized notions of the Hopf invariant.

Definition 5.1 (**Unstable Hopf invariant**)**.** Given α in $\pi_i(\underline{S})$, suppose α has sphere of origin S^{m+1}. Let $\widetilde{\alpha} \in \pi_{i+m+1}(S^{m+1})$ be an unstable element whose stabilization is α. Then we define the *unstable Hopf invariant* $HI(\alpha)$ to be the coset of elements $H(\widetilde{\alpha}) \in \pi_{i+m+1}(S^{2m+1})$ for all choices of $\widetilde{\alpha}$.

The definition of the unstable Hopf invariant is summarized by the following diagram.

$$\begin{array}{ccc} \pi_i(\Omega^{m+1}S^{m+1}) & \xrightarrow{H_*} & \pi_i(\Omega^{m+1}S^{2m+1}) \\ \downarrow{\scriptstyle E^\infty} & \overset{HI}{\nearrow} & \downarrow{\scriptstyle E^\infty} \\ \pi_i(\underline{S}) & & \pi_i(\underline{S}^m) \end{array}$$

The unstable Hopf invariant is the coset of elements of the E_1 term of the EHPSS for $\pi_*(\underline{S})$ that detect α.

Let $SH : QS^0 \to Q\mathbb{R}P^\infty$ be the James-Hopf map associated to the Snaith splitting of QS^0. The unstable Hopf invariant should be compared to the following 'stable' analog. Let $\mathbb{R}P^\infty_m$ be the space $\mathbb{R}P^\infty/\mathbb{R}P^{m-1}$.

Definition 5.2 (**Stable Hopf invariant**)**.** Given α in $\pi_i(\underline{S})$, suppose that m is maximal so that the image of $SH(\alpha) \in \pi_i(Q\mathbb{R}P^\infty)$ in $\pi_i(Q\mathbb{R}P^\infty_m)$ is nontrivial. Let $\beta \in \pi_i(QS^m)$ be a lift of the image of $SH(\alpha)$ under the inclusion of the bottom cell of $Q\mathbb{R}P^\infty_m$. Then we define the *stable Hopf invariant* $SHI(\alpha)$ to be the coset of elements $\beta \in \pi_i(\underline{S}^m)$ for all choices of lifts β.

The definition of the stable Hopf invariant is summarized by the following diagram.

$$\begin{array}{ccc} \pi_i(Q\mathbb{R}P^\infty) & \longrightarrow & \pi_i(Q\mathbb{R}P^\infty_m) \\ \uparrow{\scriptstyle SH} & & \uparrow \\ \pi_i(\underline{S}) & \underset{SHI}{\rightsquigarrow} & \pi_i(\underline{S}^m) \end{array}$$

The stable Hopf invariant is the coset of elements of the E_1 term of the AHSS for $\pi_*(\Sigma^\infty\mathbb{R}P^\infty)$ that detect $SH(\alpha)$.

The relationship between the two follows from the relationship of the EHP sequence for $\pi_*(\underline{S})$ to the AHSS for $\pi_*(\Sigma^\infty\mathbb{R}P^\infty)$, and is given below.

Proposition 5.3. Let α be an element of $\pi_i(\underline{S})$. Suppose that the stabilization $E^\infty(HI(\alpha))$ does not contain 0. Then we have a containment
$$E^\infty HI(\alpha) \subseteq SHI(\alpha).$$

The GSS d_1 differentials will turn out to be related to the stable Hopf invariant when it agrees with the stabilization of the unstable Hopf invariant. The presence of higher GSS differentials will reflect when the coset $HI(\alpha)$ contains an unstable element. We define a generalized Hopf invariant that detects the Hopf invariant even when the stable Hopf invariant does not.

Definition 5.4 (**Generalized Hopf invariant).** Given α in $\pi_i(\underline{S})$, suppose α has sphere of origin S^{m+1}. Let $\widetilde{\alpha} \in \pi_{i+m+1}(S^{m+1})$ be an unstable element whose stabilization is α such that the the Goodwillie filtration of $H(\widetilde{\alpha})$ is maximal. Let 2^r be this maximal Goodwillie filtration. We shall call r the *Hopf depth* of α. Consider the following diagram

$$\begin{array}{ccccc}
\pi_i(\Omega^{m+1}S^{m+1}) & \xrightarrow{H_*} & \pi_i(\Omega^{m+1}S^{2m+1}) & \xrightarrow{p_{2^r}} & \pi_i(\Omega^{m+1}P_{2^r}(S^{2m+1})) \\
{\scriptstyle E^\infty}\downarrow & & & & \uparrow \\
\pi_i(\underline{S}^0) & & \overset{GHI}{\rightsquigarrow} & & \pi_i(\Omega^{m+1}D_{2^r}(S^{2m+1}))
\end{array}$$

where p_{2^r} is projection onto the (2^r)th term of the Goodwillie tower. Then the *generalized Hopf invariant*

$$GHI(\alpha) \subseteq \pi_i(\Omega^{m+1}D_{2^r}(S^{2m+1})) = \pi_{i-m+r}(L(r)_{2m+1})$$

is the set of all lifts of $p_{2^r}H(\widetilde{\alpha})$ to the Goodwillie derivative $D_{2^r}(S^{2m+1})$ for all such $\widetilde{\alpha}$.

We shall index the GSS as follows

$$E_1^{k,l}(S^n) = \pi_l(L(k)_n) \Rightarrow \pi_{n+l-k}(S^n).$$

Let $\alpha[j_1, \dots, j_k]$ be a permanent cycle in the iterated AHSS for $\pi_*(L(k)_n)$, where $J = (j_1, \dots, j_k)$ is CU and α is an element of $\pi_l(\underline{S}^{\|J\|})$. Then $\alpha[J]$ detects an element $\beta \in \pi_l(L(k)_n) = E_1^{k,l}(S^n)$.

Theorem 5.5 (**Relation of GSS differentials to Hopf invariants).** Suppose that β survives to $E_r^{k,l}$. Then there exists an element $\alpha[J]$ which detects β in the iterated AHSS for $\pi_*(L(k)_n)$ so that one of the following possibilities occurs.

Case I: *The Hopf depth of* α *is greater than* $r-1$.
Then $d_r(\beta)$ is detected in the iterated AHSS for $\pi_*(L(k+r)_n)$ by an element $\gamma[h_1, \dots, h_{r+s}]$ where $h_{r+s} < j_k$.

Case II: *The Hopf depth of* α *equals* $r-1$.
Then we have $GHI(\alpha) \subseteq \pi_{l-m+r-1}(\Sigma^{\|J\|}L(r-1)_{2m+1})$.

Subcase (a): $m < 2j_1 + 1$
Then the same conclusion for Case I holds.

Subcase (b): $m \geq 2j_1 + 1$
Then there exists an element $\nu[i_1, \dots, i_{r-1}]$ in the iterated AHSS for $\pi_*(L(r-1)_{2m+1})$ which detects an element of $GHI(\alpha)$ and such that the element

$$\nu[i_1, \dots, i_{r-1}, m, j_1, \dots, j_k]$$

of the iterated AHSS detects $d_r(\beta) \in \pi_{l+r-1}(L(k+r)_n)$.

Theorem 5.5 may be summarized (in Case II, Subcase (b)) by saying that the following 'diagram' commutes.

$$\begin{array}{ccc} \pi_*(L(k)_n) & \overset{d_r^{GSS}}{\rightsquigarrow} & \pi_*(L(k+r)_n) \\ \Uparrow \scriptstyle{\text{iterated AHSS}} & & \Uparrow \scriptstyle{\text{iterated AHSS}} \\ \pi_*(S^{\|J\|}) & \underset{GHI}{\rightsquigarrow} & \pi_*(\Sigma^{\|J\|} L(r-1)_{2m+1}) \end{array}$$

We end this discussion with an indication of how to proceed if Theorem 5.5 fails to give any information.

Let us first consider the GSS for S^1.

$$E_1^{k,l}(S^1) = \pi_l(L(k)_1) \Rightarrow \pi_{l-k+1}(S^1)$$

Kuhn's theorem implies that this spectral sequence collapses at E_2. Assume that we are able to compute $\pi_*(L(k)_1)$ through a range, and that we know all of the stable Hopf invariants of all of the elements of the stable stems through a range. Then one has a good chance of knowing the complete structure of this spectral sequence. Theorem 5.5 implies that most of the d_1's are given as stable Hopf invariants. Since $\pi_*(S^1)$ is concentrated in the zero stem, the remaining d_1's may be deduced. We shall use this methodology to compute the S^1 GSS in the Toda range in Section 7.

We shall now explain how to compute the differentials in the GSS for S^n that are not covered by Theorem 5.5. We shall relate these more difficult differentials to differentials in the GSS for S^1, which by the last paragraph may be deduced in favorable cases from Kuhn's theorem, and other certain generalized Hopf invariants. The following proposition covers all of the cases found in the Toda range (see Section 7. Let $q(k)_n : L(k)_1 \to L(k)_n$ be the quotient map.

Theorem 5.6. Suppose that β is an element of the GSS E_1-term $E_1^{k,l}(S^n) = \pi_l(L(k)_n)$. Suppose that $\alpha[J]$ detects β in the iterated AHSS for $\pi_*(L(k)_n)$. Then there are two possibilities.

- **Case I:** *The element $\alpha[J]$ is a permanent cycle in the AHSS for $L(k)_1$.* Then there exists an element $\widetilde{\beta} \in \pi_l(L(k)_1)$ for which $q(k)_n(\widetilde{\beta})$ equals β modulo elements of lower iterated Atiyah-Hirzebruch filtration, and we have a formula of GSS differentials

$$d_r(\beta) \equiv q(k+r)_n(d_r(\widetilde{\beta}))$$

  modulo d_r(elements of lower Atiyah-Hirzebruch filtration).
- **Case II:** *The element $\alpha[J]$ is not a permanent cycle in the AHSS for $L(k)_1$* Then there is a non-trivial AHSS differential $d(\alpha[J]) = \alpha'[J']$. Then there exists an element $\gamma[H]$ in the iterated AHSS for $\pi_*(L(k+r)_n)$ which detects $d_r(\beta)$, and such that there is an AHSS differential

$$d(\gamma[H]) = \gamma'[I, m, J']$$

  where α' is born on S^{m+1} with $\gamma'[I] \in GHI(\alpha')$.

