NOTES ON THE CONSTRUCTION OF *tmf*

MARK BEHRENS

Contents

1. Introduction

In these notes I will sketch the construction of *tmf* using Goerss-Hopkins obstruction theory. These notes are the result of my attempts to understand the material surrounding a talk I gave at the Talbot workshop in 2007. There is no claim to originality in this approach. All of the results are the results of other people, namely: Paul Goerss, Mike Hopkins, and Haynes Miller. I benefited from conversations with Niko Naumann and Charles Rezk, and from Mike Hill's talk at the Talbot workshop. I am especially grateful for numerous corrections and suggestions which Tyler Lawson, Aaron Mazel-Gee, Lennart Meier, Niko Naumann, and Markus Szymik supplied me with. The remaining mathematical errors, inconsistencies, and points of inelegance in these notes are mine and mine alone.

Let $\overline{\mathcal{M}}_{ell}$ denote the moduli stack of generalized elliptic curves over $\mathrm{Spec}(\mathbb{Z})$. For us, unless we specifically specify otherwise, a generalized elliptic curve is implicitly assumed to have irreducible geometric fibers (i.e. no Néron n-gons for $n > 1$). That is to say, $\overline{\mathcal{M}}_{ell}$ is the moduli stack of pointed curves whose fibers are either elliptic curves, or possess a nodal singularity. Our aim is to prove the following theorem.

Theorem 1.1. *There is a presheaf $\mathcal{O}^{top}$ of E_∞-ring spectra on the site $(\overline{\mathcal{M}}_{ell})_{et}$, which is fibrant as a presheaf of spectra in the Jardine model structure. Given an affine étale open*

$$\mathrm{Spec}(R) \xrightarrow{C} \overline{\mathcal{M}}_{ell}$$

classifying a generalized elliptic curve C/R, the spectrum of sections $E = \mathcal{O}^{top}(\mathrm{Spec}(R))$ is a weakly even periodic ring spectrum satisfying:

(1) $\pi_0(E) \cong R$,
(2) $\mathbb{G}_E \cong \widehat{C}$.

Here, $\widehat{C}$ is the formal group of C.

Date: July 26, 2013.

Remark 1.2. A ring spectrum E is *weakly even periodic* if $\pi_* E$ is concentrated in even degrees, $\pi_2 E$ is an invertible $\pi_0 E$-module, and the natural map

$$\pi_2 E \otimes \pi_{2t} E \cong \pi_{2t+2} E$$

is an isomorphism. The spectrum E is automatically complex orientable, and we let $\mathbb{G}_E$ denote the formal group over $\pi_0 E$ associated to E. It then follows that there is a canonical isomorphism

$$\pi_{2t} E \cong \Gamma \omega_{\mathbb{G}_E}^{\otimes t}$$

where $\omega_{\mathbb{G}_E}$ is the line bundle (over $\mathrm{Spec}(R)$) of invariant 1-forms on $\mathbb{G}_E$.

Remark 1.3. The properties of the spectrum of sections of $E = \mathcal{O}^{top}(\mathrm{Spec}(R))$ enumerated in Theorem 1.1 make E an *elliptic spectrum* associated to the generalized elliptic curve C/R in the sense of Hopkins and Miller [Hop95]. Thus Theorem 1.1 gives a functorial collection of E_∞-elliptic spectra associated to the collection of generalized elliptic curves whose classifying maps are étale.

Remark 1.4. This theorem practically determines $\mathcal{O}^{top}$, at least as a diagram in the stable homotopy category. Given an affine étale open $\mathrm{Spec}(R) \xrightarrow{C} \overline{\mathcal{M}}_{ell}$, the composite

$$\mathrm{Spec}(R) \xrightarrow{C} \overline{\mathcal{M}}_{ell} \to \mathcal{M}_{FG}$$

is flat, since the map $\overline{\mathcal{M}}_{ell} \to \mathcal{M}_{FG}$ classifying the formal group of the universal generalized elliptic curve is flat (this can be verified using Serre-Tate theory, see [BL10, Lemma 9.1.6]). Thus the spectrum of sections $E = \mathcal{O}^{top}(R)$ is Landweber exact [Nau07]. Fibrant presheaves of spectra satisfy homotopy descent, and so the values of the presheaf are determined by values on the affine opens using étale descent.