Theorem 5.6, Case I, may be summarized by saying that the following diagram commutes.

$$\begin{array}{ccc}
\pi_*(L(k)_n) & \overset{d_r^{GSS}}{\rightsquigarrow} & \pi * (L(k+r)_n) \\
{\scriptstyle q(k)_n}\uparrow & & \uparrow{\scriptstyle q(k+r)_n} \\
\pi_*(L(k)_1) & \underset{d_r^{GSS}}{\rightsquigarrow} & \pi_*(L(k+r)_1)
\end{array}$$

Theorem 5.6, Case II, may be summarized by saying that the following 'diagram' commutes.

$$\begin{array}{ccc}
\pi_*(L(k)_n) & \overset{d_r^{GSS}}{\rightsquigarrow} & \pi_*(L(k+r)_n) \\
\Uparrow {\scriptstyle AHSS} & & \Uparrow {\scriptstyle AHSS} \\
\pi_*(L(k-1)_{2j_k+1}) & & \pi_*(L(k+r-1)_{2h_{k+r}+1}) \\
{\scriptstyle d^{AHSS}_{j_k-j'_k}} \wr\downarrow & & \wr\downarrow {\scriptstyle d^{AHSS}_{h_{k+r}-j'_k}} \\
\pi_*(L(k-1)_{2j'_k+1}) & & \pi_*(L(k+r-1)_{2j'_k+1}) \\
\Uparrow {\scriptstyle \substack{iterated \\ AHSS}} & & \Uparrow {\scriptstyle \substack{iterated \\ AHSS}} \\
\pi_*(S^{\|J'\|}) & \underset{GHI}{\rightsquigarrow} & \pi_*(\Sigma^{\|J'\|} L(r-1)_{2m+1})
\end{array}$$

6. EHP differentials and attaching maps

In this section we will describe how differentials in the EHPSS may be deduced from differentials in the iterated AHSS.

The EHPSS takes the form

$$E_1^{i,j} = \pi_{j+1}(S^{2i+1}) \Rightarrow \pi_{j-i}(QS^0).$$

We shall refer to elements γ in the E_1-term $\pi_{j+1}(S^{2i+1})$ of the EHPSS by elements β in $\pi_{j+k+1}(L(k)_{2i+1})$ which detect them in the GSS.

We observe that these representatives may actually be read off from the data of the EHPSS while executing the Curtis algorithm by Theorem 4.2.

In the following theorem, we shall be working with the AHSS indexed by

$$E_1^{i,j}(L(k+1)_1) = \pi_j(L(k)_{2i+1}) \Rightarrow \pi_{j+i}(L(k+1)_1).$$

Elements of the $E_1^{i,j}(L(k+1)_1)$ will be given by the notation $\beta[i]$, where β is an element of $\pi_j(L(k)_{2i+1})$.

Theorem 6.1 (Relation of EHPSS differentials to AHSS differentials). Suppose that $\gamma \in \pi_{j+1}(S^{2i+1})$ survives to the E_r-term in the EHPSS and let $\beta \in \pi_{j+k+1}(L(k)_{2i+1})$ be an element of the GSS which detects γ. Suppose that there is a differential

$$d_r(\beta[i]) = \beta'[i-r]$$

in the AHSS for $L(k+1)_1$. Suppose that β' is a permanent cycle in the GSS for $S^{2(i-r)+1}$. Then modulo elements of higher Goodwillie filtration, the EHPSS differential $d_r(\gamma)$ is detected by β' in the GSS.

Theorem 6.1 may be summarized by saying that the following diagram commutes.

$$\begin{array}{ccc} \pi_*(S^{2i+1}) & \overset{d_r^{EHP}}{\rightsquigarrow} & \pi_*(S^{2(i-r)+1}) \\ \Uparrow {\scriptstyle GSS} & & \Uparrow {\scriptstyle GSS} \\ \pi_*(L(k)_{2i+1}) & \underset{d_r^{AHSS}}{\rightsquigarrow} & \pi_*(L(k)_{2(i-r)+1}) \end{array}$$

We shall now address what happens when the GSS element β' in Theorem 6.1 is not a permanent cycle in the GSS.

Theorem 6.2. Suppose that γ, β, β' are the same as in the statement of Theorem 6.1, except that β' is not a permanent cycle in the GSS. Suppose that there is a non-trivial GSS differential

$$d_{r'}(\beta') = \nu'$$

for $\nu' \in \pi_*(L(k+r')_{2(i-r)+1})$. Then there exists an element

$$\nu \in \pi_*(L(k+r')_{2(i-r+r'')+1})$$

so that in the AHSS for $\pi_*(L(k+r'+1)_1)$ we have $d_{r''}(\nu) = \nu'$ and if ν is a permanent cycle in the GSS for $S^{2(i-r+r'')+1}$ and γ persists to $E_{r-r''}$, then ν detects the EHPSS differential $d_{r-r''}(\gamma)$.

Theorem 6.2 may be summarized by saying that the following 'diagram' commutes.

$$\begin{array}{ccc} \pi_*(S^{2i+1}) & \overset{d_{r-r''}^{EHP}}{\rightsquigarrow} & \pi_*(S^{2(i-r+r'')+1}) \\ \Uparrow {\scriptstyle GSS} & & \Uparrow {\scriptstyle GSS} \\ \pi_*(L(k)_{2i+1}) & & \pi_*(L(k+r')_{2(i-r+r'')+1}) \\ {\scriptstyle d_r^{AHSS}} \wr\downarrow & & \wr\downarrow {\scriptstyle d_{r''}^{AHSS}} \\ \pi_*(L(k)_{2(i-r)+1}) & \underset{d_{r'}^{GSS}}{\rightsquigarrow} & \pi_*(L(k+r')_{2(i-r)+1}) \end{array}$$

We shall now address what happens when the GSS element β' in Theorem 6.1 is killed in the GSS.

Theorem 6.3. Suppose that γ, β, and β' are the same as in the statement of Theorem 6.1, and that β' is killed in the GSS for $S^{2(i-r)+1}$ by a differential

$$d_{r'}(\nu) = \beta'$$

for $\nu \in \pi_*(L(k-r')_{2(i-r)+1})$. Suppose that ν supports a nontrivial $d_{r''}$ in the AHSS for $\pi_*(L(k-r'+1))$. Then if $d_{r''}(\nu) \in \pi_*(L(k-r')_{2(i-r-r'')+1}$ is a permanent cycle in the GSS for $\pi_*(S^{2(i-r-r'')+1})$, then it detects the EHPSS differential $d_{r+r''}(\gamma) \in \pi_*(S^{2(i-r-r'')+1})$.

Theorem 6.3 is summarized by saying that the following 'diagram' commutes.

$$
\begin{array}{ccc}
\pi_*(S^{2i+1}) & \overset{d^{EHP}_{r+r''}}{\rightsquigarrow} & \pi_*(S^{2(i-r-r'')+1}) \\
\Uparrow{\scriptstyle GSS} & & \Uparrow{\scriptstyle GSS} \\
\pi_*(L(k)_{2i+1}) & & \pi_*(L(k-r')_{2(i-r-r'')+1}) \\
{\scriptstyle d_r^{AHSS}}\big\downarrow & & \big\uparrow{\scriptstyle d_{r''}^{AHSS}} \\
\pi_*(L(k)_{2(i-r)+1}) & \underset{d^{GSS}_{r'}}{\leftsquigarrow} & \pi_*(L(k-r')_{2(i-r)+1})
\end{array}
$$

Finally, we address a possible source of differentials when β is a permanent cycle in the AHSS for $L(k+1)$.

Theorem 6.4. Suppose γ and β are the same as in the statements of Theorems 6.1 and 6.2, but that β is a permanent cycle in the AHSS for $L(k+1)$. Then there exists an element $\mu \in \pi_{i+j+k+1}(L(k+1))$ such that β detects μ in the AHSS, and an element $\beta' \in \pi_{i+j+k-l+1}(L(k+1)_{2l+1})$ which detects the GSS differential $d_1(\mu)$ in the AHSS for $L(k+2)$, so that if β' is a permanent cycle in the GSS for S^{2l+1}, then β' detects the EHPSS differential $d_{i-l}(\gamma)$.

Note that the GSS differential $d_1(\mu)$ in Theorem 6.4 is in the GSS for S^1, and by Kuhn's theorem, there are no higher differentials. Theorem 6.4 may be summarized by saying that the following 'diagram' commutes.

$$
\begin{array}{ccc}
\pi_*(S^{2i+1}) & \overset{d^{EHP}_{i-l}}{\rightsquigarrow} & \pi_*(S^{2l+1}) \\
\Uparrow{\scriptstyle GSS} & & \Uparrow{\scriptstyle GSS} \\
\pi_*(L(k)_{2i+1}) & & \pi_*(L(k+1)_{2l+1}) \\
\Downarrow{\scriptstyle AHSS} & & \Downarrow{\scriptstyle AHSS} \\
\pi_*(L(k+1)) & \xrightarrow[d_1^{GSS}]{} & \pi_*(L(k+2))
\end{array}
$$

7. Some low dimensional calculations

In this section we will apply our observations to give a complete understanding of the unstable 2-primary homotopy groups of spheres in the Toda range (up to the 20-stem).

In Section 7.1, we give low dimensional calculations of the AHSS's which compute $\pi_*(L(k)_n)$ in terms of $\pi_*(L(k-1)_{2m+1})$. This data will be used to give the E_1-terms of the GSS's, and to give the differentials in the EHPSS. In Section 7.2 we compute the GSS for S^1. We use Kuhn's theorem to deduce from this the stable Hopf invariants. In Section 7.3 we compute the EHPSS in the Toda range, using our AHSS differentials, together with some stable Hopf invariants. In Section 7.4 we compute the GSS for S^2 to illustrate our methods. We use the knowledge of generalized Hopf invariants given to us by the EHPSS to compute the differentials.

7.1. **AHSS calculations.** The AHSS for $\pi_*(L(s))$ is given $1 \le s \le 3$ in Tables 1, 2, and 3. These AHSS's take the form

$$E^1_{k,n} = \pi_k(L(s-1)_{2n+1}) \Rightarrow \pi_{k+n}(L(s)).$$

Thus, the E_1-terms of these spectral sequences are the E_∞-terms of the previous spectral sequences. These spectral sequences may be truncated to converge to $\pi_*(L(s)_m)$ by setting $E^1_{*,n} = 0$ for $n < m$. Note that $L(0)_{2n+1} = \underline{S}$.

The terms are given in terms of an $\mathbb{F}_2$-basis of the associated graded with respect to the 2-adic filtration. The notation

$$x(2^m)[n_1, \dots, n_s]$$

represents the span of the elements

$$\{x[N], 2x[N], 4x[N], \dots, 2^{m-1}[N]\}$$

where N is a completely unadmissible sequence of length s, and x is an element of the stable stems. If $m = 1$, then the "(2^m)" in the notation is omitted.

We list all elements which are not the targets of differentials. If an element supports a non-trivial differential, then the differential is represented by an arrow which points toward the target.

We use standard terminology for the generators of the stable stems, most of which originates with Toda [15]. The generators of the v_1-periodic homotopy groups are given using a variation on the notation of Ravenel [13]. The element $\alpha_{k/l}$ is a v_1-periodic element of order 2^l in the $(2k-1)$-stem.

The most difficult of these computations is the computation of $\pi_*(L(1))$ given in Table 1. The differentials on the v_1-periodic elements are mostly deduced from the J-homology AHSS for P^∞, as computed in [10].

The remaining differentials may be deduced in this range from the dual action of the Steenrod algebra on $H^*(L(1))$. For instance, in the $k = 5$ list, the formula

$$Sq^2_* Q^4 = Q^2$$

implies the AHSS differential

$$d_2(\eta[4]) = \eta \cdot \eta[2] = \eta^2[2]$$

and in $k = 12$, the formula

$$Sq^2_* Sq^1_* Q^6 = Q^4 Sq^1_* + Q^3$$

implies the AHSS differential

$$d_3(\nu^2[6]) = \langle \eta, 2, \nu^2 \rangle [3] = \epsilon[3].$$

The other AHSS's are actually much less complicated. The Steenrod operations on $H_*(L(s))$ are given by the Nishida relations. There are subtleties associated with the fact that we are working with an iterated associated graded as the input of the E_1 terms at every stage. We give an example that nicely illustrates how to interpret things. We look at the AHSS for $L(2)$, $k = 19$. The Nishida relations give

$$Sq^4_* Q^9 Q^4 = Q^7 Q^3 Sq^1_* + Q^7 Q^2 + Q^5 Q^4.$$

The last term should be ignored because it is not completely unadmissible. We would initially think that this should give an AHSS differential

$$d_4(\nu^2[9,4]) = \nu^3[7,2] = \sigma\eta^2[7,2]$$

except that $\sigma\eta^2[7]$ is null in $\pi_*(L(1)_7)$. In the AHSS for $L(1)$, $\sigma\eta^2[7]$ is killed by the differential $d(\sigma\eta[9])$ induced from

$$Sq^2_* Q^9 = Q^8 Sq^1_* + Q^7.$$

However, something more is going on in $L(2)$. The Nishida relations give

$$Sq^2_* Q^9 Q^2 = Q^8 Q^1 + Q^7 Q^2$$

which translate into the fact that in $L(2)$, we have

$$d(\sigma\eta[9,2]) = \sigma\eta^2[8,1] + \sigma\eta^2[7,2]$$

so the terms in the RHS are equated. Thus we actually have

$$d(\nu^2[9,4]) = \sigma\eta^2[7,2] = \sigma\eta^2[8,1].$$

Now, actually, since we are using Steenrod operations, all of the above formulas are really only true modulo higher Adams filtration. We actually have

$$d(\nu^2[9,4]) = (\sigma\eta^2 + \epsilon\eta)[8,1].$$

7.2. **Calculation of the GSS for S^1 and Kuhn's theorem.** The GSS for $\pi_*(S^1)$ is given in Table 5.

$$E_1^{k,l} = \pi_l(L(k)) \Rightarrow \pi_{l-k+1}(S^1)$$

We know that $\pi_*(S^1)$ is concentrated in degree 1, and by Kuhn's theorem, the only differentials it has are d_1's.

The E_1 term may be read off of Tables 1, 2 and 3. The other $L(k)$'s are too highly connected to contribute anything in our range of computation. The d_1's originating in the first column of Table 5 are exactly the stable Hopf invariants. For example, we have $SHI(\eta) = 1$ and

$$d_1(\eta) = 1[1].$$

Most of the other d_1's are also given by stable Hopf invariants. The ones that are not are indicated by dashed arrows in Table 5. They are deduced from Kuhn's theorem — they are necessary to make the spectral sequence acyclic.

Actually, in this range, these stable Hopf invariants are rather easy to deduce from the Hopf invariants of η, ν, and σ and the attaching map structure in $L(1)$. We also used the relation of Hopf invariants to root invariants [11].

7.3. **Calculation of the EHPSS.** Table 4 displays the EHPSS in the Toda range.

$$E^1_{n,k} = \pi_{k+n+1}(S^{2n+1}) \Rightarrow \pi_k(\underline{S})$$

Table 4 consists of lists of the permanent cycles in the $E^1_{*,k}$-term of the EHPSS. If an element is the target of a differential, then this is displayed with an left arrow followed by the source of the differential. If an element is not the target of a differential, then it is boxed, and a double right arrow gives the name of the element of the stable stems that it detects in the EHPSS. One can read the chart backwards to read off the (unstable) Hopf invariants of elements of the elements of the stable stems.

The elements in the $E^1_{n,k}$-term of the EHPSS are given with the notation

$$x(2^m)[n_1, \ldots, n_s, n]$$

where x is an element of the stable stems, and $N = (n_1, n_2, \ldots, n_s, n)$ is a completely unadmissible sequence. The parenthetical (2^m) is the order of the element, and this notation is omitted if $m = 1$. There are two interpretations of this notation. Let $\overline{N}$

be the subsequence $(n_1, \ldots, n_s)$. Then one could use the iterated AHSS followed by the GSS to compute the unstable homotopy groups give the $E^1_{n,k}$-term.

$$\pi_{k-n+s}(\underline{S}^{\|\overline{N}\|}) \xrightarrow[AHSS]{iterated} \pi_{k-n+s}(L(s)_{2n+1}) \xRightarrow[GSS]{} \pi_{k+n+1}(S^{2n+1})$$

Then the element $x \in \pi_{k-n+s}(\underline{S}^{\|\overline{N}\|})$ detects the element $x[N]$ of $E^1_{n,k}$.

Here is an alternative point of view on the notation. One can truncate the EHPSS to make it converge to the unstable homotopy groups of a sphere. Setting $E^1_{n,k}(S^m)$ equal to $E^1_{n,k}$ for $n < m$ and zero for $n \geq m$ gives the S^m-EHPSS

$$E^1_{n,k}(S^m) \Rightarrow \pi_{k+m}(S^m).$$

Then the element $x[N] \in \pi_{k+n+1}(S^{2n+1})$ above has the following meaning. If $s = 0$, then $x[N]$ has a nontrivial image in the stable stems, and we have, under the stabilization map

$$\begin{array}{c} \pi_{k+n+1}(S^{2n+1}) \xrightarrow{E^\infty} \pi_{k-n}(\underline{S}) \\ x[n] \mapsto x \end{array}$$

Otherwise, $x[N]$ is unstable, and and is detected by an element of the EHPSS which is the target of a differential. Then there is a zig-zag

$$\begin{array}{cccccccc}
x[n_1, \ldots, n_s, n] & \xLeftarrow[EHPSS]{S^{2n+1}} & y_1[J_1] & & & & & \\
 & & \uparrow d^{EHP} & & & & & \\
 & & x[n_1, \ldots, n_s] & \xLeftarrow[EHPSS]{S^{2n_s+1}} & y_2[J_2] & & & \\
 & & & & \uparrow d^{EHP} & & & \\
 & & & & & \ddots & & \\
 & & & & & & \xLeftarrow[EHPSS]{S^{2n_2+1}} & y_s[J_s] \\
 & & & & & & & \uparrow d^{EHP} \\
 & & & & & & & x[n_1]
\end{array}$$

where $x[n_1]$ is stable. That this description agrees with the previous description is the content of Theorem 4.2.

We give an example. In $k = 15$ there is an element

$$\nu[9, 4, 1] \in E^1_{1,15} = \pi_{17}(S^3).$$

One can compute the GSS for S^3 using the techniques of Section 7.4, and one finds that this element is detected by the element

$$\nu[9, 4] \in \pi_{16}(\underline{S}^{9+4})$$

in the iterated AHSS followed by the GSS. On the other hand, there is a there is the following zig-zag.

$$\begin{array}{ccccc}
\nu[9,4,1] & \xLeftarrow[EHPSS]{S^3} & \sigma\eta[5,2] & & \\
 & & \uparrow{\scriptstyle d^{EHP}} & & \\
 & & \nu[9,4] & \xLeftarrow[EHPSS]{S^9} & \nu^2[5] \\
 & & & & \uparrow{\scriptstyle d^{EHP}} \\
 & & & & \nu[9]
\end{array}$$

The EHPSS is completed by means of the Curtis algorithm. A nice description of this algorithm appears in [13]. The algorithm says that you can compute the entire spectral sequence if you can compute differentials. This is where Theorems 6.1, 6.2, 6.3 and 6.4 come into play. All of the EHPSS differentials in the Toda range are given by one of these theorems. In Table 4, differentials with no markings arise from the application of Theorem 6.1, whereas differentials marked with $(*)$, $(**)$, or $(***)$ arise from the application of Theorems 6.2, 6.3, or 6.4, respectively.