Remark 1.5. The spectrum *tmf* is defined to be the connective cover of the global sections of this sheaf:

$$tmf = \tau_{\geq 0} \mathcal{O}^{top}(\overline{\mathcal{M}}_{ell}).$$

We give an outline of the argument we shall give. Consider the substacks

$$(\overline{\mathcal{M}}_{ell})_p \xrightarrow{\iota_p} \overline{\mathcal{M}}_{ell},$$
$$(\overline{\mathcal{M}}_{ell})_{\mathbb{Q}} \xrightarrow{\iota_{\mathbb{Q}}} \overline{\mathcal{M}}_{ell},$$

where:

$$(\overline{\mathcal{M}}_{ell})_p = p\text{-completion of } \overline{\mathcal{M}}_{ell},$$
$$(\overline{\mathcal{M}}_{ell})_{\mathbb{Q}} = \overline{\mathcal{M}}_{ell} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Remark 1.6. We pause to make two important comments on our use of formal geometry in this paper.

1. The object $(\overline{\mathcal{M}}_{ell})_p$ is a *formal Deligne-Mumford stack*. We shall use these throughout this paper — we refer the reader to the appendix of [Har05] for some of the basic definitions. Given a formal Deligne-Mumford stack $\mathcal{X}$ and a ring R complete with respect to an ideal I, we define the R-points of $\mathcal{X}$ by $\mathcal{X}(R) = \varprojlim_i \mathcal{X}(R/I^i)$.
2. If R is complete with respect to an ideal I, a generalized elliptic curve $C/\mathrm{Spf}(R)$ is a compatible ind-system $C_m/\mathrm{Spec}(R/I^m)$. There is, however, a canonical “algebraization” $C^{alg}/\mathrm{Spec}(R)$ where C^{alg} is a generalized elliptic curve which restricts to C_m over $\mathrm{Spec}(R/I^m)$ [Con07, Cor. 2.2.4]. With this in mind, we shall in these notes always regard $C/\mathrm{Spf}(R)$ as being represented by an honest generalized elliptic curve over the ring R.

We shall construct $\mathcal{O}^{top}$ as the homotopy pullback of an arithmetic square of presheaves of E_∞-ring spectra

$$\begin{tikzcd}
\mathcal{O}^{top} \arrow[r] \arrow[d] & \prod_{p \text{ prime}} (\iota_p)_* \mathcal{O}_p^{top} \arrow[d] \\
(\iota_{\mathbb{Q}})_* \mathcal{O}_{\mathbb{Q}}^{top} \arrow[r, "\alpha_{\mathrm{arith}}"'] & \left(\prod_{p \text{ prime}} (\iota_p)_* \mathcal{O}_p^{top} \right)_{\mathbb{Q}}
\end{tikzcd}$$

Here, $\mathcal{O}_p^{top}$ is a presheaf on $(\overline{\mathcal{M}}_{ell})_p$, and $\mathcal{O}_{\mathbb{Q}}^{top}$ is a presheaf on $(\overline{\mathcal{M}}_{ell})_{\mathbb{Q}}$. The presheaf

$$\left(\prod_{p \text{ prime}} (\iota_p)_* \mathcal{O}_p^{top} \right)_{\mathbb{Q}}$$

is the (sectionwise) rationalization of the presheaf $\prod_{p \text{ prime}} (\iota_p)_* \mathcal{O}_p^{top}$. The presheaf $\mathcal{O}_{\mathbb{Q}}^{top}$ will be constructed using rational homotopy theory, as will the map α_{arith}.