We illustrate the applications of these theorems with specific examples. In $k = 5$, there is an EHPSS differential

$$d^{EHP}(1[5,2]) = \eta[4,1].$$

This differential arises from the differential in the $L(2)$-AHSS (see Table 2, $k = 7$)

$$d^{AHSS}(1[5,2]) = \eta[4,1]$$

by Theorem 6.1

In $k = 12$ there is an EHPSS differential

$$d^{EHP}(\eta\epsilon[4]) = \eta^3[8,2].$$

This differential arises from the system of AHSS and GSS differentials

$$\begin{array}{ccc}
\eta\epsilon[4] & & \eta^3[8,2] \\
{\scriptstyle d^{AHSS}}\downarrow & & \downarrow{\scriptstyle d^{AHSS}} \\
\alpha_{6/3}[1] & \xrightarrow[d^{GSS}]{} & 8\sigma[4,1]
\end{array}$$

by Theorem 6.2.

In $k = 19$, there is an EHPSS differential

$$d^{EHP}(\eta^2[13,6]) = \eta\alpha_{8/5}[3].$$

This differential arises from the system of AHSS and GSS differentials

$$\begin{array}{ccc}
\eta^2[13,6] & & \eta\alpha_{8/5}[3] \\
{\scriptstyle d^{AHSS}}\downarrow & & \uparrow{\scriptstyle d^{AHSS}} \\
\eta^3[12,5] & \xleftarrow[d^{GSS}]{} & \alpha_{8/5}[5]
\end{array}$$

by Theorem 6.3. This is actually the only occurrence of a differential arising in this manner in the Toda range, and it also represents the only differential in this range that *decreases* Goodwillie filtration.

In $k = 17$ there is an EHPSS differential

$$d^{EHP}(\theta_3[4]) = 1[15,3].$$

This differential arises from the S^1-GSS differential

$$d^{GSS}(\theta_3[4]) = 1[15,3]$$

by Theorem 6.4.

7.4. **GSS calculations.** Table 6 displays a calculation of the GSS for S^2

$$E_1^{k,l} = \pi_l(L(k)_2) \Rightarrow \pi_{l-k+2}(S^2).$$

in the Toda range.

The E_1-term may be read off of truncated versions of Tables 1, 2, and 3.

The differentials all arise from the application of Theorems 5.5 (no marks) and 5.6 (indicated by $(**)$ for Case I, and $(*)$ for Case II). We must use our EHPSS calculations to compute generalized Hopf invariants.

We give examples of the application of each of these theorems. In $n = 16$ there is a GSS differential

$$d_2^{GSS}(\eta\alpha_{8/5}) = \eta^2[13,2].$$

This arises from the fact that the unstable Hopf invariant $HI(\eta\alpha_{8/5})$ is detected by $\eta^2[13] \in \pi_*(L(1)_5)$ in the GSS for S^5 by Theorem 5.5. The stable element $\eta\alpha_{8/5}$ has an *unstable* Hopf invariant, and this is precisely the mechanism which results in higher GSS differentials.

In $n = 17$ there is a GSS differential

$$d_1^{GSS}(\theta_3[4]) = 1[15,3].$$

This arises from the same differential happening in the S^1 GSS, by Theorem 5.6, Case I. Note that there was a similar anomalous EHPSS differential given in Section 7.3 in the sample application of Theorem 6.4.

In $n = 12$ there is a GSS differential

$$d_1^{GSS}(\epsilon\eta[4]) = \eta^3[8,2].$$

This arises from the application of Theorem 5.6, Case II to the same system of AHSS and GSS differentials that gave us

$$d^{EHP}(\epsilon\eta[4]) = \eta^3[8,2]$$

in Section 7.3. In general the same input goes into both Theorem 5.6, Case II, and Theorem 6.2.

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Table 1. The AHSS for $\pi_k(L(1))$

$\underline{k=1}$
$1(\infty)[1]$

$\underline{k=2}$
$\eta[1]$
$1(\infty)[2] \to 2(\infty)[1]$

$\underline{k=3}$
$\eta^2[1]$
$\eta[2]$
$1[3]$

$\underline{k=4}$
$\nu[1]$
$1(\infty)[4] \to 2(\infty)[3]$

$\underline{k=5}$
$\nu(4)[2] \to 2\nu(4)[1]$
$\eta[4] \to \eta^2[2]$
$1[5] \to \eta[3]$

$\underline{k=6}$
$\nu[3]$
$\eta^2[4] \to 4\nu[2]$
$\eta[5] \to \eta^2[3]$
$1(\infty)[6] \to 2(\infty)[5]$

$\underline{k=7}$
$\nu^2[1]$
$\nu(4)[4] \to 2\nu(4)[3]$
$4\nu[4]$
$\eta^2[5]$
$\eta[6]$
$1[7]$

$\underline{k=8}$
$\sigma[1]$
$\nu^2[2]$
$\nu[5]$
$1(\infty)[8] \to 2(\infty)[7]$

$\underline{k=9}$
$\sigma\eta[1]$
$\epsilon[1]$
$\sigma(8)[2] \to 2\sigma(8)[1]$
$8\sigma[2]$
$\nu^2[3]$
$\nu(4)[6] \to 2\nu(4)[5]$
$\eta[8] \to \eta^2[6]$
$1[9] \to \eta[7]$

$\underline{k=10}$
$\sigma\eta^2[1]$
$\alpha_5[1]$
$\sigma\eta[2]$
$\sigma[3]$
$\eta^2[8] \to 4\nu[6]$
$\eta[9] \to \eta^2[7]$
$1(\infty)[10] \to 2(\infty)[9]$

$\underline{k=11}$
$\eta\alpha_5[1]$
$\alpha_5[2]$
$\sigma(8)[4] \to 2\sigma(8)[3]$
$8\sigma[4]$
$\nu(4)[8] \to 2\nu(4)[7]$
$4\nu[8] \to \epsilon\eta[1]$
$\eta^2[9] \to \epsilon[2]$
$\eta[10] \to \nu^2[4]$
$1[11] \to \nu[7]$

$\underline{k=12}$
$\sigma\eta[4] \to \sigma\eta^2[2]$
$\epsilon[4] \to \epsilon\eta[2]$
$\sigma[5] \to \sigma\eta[3]$
$\nu^2[6] \to \epsilon[3]$
$\nu[9] \to \nu^2[5]$
$1(\infty)[12] \to 2(\infty)[11]$

$\underline{k=13}$
$\alpha_{6/3}(4)[2] \to \alpha_{6/2}(4)[1]$
$\epsilon\eta[4] \to \alpha_{6/3}[1]$
$\alpha_5[4] \to \alpha_5\eta[2]$
$\sigma\eta[5] \to \sigma\eta^2[3]$
$\epsilon[5] \to \epsilon\eta[3]$
$\sigma(8)[6] \to 2\sigma(8)[5]$
$8\sigma[6] \to \alpha_5[3]$
$\nu(4)[10] \to 2\nu(4)[9]$
$\eta[12] \to \eta^2[10]$
$1[13] \to \eta[11]$

$\underline{k=14}$
$\alpha_5\eta[4] \to \alpha_6[2]$
$\alpha_5[5] \to \alpha_5\eta[3]$
$\epsilon[6]$
$\sigma[7]$
$\nu^2[8] \to \sigma\eta^2[4]$
$\nu[11] \to \nu^2[7]$
$\eta^2[12] \to 4\nu[10]$
$\eta[13] \to \eta^2[11]$
$1(\infty)[14] \to 2(\infty)[13]$

$\underline{k=15}$
$\theta_3[1]$
$\kappa[1]$
$\alpha_{6/3}(4)[4] \to \alpha_{6/2}(4)[3]$
$\alpha_6[4]$
$\alpha_5\eta[5]$
$\alpha_5[6]$
$\sigma(8)[8] \to 2\sigma(8)[7]$
$8\sigma[8]$
$\nu^2[9] \to \sigma\eta^2[5]$
$\nu(4)[12] \to 2\nu(4)[11]$
$4\nu[12]$
$\eta^2[13] \to \alpha_{6/3}[3]$
$\eta[14] \to \epsilon\eta[5]$
$1[15] \to (\sigma\eta + \epsilon?)[6]$

Table 1, cont'd

$\underline{k=16}$
$\alpha_{8/5}[1]$
$\theta_3[2]$
$\kappa[2]$
$\sigma\eta[8] \to \sigma\eta^2[6]$
$\epsilon[8] \to \epsilon\eta[6]$
$\sigma[9] \to \sigma\eta[7]$
$\nu^2[10] \to \epsilon[7]$
$\nu[13]$
$1(\infty)[16] \to 2(\infty)[15]$

$\underline{k=17}$
$\eta_4[1]$
$\alpha_{8/5}\eta[1]$
$\alpha_{8/5}(16)[2] \to \alpha_{8/4}(16)[1]$
$\alpha_8[2]$
$\theta_3[3]$
$\kappa[3]$
$\alpha_{6/3}(4)[6] \to \alpha_{6/2}(4)[5]$
$\left.\begin{matrix}\sigma\eta^2[8]\\ \epsilon\eta[8]\end{matrix}\right\} \to \alpha_{6/3}[5]$
$(\sigma\eta^2+\epsilon\eta)[8] \to \kappa\eta[1]$
$\alpha_5[8] \to \alpha_5\eta[6]$
$\sigma\eta[9] \to \sigma\eta^2[7]$
$\epsilon[9] \to \epsilon\eta[7]$
$\sigma(8)[10] \to 2\sigma(8)[9]$
$8\sigma[10] \to \alpha_5[7]$
$\nu^2[11]$
$\nu(4)[14] \to 2\nu(4)[13]$
$\eta[16] \to \eta^2[14]$
$1[17] \to \eta[15]$

$\underline{k=18}$
$\eta_4\eta[1]$
$\alpha_9[1]$
$\eta_4[2]$
$\theta_3[4]$
$\kappa[4] \to \kappa\eta[2]$
$\alpha_5\eta[8] \to \alpha_6[6]$
$\sigma\eta^2[9]$
$\alpha_5[9] \to \alpha_5\eta[7]$
$\sigma\eta[10]$
$\sigma[11]$
$\eta^2[16] \to 4\nu[14]$
$\eta[17] \to \eta^2[15]$
$1(\infty)[18] \to 2(\infty)[17]$

$\underline{k=19}$
$\nu^*[1]$
$\alpha_9\eta[1]$
$\kappa\nu[2]$
$\alpha_9[2]$
$\alpha_{8/5}(16)[4] \to \alpha_{8/4}(16)[3]$
$16\alpha_{8/5}[4]$
$\kappa\eta[4] \to \kappa\nu[1]$
$\theta_3[5]$
$\kappa[5] \to \kappa\eta[3]$
$\alpha_{6/3}(4)[8] \to \alpha_{6/2}(4)[7]$
$\alpha_6[8] \to \alpha_{8/5}\eta^2[1]$
$\alpha_5\eta[9] \to \alpha_{8/5}\eta[2]$
$\alpha_5[10] \to \alpha_{8/5}[3]$
$\sigma(8)[12] \to 2\sigma(8)[11]$
$8\sigma[12] \to \alpha_{6/3}[7]$
$\nu(4)[16] \to 2\nu(4)[15]$
$4\nu[16] \to \epsilon\eta[9]$
$\eta^2[17] \to \epsilon[10]$
$\eta[18] \to \nu^2[12]$
$1[19] \to \nu[15]$

$\underline{k=20}$
$\overline{\sigma}[1]$
$\nu^*(4)[2] \to 2\nu^*(4)[1]$
$\kappa\nu[3]$
$\eta_4[4] \to \eta_4\eta[2]$
$\alpha_{8/5}\eta[4] \to \alpha_{8/5}\eta^2[2]$
$\alpha_{8/5}[5] \to \alpha_{8/5}\eta[3]$
$\kappa\eta[5]$
$\theta_3[6] \to \eta_4[3]$
$\kappa[6]$
$\sigma\eta[12] \to \sigma\eta^2[10]$
$\epsilon[12] \to \epsilon\eta[10]$
$\sigma[13] \to \sigma\eta[11]$
$\nu^2[14] \to \epsilon[11]$
$\nu[17] \to \nu^2[13]$
$1(\infty)[20] \to 2(\infty)[19]$

$\underline{k=21}$
$\overline{\kappa}[1]$
$\overline{\sigma}[2]$
$\alpha_{10/3}(4)[2] \to \alpha_{10/2}(4)[1]$
$\nu^*[3]$
$\eta_4\eta[4] \to 4\nu^*[2]$
$\alpha_{8/5}\eta^2[4] \to \alpha_{10/3}[1]$
$\kappa\nu[4]$
$\alpha_9[4] \to \alpha_9\eta[2]$
$\eta_4[5] \to \eta_4\eta[3]$
$\alpha_{8/5}\eta[5] \to \alpha_{8/5}\eta^2[3]$
$\alpha_{8/5}(16)[6] \to \alpha_{8/4}(16)[5]$
$\alpha_8[6] \to \alpha_9[3]$
$\theta_3[7]$
$\alpha_{6/3}(4)[10] \to \alpha_{6/2}(4)[9]$
$\left.\begin{matrix}\sigma\eta^2[12]\\ \epsilon\eta[12]\end{matrix}\right\} \to \alpha_{6/3}[9]$
$\alpha_5[12] \to \alpha_5\eta[10]$
$\sigma\eta[13] \to \sigma\eta^2[11]$
$\epsilon[13] \to \epsilon\eta[11]$
$\sigma(8)[14] \to 2\sigma(8)[13]$
$8\sigma[14] \to \alpha_5[11]$
$\nu(4)[18] \to 2\nu(4)[17]$
$\eta[20] \to \eta^2[18]$
$1[21] \to \eta[19]$

Table 1, cont'd

$\underline{k = 22}$	$\underline{k = 23 \text{(outgoing diffs only)}}$
$\sigma^3[1]$	$\alpha_{10/3}(4)[4] \to \alpha_{10/2}(4)[3]$
$\overline{\kappa}\eta[1]$	$\alpha_{8/5}(16)[8] \to \alpha_{8/4}(16)[7]$
$\overline{\kappa}(4)[2] \to 2\overline{\kappa}(4)[1]$	$\kappa\eta[8] \to \kappa\nu[5]$
$4\overline{\kappa}[2]$	$\kappa[9] \to \kappa\eta[7]$
$\overline{\sigma}[3]$	$\alpha_{6/3}(4)[12] \to \alpha_{6/2}(4)[11]$
$\nu^*(4)[4] \to 2\nu^*(4)[3]$	$\alpha_6[12] \to \alpha_{10/3}[3]$
$4\nu^*[4]$	$\alpha_5\eta[13] \to \alpha_{8/5}\eta^2[5]$
$\alpha_9\eta[4] \to \alpha_{10}[2]$	$\alpha_5[14] \to \alpha_{8/5}\eta[6]$
$\eta_4\eta[5]$	$\sigma(8)[16] \to 2\sigma(8)[15]$
$\alpha_9[5] \to \alpha_9\eta[3]$	$8\sigma[16] \to \alpha_{8/5}[7]$
$\eta_4[6]$	$\nu^2[17] \to \sigma\eta^2[13]$
$\alpha_{8/5}[7]$	$\nu(4)[20] \to 2\nu(4)[19]$
$\theta_3[8]$	$4\nu[20] \to \alpha_{6/3}[11]$
$\kappa[8] \to \kappa\eta[6]$	$\eta^2[21] \to \epsilon\eta[13]$
$\alpha_5\eta[12] \to \alpha_6[10]$	$\eta[22] \to \sigma\eta[14]$
$\alpha_5[13] \to \alpha_5\eta[11]$	$1[23] \to \sigma[15]$
$\epsilon[14] \to \kappa[7]$	
$\nu^2[16] \to (\sigma\eta^2 + \epsilon\eta)[12]$	
$\nu[19] \to \nu^2[15]$	
$\eta^2[20] \to 4\nu[18]$	
$\eta[21] \to \eta^2[19]$	
$1(\infty)[22] \to 2(\infty)[21]$	

TABLE 2. The AHSS for $\pi_k(L(2))$

$\underline{k = 4}$	$\underline{k = 7}$	$\underline{k = 8}$
$1[3,1]$	$\nu[3,1]$	$1[7,1]$
	$1[5,2] \to \eta[4,1]$	$\eta[5,2] \to \eta^2[4,1]$

$\underline{k = 9}$	$\underline{k = 10}$	$\underline{k = 11}$
$\nu[5,1]$	$\nu^2[3,1]$	$\sigma[3,1]$
$\eta^2[5,2] \to \eta^3[4,1]$	$\nu[5,2]$	
$\eta[6,2] \to \eta^2[5,1]$	$1[7,3]$	
$1[7,2] \to \eta[6,1]$		

$\underline{k = 13}$	$\underline{k = 14}$	$\underline{k = 15}$
$\eta^3[8,2] \to 8\sigma[4,1]$	$\sigma[5,2] \to \sigma\eta[4,1]$	$\sigma[7,1]$
$\eta^2[9,2] \to \eta^3[8,1]$	$\nu^2[6,2] \to \epsilon[4,1]$	$\epsilon[5,2] \to \epsilon\eta[4,1]$
$\eta[10,2] \to \eta^2[9,1]$	$\eta[9,4] \to \eta^2[8,3]$	$8\sigma[6,2] \to \alpha_5[4,1]$
$1[9,4] \to \eta[8,3]$		$\eta^2[9,4] \to \eta^3[8,3]$
		$\eta[10,4] \to \eta^2[9,3]$
		$1[11,4] \to \eta[10,3]$

Table 2, cont'd

$\underline{k=16}$
$\alpha_5[5,2] \to \eta\alpha_5[4,1]$
$\epsilon[6,2]$
$\sigma[7,2]$
$\nu^2[8,2] \to \epsilon[6,1]$
$\nu[9,4] \to \sigma\eta[5,2]?$
$1[11,5] \to \nu[9,3]$

$\underline{k=17}$
$\nu[13,1]$
$\eta\alpha_5[5,2] \to \alpha_6[4,1]$
$\alpha_5[6,2] \to \eta\alpha_5[5,1]$
$8\sigma[8,2] \to \alpha_5[6,1]$
$\eta^3[12,2] \to 8\sigma[8,1]$
$\eta^2[13,2] \to \eta^3[12,1]$
$\sigma[7,3]$

$\underline{k=18}$
$\theta_3[3,1]$
$\nu^2[11,1]$
$\nu[13,2]$
$1[15,3]$
$\nu[11,4] \to \nu^2[8,3]$

$\underline{k=19}$
$\theta_3[4,1]$
$\kappa[4,1]$
$\sigma[11,1]$
$(\sigma\eta^2+\epsilon\eta)[8,2] \to \kappa[3,1]?$
$\nu^2[9,4] \to (\sigma\eta^2+\epsilon\eta)[8,1]$
$\eta^3[12,4] \to 8\sigma[8,3]$
$\eta^2[13,4] \to \eta^3[12,3]$
$\eta[14,4] \to \eta^2[13,3]$
$1[15,4] \to \eta[14,3]$
$\nu[11,5] \to \nu^2[9,3]$
$1[13,6] \to \eta[12,5]$

$\underline{k=20}$
$\theta_3[5,1]$
$\sigma\eta[10,2] \to \sigma\eta^2[9,1]$
$\sigma[11,2] \to \sigma\eta[10,1]$
$\sigma[9,4] \to \sigma\eta[8,3]$
$\nu^2[10,4] \to \epsilon[8,3]$
$\nu[13,4] \to \nu^2[11,2]$
$1[15,5] \to \nu[13,3]$
$\eta[13,6] \to \eta^2[12,5]$

$\underline{k=21}$
$\kappa[6,1]$
$\theta_3[5,2]$
$\kappa[5,2] \to \kappa\eta[4,1]$
$\alpha_6[8,2] \to \alpha_8[4,1]?$
$\eta\alpha_5[9,2] \to \alpha_6[8,1]$
$\alpha_5[10,2] \to \eta\alpha_5[9,1]$
$\sigma[11,3]$
$\sigma\eta[9,4] \to \sigma\eta^2[8,3]$
$\epsilon[9,4] \to \epsilon\eta[8,3]$
$8\sigma[10,4] \to \alpha_5[8,3]$
$\nu^2[11,4] \to \sigma\eta^2[9,2]$
$\nu[13,5] \to \nu^2[11,3]$
$\eta^2[13,6] \to 4\nu[12,5]$
$\eta[14,6] \to \eta^2[13,5]$
$1[15,6] \to \eta[14,5]$