It remains to construct the presheaves $\mathcal{O}_p^{top}$ for each prime p. Define

$$(\overline{\mathcal{M}}_{ell})_{\mathbb{F}_p} = \overline{\mathcal{M}}_{ell} \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

Let

$$(\mathcal{M}_{ell}^{ord})_{\mathbb{F}_p} \subset (\overline{\mathcal{M}}_{ell})_{\mathbb{F}_p}$$

denote the locus of ordinary generalized elliptic curves in characteristic p, and let

$$(\mathcal{M}_{ell}^{ss})_{\mathbb{F}_p} = (\overline{\mathcal{M}}_{ell})_{\mathbb{F}_p} - (\mathcal{M}_{ell}^{ord})_{\mathbb{F}_p}$$

denote the locus of supersingular elliptic curves in characteristic p. Consider the substacks

$$\mathcal{M}_{ell}^{ord} \xrightarrow{\iota_{ord}} (\overline{\mathcal{M}}_{ell})_p, \tag{1.1}$$

$$\mathcal{M}_{ell}^{ss} \xrightarrow{\iota_{ss}} (\overline{\mathcal{M}}_{ell})_p, \tag{1.2}$$

where

$\mathcal{M}_{ell}^{ord}$ = moduli stack of generalized elliptic curves over p-complete rings with ordinary reduction,

$\mathcal{M}_{ell}^{ss}$ = completion of $\overline{\mathcal{M}}_{ell}$ at $(\mathcal{M}_{ell}^{ss})_{\mathbb{F}_p}$.

The presheaves $\mathcal{O}_p^{top}$ will be constructed as homotopy pullbacks:

$$\begin{tikzcd}
\mathcal{O}_p^{top} \arrow[r] \arrow[d] & (\iota_{ss})_* \mathcal{O}_{K(2)}^{top} \arrow[d] \\
(\iota_{ord})_* \mathcal{O}_{K(1)}^{top} \arrow[r, "\alpha_{\mathrm{chrom}}"'] & \left((\iota_{ss})_* \mathcal{O}_{K(2)}^{top} \right)_{K(1)}
\end{tikzcd}$$

Here,

$$\left((\iota_{ss})_* \mathcal{O}_{K(2)}^{top} \right)_{K(1)}$$

denotes the (sectionwise) $K(1)$-localization of the presheaf $(\iota_{ss})_* \mathcal{O}_{K(2)}^{top}$. (The reader wondering at this point why these localizations are related to the ordinary and supersingular loci is invited to glance at Lemma 8.1.)

The presheaf $\mathcal{O}_{K(2)}^{top}$ will be constructed using the Goerss-Hopkins-Miller Theorem — its spectra of sections are given by homotopy fixed points of Morava E-theories with respect to finite group actions.

The presheaf $\mathcal{O}_{K(1)}^{top}$ will be constructed using explicit Goerss-Hopkins obstruction theory. The map α_{chrom} will be be produced from an analysis of the $K(1)$-local mapping spaces, and the θ-algebra structure inherent in certain rings of p-adic modular forms.

$$\begin{tikzcd}
\text{height 2} & & \mathcal{O}^{top} \arrow[r] \arrow[dd] \arrow[dr, phantom, "h\,\lrcorner", very near start] & \prod_p (\iota_p)_* \mathcal{O}_p^{top} \arrow[dd, "L_{\mathbb{Q}}"] & & \mathcal{O}_p^{top} \arrow[ll, rightsquigarrow] \arrow[r] \arrow[d] \arrow[dr, phantom, "h\,\lrcorner", very near start] & (\iota_{ss})_* \mathcal{O}_{K(2)}^{top} \arrow[d, "L_{K(1)}"] & \mathcal{O}_{K(2)}^{top} \arrow[l, rightsquigarrow] \\
\text{height 1} & & & & \mathcal{O}_{K(1)}^{top} \arrow[r, rightsquigarrow] & (\iota_{ord})_* \mathcal{O}_{K(1)}^{top} \arrow[r, "\alpha_{\mathrm{chrom}}"'] & \left((\iota_{ss})_* \mathcal{O}_{K(2)}^{top} \right)_{K(1)} & \\
\text{height 0} & \mathcal{O}_{\mathbb{Q}}^{top} \arrow[r, rightsquigarrow] & (\iota_{\mathbb{Q}})_* \mathcal{O}_{\mathbb{Q}}^{top} \arrow[r, "\alpha_{\mathrm{arith}}"'] & \left(\prod_p (\iota_p)_* \mathcal{O}_p^{top} \right)_{\mathbb{Q}} & & & & \\
 & & & \overline{\mathcal{M}}_{ell} & & & & \\
 & (\overline{\mathcal{M}}_{ell})_{\mathbb{Q}} \arrow[urr, "\iota_{\mathbb{Q}}"] & & & & & (\overline{\mathcal{M}}_{ell})_p \arrow[ulll, "\iota_p"'] & \\
 & & & & \mathcal{M}_{ell}^{ord} \arrow[urr, "\iota_{ord}"] & & & \mathcal{M}_{ell}^{ss} \arrow[ul, "\iota_{ss}"']
\end{tikzcd}$$

Figure 1. Summary of the construction of tmf [courtesy of Aaron Mazel-Gee]

Figure 1 shows a diagram which summarizes the above discussion. Many thanks to Aaron Mazel-Gee for creating this diagram, and making it available for inclusion here.

2. DESCENT LEMMAS FOR PRESHEAVES OF SPECTRA

For a small Grothendieck site $\mathcal{C}$ with enough points, let $\mathrm{PreSp}_{\mathcal{C}}$ denote the category of presheaves of symmetric spectra of simplicial sets. The category $\mathrm{PreSp}_{\mathcal{C}}$ has a Jardine model category structure [Jar00], where

1. The cofibrations are the sectionwise cofibrations of symmetric spectra,
2. The weak equivalences are the stalkwise stable equivalences of symmetric spectra,
3. The fibrant objects are those objects which are fibrant in the injective model structure of the underlying diagram model category structure, and which satisfy descent with respect to hypercovers [DHI04].

The following lemma will be useful.

Lemma 2.1.

1. *If* $\mathcal{F} \in \mathrm{PreSp}_{\mathcal{C}}$ *satisfies homotopy descent with respect to hypercovers, then the fibrant replacement in the Jardine model structure*
$$\mathcal{F} \to \mathcal{F}'$$
is a sectionwise weak equivalence.
2. *If* $f : \mathcal{F} \to \mathcal{G}$ *is a stalkwise weak equivalence in* $\mathrm{PreSp}_{\mathcal{C}}$*, and* $\mathcal{F}$ *and* $\mathcal{G}$ *satisfy homotopy descent with respect to hypercovers, then* f *is a sectionwise weak equivalence.*

Proof. (1) The Jardine model category structure is a localization of the injective model category structure on $\mathrm{PreSp}_{\mathcal{C}}$. In the injective model structure, weak equivalences are sectionwise. Let

$$\mathcal{F} \to \mathcal{F}'$$

be the fibrant replacement in the injective model category structure. This map is necessarily a sectionwise weak equivalence. By the Dugger-Hollander-Isaksen criterion, to see that $\mathcal{F}'$ is fibrant in the Jardine model structure, it suffices to show that $\mathcal{F}'$ satisfies homotopy descent with respect to hypercovers. Let $U \in \mathcal{C}$ and let $U_\bullet$ be a hypercover of U. Consider the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \operatorname{holim}_{\Delta} \mathcal{F}(U_\bullet) \arrow[d, "\simeq"] \\
\mathcal{F}'(U) \arrow[r] & \operatorname{holim}_{\Delta} \mathcal{F}'(U_\bullet)
\end{tikzcd}$$

We deduce that the bottom arrow is an equivalence. Thus $\mathcal{F}'$ satisfies descent with respect to hypercovers, and is fibrant in the Jardine model category structure.