$\underline{k=22}$
$\nu^*[3,1]$
$\nu\kappa[4,1]$
$\theta_3[7,1]$
$\alpha_{8/5}[5,2] \to \eta\alpha_{8/5}[4,1]$
$\kappa\eta[5,2] \to \nu\kappa[3,1]$
$\theta_3[6,2] \to \eta_4[4,1]$
$\kappa[6,2] \to \kappa\eta[5,1]$
$\sigma\eta^2[9,4]$
$\alpha_5[9,4] \to \eta\alpha_5[8,3]$
$\sigma\eta[10,4] \to \sigma\eta^2[9,3]$
$\sigma[11,4] \to \sigma\eta[10,3]$
$\nu^2[11,5]$
$\nu[13,6]$
$1[15,7]$

$\underline{k=23\text{(outgoing diffs only)}}$
$\eta_4[5,2] \to \eta\eta_4[4,1]$
$\eta\alpha_{8/5}[5,2] \to \eta^2\alpha_{8/5}[4,1]$
$\alpha_8[6,2] \to \alpha_9[4,1]$
$\eta\alpha_5[9,4] \to \alpha_6[8,3]$
$\alpha_5[10,4] \to \eta\alpha_5[9,3]$
$8\sigma[12,4] \to \alpha_5[10,3]$

TABLE 3. The AHSS for $\pi_k(L(3))$

$\underline{k=11}$
$1[7,3,1]$

$\underline{k=18}$
$\sigma[7,3,1]$
$1[11,5,2] \to \nu[9,4,1]$

$\underline{k=19}$
$1[15,3,1]$

$\underline{k=21}$
$\nu[11,5,2] \to \nu^2[9,4,1]$

$\underline{k=22}$
$\sigma[11,3,1]$
$1[15,5,2] \to \nu[13,4,1]$

$\underline{k=23}$
$1[15,7,1]$
$\nu[13,5,2] \to \nu^2[11,4,1]$

TABLE 4. The EHPSS

$\underline{k=0}$
$\boxed{1(\infty)[0]} \Rightarrow 1(\infty)$

$\underline{k=1}$
$\boxed{1[1]} \Rightarrow \eta$
$2(\infty)[1] \leftarrow 1(\infty)[2]$

$\underline{k=2}$
$\boxed{\eta[1]} \Rightarrow \eta^2$

$\underline{k=3}$
$\boxed{\eta^2[1]} \Rightarrow 4\nu$
$\boxed{\eta[2]} \Rightarrow 2\nu$
$\boxed{1[3]} \Rightarrow \nu$
$2(\infty)[3] \leftarrow 1(\infty)[4]$

$\underline{k=4}$
$2\nu(4)[1] \leftarrow \nu(4)[2]$
$\eta^2[2] \leftarrow \eta[4]$
$\eta[3] \leftarrow 1[5]$

$\underline{k=5}$
$\eta[4,1] \leftarrow 1[5,2]$
$4\nu[2] \leftarrow \eta^2[4]$
$\eta^2[3] \leftarrow \eta[5]$
$2(\infty)[5] \leftarrow 1(\infty)[6]$

$\underline{k=6}$
$\eta^2[4,1] \leftarrow \eta[5,2]$
$\boxed{\nu[3]} \Rightarrow \nu^2$
$2\nu(4)[3] \leftarrow \nu(4)[4]$

$\underline{k=7}$
$\boxed{4\nu[4]} \Rightarrow 8\sigma$
$\boxed{\eta^2[5]} \Rightarrow 4\sigma$
$\boxed{\eta[6]} \Rightarrow 2\sigma$
$\boxed{1[7]} \Rightarrow \sigma$
$2(\infty)[7] \leftarrow 1(\infty)[8]$

$\underline{k=8}$
$\boxed{\nu^2[2]} \Rightarrow \epsilon$
$\boxed{\nu[5]} \Rightarrow \sigma\eta$
$2\nu(4)[5] \leftarrow \nu(4)[6]$
$\eta^2[6] \leftarrow \eta[8]$
$\eta[7] \leftarrow 1[9]$

$\underline{k=9}$
$\boxed{\epsilon[1]} \Rightarrow \epsilon\eta$
$\boxed{8\sigma[2]} \Rightarrow \alpha_5$
$\boxed{\nu^2[3]} \Rightarrow \sigma\eta^2$
$4\nu[6] \leftarrow \eta^2[8]$
$\eta^2[7] \leftarrow \eta[9]$
$2(\infty)[9] \leftarrow 1(\infty)[10]$

$\underline{k=10}$
$\eta\epsilon[1] \leftarrow 4\nu[8]$
$\boxed{\alpha_5[1]} \Rightarrow \eta\alpha_5$
$\epsilon[2] \leftarrow \eta^2[9]$
$2\sigma(8)[3] \leftarrow \sigma(8)[4]$
$\nu^2[4] \leftarrow \eta[10]$
$\nu[7] \leftarrow 1[11]$
$2\nu(4)[7] \leftarrow \nu(4)[8]$

$\underline{k=11}$
$\boxed{\eta\alpha_5[1]} \Rightarrow \alpha_6$
$\eta^3[8,1] \leftarrow \eta^2[9,2]$
$\eta^2[9,1] \leftarrow \eta[10,2]$
$\epsilon\eta[2] \leftarrow \epsilon[4]$
$\boxed{\alpha_5[2]} \Rightarrow \alpha_{6/2}$
$\sigma\eta^2[2] \leftarrow \sigma\eta[4]$
$\epsilon[3] \leftarrow \nu^2[6]$
$\sigma\eta[3] \leftarrow \sigma[5]$
$\eta[8,3] \leftarrow 1[9,4]$
$\boxed{8\sigma[4]} \Rightarrow \alpha_{6/3}$
$\nu^2[5] \leftarrow \nu[9]$
$2(\infty)[11] \leftarrow 1(\infty)[12]$

Table 4, cont'd

$\underline{k=12}$
$\alpha_{6/2}(4)[1] \leftarrow \alpha_{6/3}(4)[2]$
$\epsilon[4,1] \leftarrow \nu^2[6,2]$
$\sigma\eta[4,1] \leftarrow \sigma[5,2]$
$\eta^3[8,2] \leftarrow \epsilon\eta[4]\ (*)$
$\eta\alpha_5[2] \leftarrow \alpha_5[4]$
$\epsilon\eta[3] \leftarrow \epsilon[5]$
$\alpha_5[3] \leftarrow 8\sigma[6]$
$\sigma\eta^2[3] \leftarrow \sigma\eta[5]$
$\eta^2[8,3] \leftarrow \eta[9,4]$
$2\sigma(8)[5] \leftarrow \sigma(8)[6]$
$2\nu(4)[9] \leftarrow \nu(4)[10]$
$\eta^2[10] \leftarrow \eta[12]$
$\eta[11] \leftarrow 1[13]$

$\underline{k=13}$
$\epsilon\eta[4,1] \leftarrow \epsilon[5,2]$
$\alpha_5[4,1] \leftarrow 8\sigma[6,2]$
$\alpha_6[2] \leftarrow \eta\alpha_5[4]$
$\eta^3[8,3] \leftarrow \eta^2[9,4]$
$\eta\alpha_5[3] \leftarrow \alpha_5[5]$
$\eta^2[9,3] \leftarrow \eta[10,4]$
$\eta[10,3] \leftarrow 1[11,4]$
$\sigma\eta^2[4] \leftarrow \nu^2[8]$
$\nu^2[7] \leftarrow \nu[11]$
$4\nu[10] \leftarrow \eta^2[12]$
$\eta^2[11] \leftarrow \eta[13]$
$2(\infty)[13] \leftarrow 1(\infty)[14]$

$\underline{k=14}$
$\eta\alpha_5[4,1] \leftarrow \alpha_5[5,2]$
$\sigma\eta[5,2] \leftarrow \nu[9,4]$
$\alpha_{6/3}[3] \leftarrow \eta^2[13]$
$\alpha_{6/2}(4)[3] \leftarrow \alpha_{6/3}(4)[4]$
$\nu[9,3] \leftarrow 1[11,5]$
$\sigma\eta^2[5] \leftarrow \nu^2[9]$
$\epsilon\eta[5] \leftarrow \eta[14]$
$\sigma\eta[6] \leftarrow 1[15]$
$\boxed{\epsilon[6]} \Rightarrow \kappa$
$\boxed{\sigma[7]} \Rightarrow \theta_3$
$2\sigma(8)[7] \leftarrow \sigma(8)[8]$
$2\nu(4)[11] \leftarrow \nu(4)[12]$

$\underline{k=15}$
$\nu[9,4,1] \leftarrow 1[11,5,2]$
$\boxed{\nu^2[8,2]} \Rightarrow \kappa\eta$
$\boxed{\alpha_6[4]} \Rightarrow \alpha_8$
$\boxed{\eta\alpha_5[5]} \Rightarrow \alpha_{8/2}$
$\sigma\eta^2[6] \leftarrow \sigma\eta[8]$
$\epsilon\eta[6] \leftarrow \epsilon[8]$
$\boxed{\alpha_5[6]} \Rightarrow \alpha_{8/3}$
$\sigma\eta[7] \leftarrow \sigma[9]$
$\epsilon[7] \leftarrow \nu^2[10]$
$\boxed{8\sigma[8]} \Rightarrow \alpha_{8/4}$
$\boxed{4\nu[12]} \Rightarrow \alpha_{8/5}$
$2(\infty)[15] \leftarrow 1(\infty)[16]$

$\underline{k=16}$
$\kappa\eta[1] \leftarrow (\sigma\eta^2+\epsilon\eta)[8]$
$\boxed{\eta^2[13,2]} \Rightarrow \eta\alpha_{8/5}$
$\nu^2[8,3] \leftarrow \nu[11,4]$
$\alpha_{6/3}[5] \leftarrow \begin{cases} \sigma\eta^2[8] \\ \epsilon\eta[8] \end{cases}$
$\eta\alpha_5[6] \leftarrow \alpha_5[8]$
$\sigma\eta^2[7] \leftarrow \sigma\eta[9]$
$\epsilon\eta[7] \leftarrow \epsilon[9]$
$\alpha_5[7] \leftarrow 8\sigma[10]$
$2\sigma(8)[9] \leftarrow \sigma(8)[10]$
$\boxed{\nu[13]} \Rightarrow \eta_4$
$2\nu(4)[13] \leftarrow \nu(4)[14]$
$\eta^2[14] \leftarrow \eta[16]$
$\eta[15] \leftarrow 1[17]$