(2) Consider the diagram of Jardine fibrant replacements:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "f"] \arrow[d, "u"'] & \mathcal{G} \arrow[d, "v"] \\
\mathcal{F}' \arrow[r, "f'"'] & \mathcal{G}'
\end{tikzcd}$$

By (1), the maps u and v are sectionwise equivalences. The map f' is a stalkwise weak equivalence between Jardine fibrant objects. Because the Jardine model structure is a localization of the injective model structure, we deduce that f' is a sectionwise weak equivalence. We therefore conclude that f is a sectionwise weak equivalence. □

Let $\mathcal{X}$ be a Deligne-Mumford stack, and consider the site $\mathcal{X}_{et}$. Being a Deligne-Mumford stack, $\mathcal{X}$ possesses an affine étale cover. The full subcategory

$$\mathcal{X}_{et,aff} \xrightarrow{i} \mathcal{X}_{et}$$

consisting of only the affine étale opens is also a Grothendieck site. The map i induces an adjoint pair of functors

$$i^* : \mathrm{PreSp}_{\mathcal{X}_{et}} \leftrightarrows \mathrm{PreSp}_{\mathcal{X}_{et,aff}} : i_*$$

where i^* is the functor given by precomposition with i, and i_* is the right Kan extension.

Lemma 2.2.

1. *The adjoint pair (i^*, i_*) is a Quillen equivalence.*
2. *To construct a fibrant presheaf of spectra on $\mathcal{X}_{et}$, it suffices to construct a fibrant presheaf on $\mathcal{X}_{et,aff}$ and apply the functor i_*.*

Proof. By [Hov99, Cor. 1.3.16], to check (1) it suffices to check that (i^*, i_*) is a Quillen pair, that i^* reflects weak equivalences, and that the map

$$i_* L i^* X \to X$$

is a weak equivalence. The functor i^* is easily seen to preserve cofibrations, and it preserves and reflects all weak equivalences, since the sites $\mathcal{X}_{et}$ and $\mathcal{X}_{et,aff}$ have the same points. Since the functor i_* preserves stalks, the map above is a stalkwise weak equivalence, hence is an equivalence. Therefore (i^*, i_*) is a Quillen equivalence. (2) In particular, the functor i_* preserves fibrant objects. ☐

The following construction formalizes the idea that a Jardine fibrant presheaf on $\mathcal{X}_{et}$ is determined by its sections on étale affine opens.

Construction 2.3.
Input: A presheaf $\mathcal{F}$ on $\mathcal{X}_{et,aff}$ that satisfies hyperdescent.

Output: A Jardine fibrant presheaf $\mathcal{G}$ on $\mathcal{X}_{et}$, and a zig-zag of sectionwise weak equivalences between $\mathcal{F}$ and $i^*\mathcal{G}$.

We explain this construction. Let

$$u : \mathcal{F} \to \mathcal{F}'$$

be the Jardine fibrant replacement of $\mathcal{F}$. By Lemma 2.1, u is a sectionwise weak equivalence. Let $\mathcal{G}$ be the presheaf $i_*\mathcal{F}'$. By Lemma 2.2, $\mathcal{G}$ is Jardine fibrant. The counit of the adjunction

$$\epsilon : i^*\mathcal{G} = i^* i_* \mathcal{F}' \to \mathcal{F}'$$

is a stalkwise weak equivalence since, by Lemma 2.2, the adjoint pair (i^*, i_*) is a Quillen equivalence. The sheaf $i^*\mathcal{G}$ is easily seen to satisfy hyperdescent — it is the restriction of $\mathcal{G}$ to a subcategory. Therefore, by Lemma 2.1, the map ϵ is a sectionwise weak equivalence. Thus we have a zig-zag of sectionwise equivalences

$$i^*\mathcal{G} \to \mathcal{F}' \leftarrow \mathcal{F}.$$

Construction 2.3 requires a presheaf $\mathcal{F}$ on $\mathcal{X}_{et,aff}$ which satisfies homotopy descent with respect to hypercovers. The following lemma gives a useful criterion for verifying that $\mathcal{F}$ has this property.

Lemma 2.4. *Suppose that $\mathcal{F}$ is an object of $\mathrm{PreSp}_{\mathcal{X}_{et,aff}}$, and suppose that there is a graded quasi-coherent sheaf $\mathcal{A}_*$ on $\mathcal{X}$ and natural isomorphisms*

$$f_U : \mathcal{A}_*(U) \xrightarrow{\cong} \pi_*\mathcal{F}(U)$$

for all affine étale opens $U \to \mathcal{X}$. Then $\mathcal{F}$ satisfies homotopy descent with respect to hypercovers.

Proof. Suppose that $U \to \mathcal{X}$ is an affine étale open, and that $U_\bullet$ is a hypercover of U. Consider the Bousfield-Kan spectral sequence

$$E_2^{s,t} = \pi^s \mathcal{A}_t(U_\bullet) \Rightarrow \pi_{t-s} \operatorname{holim}_\Delta \mathcal{F}(U_\bullet).$$

Since $\mathcal{A}_*$ quasi-coherent, it satisfies étale hyperdescent, and we deduce that the E_2-term computes the quasi-coherent cohomology

$$E_2^{s,t} \cong H^s(U, \mathcal{A}_t)$$

and since U is affine, there is no higher cohomology. The E_2-term of this spectral sequence is therefore concentrated in $s = 0$. The spectral sequence collapses to give a diagram of isomorphisms

$$\begin{tikzcd}
\mathcal{A}_*(U) \arrow[d, "f_U"', "\cong"] \arrow[dr, "\cong"] & \\
\pi_*\mathcal{F}(U) \arrow[r] & \pi_* \operatorname{holim}_\Delta \mathcal{F}(U_\bullet)
\end{tikzcd}$$

We deduce that the map

$$\mathcal{F}(U) \to \operatorname{holim}_\Delta \mathcal{F}(U_\bullet)$$

is an equivalence. □

Remark 2.5. Construction 2.3 shows that to construct the presheaf $\mathcal{O}^{top}$, it suffices to construct $\mathcal{O}^{top}(U)$ functorially for affine étale opens $U \to \overline{\mathcal{M}}_{ell}$, as long as the resulting values $\mathcal{O}^{top}(U)$ satisfy homotopy descent with respect to affine hypercovers. This is automatic: there is an isomorphism

$$\pi_{2t}\mathcal{O}^{top}(U) \cong \omega^{\otimes t}(U)$$

for an invertible sheaf ω on $\overline{\mathcal{M}}_{ell}$. Lemma 2.4 implies that $\mathcal{O}^{top}$ satisfies the required hyperdescent conditions.

3. p-DIVISIBLE GROUPS OF ELLIPTIC CURVES

Let C be an elliptic curve over R, a p-complete ring. The p-divisible group $C(p)$ is the ind-finite group-scheme over R given by

$$C(p) = \varinjlim_k C[p^k].$$

Here, the finite group scheme $C[p^k]/R$ is the kernel of the p^k-power map on C.

Let $\widehat{C}$ be the formal group of C. If the height of the mod p-reduction of $\widehat{C}$ is constant, then over $\mathrm{Spf}(R)$ there is short exact sequence

$$0 \to \widehat{C} \to C(p) \to C(p)_{et} \to 0$$

where $C(p)_{et}$ is an ind-étale divisible group-scheme over R.

If $R = k$, a field of characteristic p, then we have

$$2 = \mathrm{height}(C(p)) = \mathrm{height}(\widehat{C}) + \mathrm{height}(C(p)_{et}).$$

The height of $\widehat{C}$ is the height of the formal group. The height of $C(p)_{et}$ is the corank of the corresponding divisible group. There are two possibilities:

1. C is *ordinary*: $\widehat{C}$ has height 1, and the divisible group $C(p)_{et}$ has corank 1.
2. C is *supersingular*: $\widehat{C}$ has height 2, and the divisible group $C(p)_{et}$ is trivial.