$\underline{k=17}$
$\boxed{\eta\alpha_{8/5}[1]} \Rightarrow \eta^2\alpha_{8/5}$
$(\sigma\eta^2+\epsilon\eta)[8,1] \leftarrow \nu^2[9,4]$
$\kappa\eta[2] \leftarrow \kappa[4]$
$\boxed{\alpha_8[2]} \Rightarrow \alpha_9$
$\eta^2[13,3] \leftarrow \eta[14,4]$
$\nu^2[9,3] \leftarrow \nu[11,5]$
$\eta[14.3] \leftarrow 1[15,4]$
$1[15,3] \leftarrow \theta_3[4]\ (***)$
$\boxed{\kappa[3]} \Rightarrow \kappa\nu$
$\eta[12,5] \leftarrow 1[13,6]$
$\alpha_6[6] \leftarrow \eta\alpha_5[8]$
$\eta\alpha_5[7] \leftarrow \alpha_5[9]$
$\boxed{\nu^2[11]} \Rightarrow \eta\eta_4$
$4\nu[14] \leftarrow \eta^2[16]$
$\eta^2[15] \leftarrow \eta[17]$
$2(\infty)[17] \leftarrow 1(\infty)[18]$

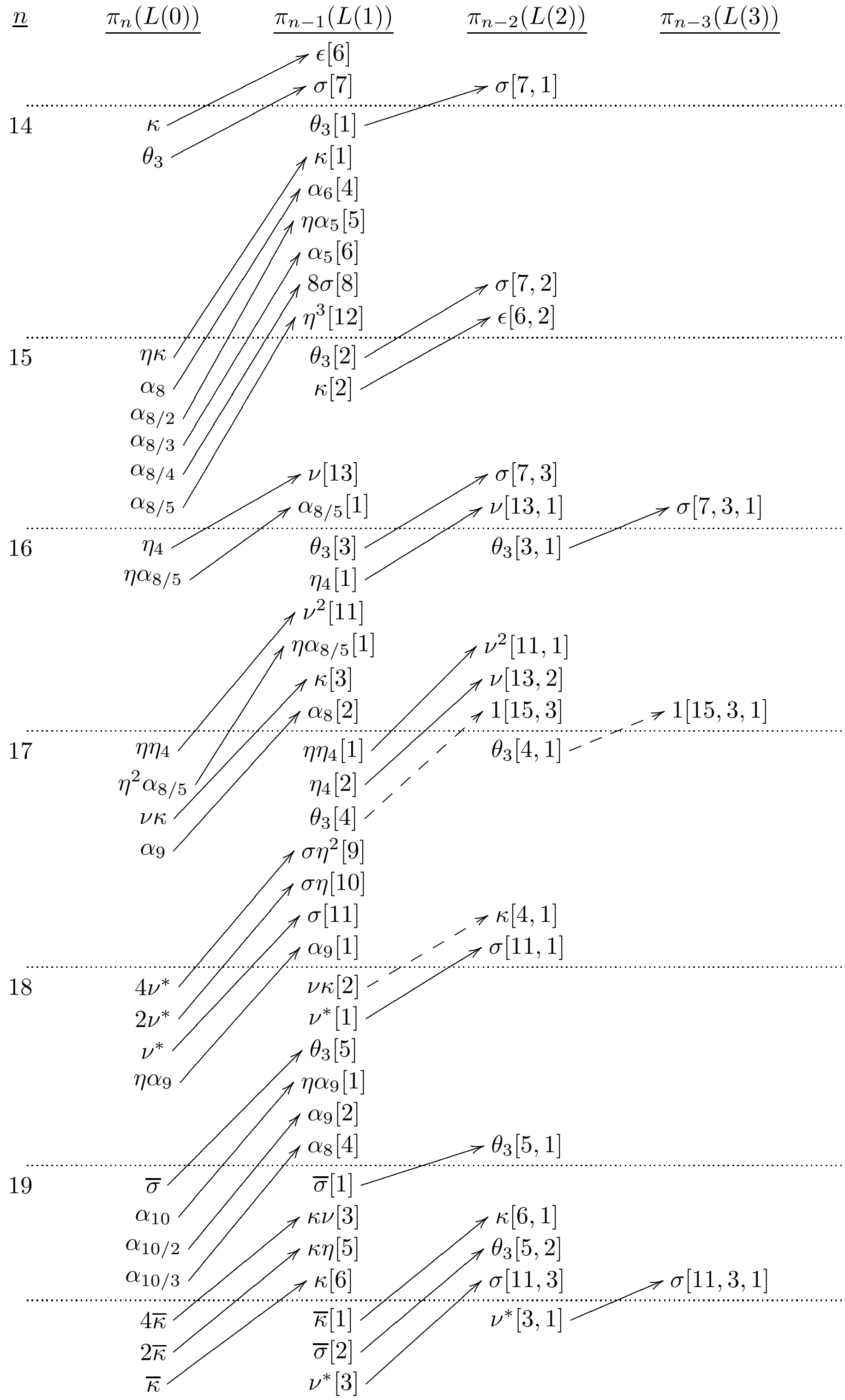
Table 4, cont'd

$\underline{k=18}$
$\eta^2\alpha_{8/5}[1] \leftarrow \alpha_6[8]$
$\boxed{\alpha_9[1]} \Rightarrow \eta\alpha_9$
$\nu^2[9,4,1] \leftarrow \nu[11,5,2]$
$\kappa[4,1] \leftarrow \kappa\nu[2]\ (***)$
$(\sigma\eta^2+\epsilon\eta)[8,2] \leftarrow \kappa\eta[4]\ (*)$
$\eta\alpha_{8/5}[2] \leftarrow \eta\alpha_5[9]$
$\kappa\eta[3] \leftarrow \kappa[5]$
$\alpha_{8/3}(8)[3] \leftarrow \alpha_{8/4}(8)[4]$
$\sigma\eta[8,3] \leftarrow \sigma[9,4]$
$\epsilon[8,3] \leftarrow \nu^2[10,4]$
$\eta^2[13,4] \leftarrow \alpha_5[10]\ (*)$
$\eta^2[12,5] \leftarrow \eta[13,6]$
$\alpha_{6/3}[7] \leftarrow 8\sigma[12]$
$\alpha_{6/2}(4)[7] \leftarrow \alpha_{6/3}(4)[8]$
$\boxed{\sigma\eta^2[9]} \Rightarrow 4\nu^*$
$\epsilon\eta[9] \leftarrow 4\nu[16]$
$\boxed{\sigma\eta[10]} \Rightarrow 2\nu^*$
$\epsilon[10] \leftarrow \eta^2[17]$
$\boxed{\sigma[11]} \Rightarrow \nu^*$
$\nu^2[12] \leftarrow \eta[18]$
$\nu[15] \leftarrow 1[19]$
$2\nu(4)[15] \leftarrow \nu(4)[16]$

$\underline{k=19}$
$\boxed{\eta\alpha_9[1]} \Rightarrow \alpha_{10}$
$\kappa\eta[4,1] \leftarrow \kappa[5,2]$
$\alpha_6[8,1] \leftarrow \eta\alpha_5[9,2]$
$\eta\alpha_5[9,1] \leftarrow \alpha_5[10,2]$
$\eta^2\alpha_{8/5}[2] \leftarrow \eta\alpha_{8/5}[4]$
$\boxed{\alpha_9[2]} \Rightarrow \alpha_{10/2}$
$\alpha_5[8,3] \leftarrow 8\sigma[10,4]$
$\eta\alpha_{8/5}[3] \leftarrow \eta^2[13,6]\ (**)$
$\sigma\eta^2[8,3] \leftarrow \sigma\eta[9,4]$
$\epsilon\eta[8,3] \leftarrow \epsilon[9,4]$
$\boxed{\alpha_8[4]} \Rightarrow \alpha_{10/3}$
$\boxed{\theta_3[5]} \Rightarrow \overline{\sigma}$
$\eta^2[13,5] \leftarrow \eta[14,6]$
$\eta[14,5] \leftarrow 1[15,6]$
$1[15,5] \leftarrow \theta_3[6]\ (*)$
$\sigma\eta^2[10] \leftarrow \sigma\eta[12]$
$\epsilon\eta[10] \leftarrow \epsilon[12]$
$\sigma\eta[11] \leftarrow \sigma[13]$
$\epsilon[11] \leftarrow \nu^2[14]$
$\nu^2[13] \leftarrow \nu[17]$
$2(\infty)[19] \leftarrow 1(\infty)[20]$

TABLE 5. The GSS for $\pi_{n+1}(S^1)$

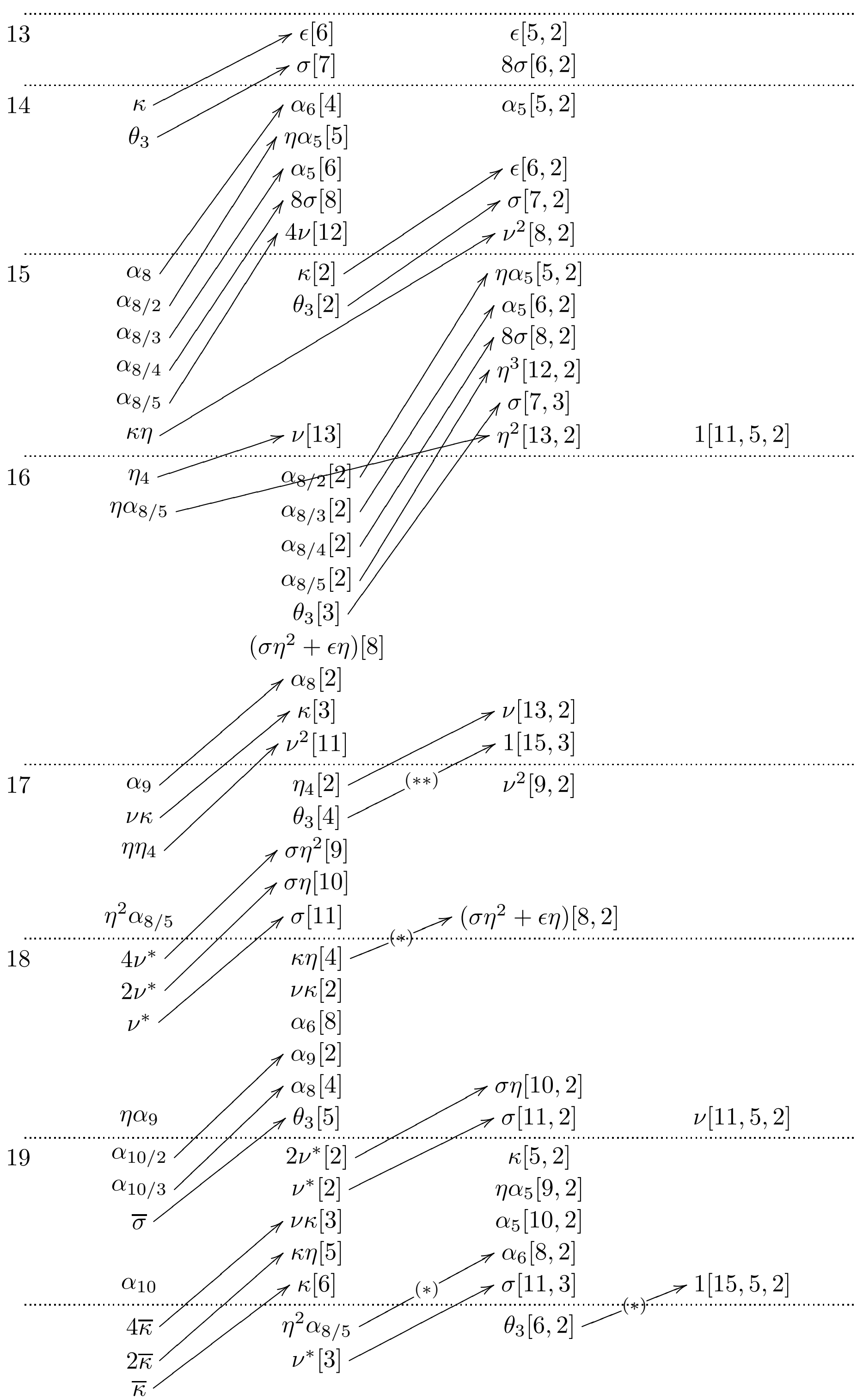
n	$\pi_n(L(0))$	$\pi_{n-1}(L(1))$	$\pi_{n-2}(L(2))$	$\pi_{n-3}(L(3))$
0	$1(\infty)$	$1[1]$		
1	η	$\eta[1]$		
2	η^2	$\eta^2[1]$		
		$\eta[2]$		
		$1[3]$	$1[3,1]$	
3	4ν	$\nu[1]$		
	2ν			
	ν			
4				
5		$\nu[3]$	$\nu[3,1]$	
6	ν^2	$\nu^2[1]$		
		$\eta^3[4]$		
		$\eta^2[5]$		
		$\eta[6]$		
		$1[7]$	$1[7,1]$	
7	8σ	$\sigma[1]$		
	4σ			
	2σ	$\nu^2[2]$		
	σ	$\nu[5]$	$\nu[5,1]$	
8	ϵ	$\sigma\eta[1]$		
	$\sigma\eta$	$\epsilon[1]$	$\nu^2[3,1]$	
		$\nu^2[3]$	$\nu[5,2]$	
		$8\sigma[2]$	$1[7,3]$	$1[7,3,1]$
9	$\epsilon\eta$	$\sigma\eta^2[1]$	$\sigma[3,1]$	
	$\sigma\eta^2$	$\sigma\eta[2]$		
	α_5	$\sigma[3]$		
		$\alpha_5[1]$		
10	$\eta\alpha_5$	$\eta\alpha_5[1]$		
		$\alpha_5[2]$		
		$8\sigma[4]$		
11	α_6			
	$\alpha_{6/2}$			
	$\alpha_{6/3}$			
12				
13		$\epsilon[6]$	$\sigma[7,1]$	
		$\sigma[7]$		
	κ	$\theta_3[1]$		
	θ_3			



n
$\pi_n(L(0))$
$\pi_{n-1}(L(1))$
$\pi_{n-2}(L(2))$
$\pi_{n-3}(L(3))$
$\epsilon[6]$
$\sigma[7]$
$\sigma[7,1]$
14
κ
$\theta_3[1]$
θ_3
$\kappa[1]$
$\alpha_6[4]$
$\eta\alpha_5[5]$
$\alpha_5[6]$
$8\sigma[8]$
$\sigma[7,2]$
$\eta^3[12]$
$\epsilon[6,2]$
15
$\eta\kappa$
$\theta_3[2]$
α_8
$\kappa[2]$
$\alpha_{8/2}$
$\alpha_{8/3}$
$\alpha_{8/4}$
$\nu[13]$
$\sigma[7,3]$
$\alpha_{8/5}$
$\alpha_{8/5}[1]$
$\nu[13,1]$
$\sigma[7,3,1]$
16
η_4
$\theta_3[3]$
$\theta_3[3,1]$
$\eta\alpha_{8/5}$
$\eta_4[1]$
$\nu^2[11]$
$\eta\alpha_{8/5}[1]$
$\nu^2[11,1]$
$\kappa[3]$
$\nu[13,2]$
$\alpha_8[2]$
$1[15,3]$
$1[15,3,1]$
17
$\eta\eta_4$
$\eta\eta_4[1]$
$\theta_3[4,1]$
$\eta^2\alpha_{8/5}$
$\eta_4[2]$
$\nu\kappa$
$\theta_3[4]$
α_9
$\sigma\eta^2[9]$
$\sigma\eta[10]$
$\sigma[11]$
$\kappa[4,1]$
$\alpha_9[1]$
$\sigma[11,1]$
18
$4\nu^*$
$\nu\kappa[2]$
$2\nu^*$
$\nu^*[1]$
ν^*
$\eta\alpha_9$
$\theta_3[5]$
$\eta\alpha_9[1]$
$\alpha_9[2]$
$\alpha_8[4]$
$\theta_3[5,1]$
19
$\overline{\sigma}$
$\overline{\sigma}[1]$
α_{10}
$\kappa\nu[3]$
$\kappa[6,1]$
$\alpha_{10/2}$
$\kappa\eta[5]$
$\theta_3[5,2]$
$\alpha_{10/3}$
$\kappa[6]$
$\sigma[11,3]$
$\sigma[11,3,1]$
$4\overline{\kappa}$
$\overline{\kappa}[1]$
$\nu^*[3,1]$
$2\overline{\kappa}$
$\overline{\sigma}[2]$
$\overline{\kappa}$
$\nu^*[3]$

TABLE 6. The GSS for $\pi_{n+2}(S^2)$

n	$\pi_n(L(0)_2)$	$\pi_{n-1}(L(1)_2)$	$\pi_{n-2}(L(2)_2)$	$\pi_{n-3}(L(3)_2)$
0	$1(\infty)$			
1	η	$1(\infty)[2]$		
2		$\eta[2]$		
	η^2	$1[3]$		
3	2ν			
	ν			
	4ν			
4		$2\nu[2]$		
		$\nu[2]$		
5		$\nu[3]$	$1[5,2]$	
6	ν^2	$\eta^3[4]$		
		$\eta^2[5]$		
		$\eta[6]$		
		$1[7]$	$\eta[5,2]$	
7	8σ			
	4σ		$\eta^2[5,2]$	
	2σ	$\nu^2[2]$	$\eta[6,2]$	
	σ	$\nu[5]$	$1[7,2]$	
8	ϵ	$4\sigma[2]$		
	$\sigma\eta$	$2\sigma[2]$		
		$\sigma[2]$		
		$8\sigma[2]$	$\nu[5,2]$	
		$\nu^2[3]$	$1[7,3]$	
9	α_5	$\sigma\eta[2]$		
	$\sigma\eta^2$	$\sigma[3]$		
	$\epsilon\eta$			
10		$4\nu[8]$		
		$\alpha_5[2]$		
	$\eta\alpha_5$	$8\sigma[4]$		
11	$\alpha_{6/2}$		$\eta^2[9,2]$	
	$\alpha_{6/3}$		$\eta[10,2]$	
	α_6		$\eta^3[8,2]$	
12		$\epsilon\eta[4]$ (*)		
		$\alpha_{6/2}[2]$	$\sigma[5,2]$	
		$\alpha_{6/3}[2]$	$\nu^2[6,2]$	


13 ϵ[6] ϵ[5, 2]
σ[7] 8σ[6, 2]
14 κ α6[4] α5[5, 2]
θ3 ηα5[5]
α5[6] ϵ[6, 2]
8σ[8] σ[7, 2]
4ν[12] ν2[8, 2]
15 α8 κ[2] ηα5[5, 2]
α8/2 θ3[2] α5[6, 2]
α8/3 8σ[8, 2]
α8/4 η3[12, 2]
α8/5 σ[7, 3]
κη ν[13] η2[13, 2] 1[11, 5, 2]
16 η4 α8/2[2]
ηα8/5 α8/3[2]
α8/4[2]
α8/5[2]
θ3[3]
(ση2 + ϵη)[8]
α8[2]
κ[3] ν[13, 2]
ν2[11] 1[15, 3]
17 α9 η4[2] ν2[9, 2]
νκ θ3[4]
(**)
ηη4 ση2[9]
ση[10]
η2α8/5 σ[11] (ση2 + ϵη)[8, 2]
(*)
18 4ν* κη[4]
2ν* νκ[2]
ν* α6[8]
α9[2]
α8[4] ση[10, 2]
ηα9 θ3[5] σ[11, 2] ν[11, 5, 2]
19 α10/2 2ν*[2] κ[5, 2]
α10/3 ν*[2] ηα5[9, 2]
σ̄ νκ[3] α5[10, 2]
κη[5] α6[8, 2]
α10 κ[6] σ[11, 3] 1[15, 5, 2]
(*)
(*)
4κ̄ η2α8/5 θ3[6, 2]
2κ̄ ν*[3]
κ̄