Theorem 3.1 (Serre-Tate)**.** *Suppose that R is a complete local ring with residue field k of characteristic p. Suppose that C is an elliptic curve over k. Then the functor*

$$\{\textit{deformations of } C \textit{ to } R\}$$
$$\downarrow$$
$$\{\textit{deformations of } C(p) \textit{ to } R\}$$

is an equivalence of categories.

4. CONSTRUCTION OF $\mathcal{O}^{top}_{K(2)}$

Lubin and Tate identified the formal neighborhood of a finite height formal group in $\mathcal{M}_{FG}$:

Theorem 4.1 (Lubin-Tate)**.** *Suppose that $\mathbb{G}$ is a formal group of finite height n over k, a perfect field of characteristic p. Then the formal moduli of deformations of $\mathbb{G}$ is given by*

$$\mathrm{Def}_{\mathbb{G}} \cong \mathrm{Spf}(B(k, \mathbb{G}))$$

where there is an isomorphism

$$B(k, \mathbb{G}) \cong \mathbb{W}(k)[[u_1, \dots, u_{n-1}]].$$

(Here, $\mathbb{W}(k)$ is the Witt ring of k.)

Let $\widetilde{\mathbb{G}}/B(k, \mathbb{G})$ denote the universal deformation of $\mathbb{G}$. The following theorem was proven by Goerss, Hopkins, and Miller [GH04].

Theorem 4.2 (Goerss-Hopkins-Miller)**.** *Let $\mathcal{C}$ be the category of pairs $(k, \mathbb{G})$ where k is a perfect field of characteristic p and $\mathbb{G}$ is a formal group of finite height over k. There is a functor*

$$\mathcal{C} \to E_\infty \text{ ring spectra}$$
$$(k, \mathbb{G}) \mapsto E(k, \mathbb{G})$$

where $E(k, \mathbb{G})$ is Landweber exact and even periodic, and

1. *$\pi_0 E(k, \mathbb{G}) = B(k, \mathbb{G})$,*
2. *$\mathbb{G}_{E(k,\mathbb{G})} \cong \widetilde{\mathbb{G}}$.*

Theorem 3.1 and Theorem 4.1 give the following.

Corollary 4.3.

1. *Suppose that C is a supersingular elliptic curve over a field k of characteristic p. There is an isomorphism*
$$\mathrm{Def}_C \cong \mathrm{Spf}(B(k, \widehat{C})).$$
2. *The substack $(\mathcal{M}^{ss}_{ell})_{\mathbb{F}_p} \subset (\overline{\mathcal{M}}_{ell})_p$ is zero dimensional.*

Proof. If C is a supersingular curve, then the inclusion of p-divisible groups $\widehat{C} \to C(p)$ is an isomorphism. Therefore, Theorem 3.1 implies that there is an isomorphism

$$\mathrm{Def}_C \cong \mathrm{Def}_{\widehat{C}}$$

and Theorem 4.1 identifies $\mathrm{Def}_{\widehat{C}}$.

To compute the dimension of $(\mathcal{M}^{ss}_{ell})_{\mathbb{F}_p}$ it suffices to do so étale locally. Let k be a finite field, and suppose that C is a supersingular elliptic curve over k. The completion of $\overline{\mathcal{M}}_{ell}$ along the map classifying C is the deformation space $\mathrm{Def}_C \cong \mathrm{Spf}(B(k, \widehat{C}))$, and there is an isomorphism

$$B(k, \widehat{C}) \cong \mathbb{W}(k)[[u_1]].$$

Since we have

$$u_1 \equiv v_1 \mod p,$$

the locus where $\widehat{C}$ has height 2 is given by the ideal (p, u_1). The quotient $B(k, \widehat{C})/(p, u_1)$ is k, and is therefore zero dimensional. □

We now construct the values of the presheaf $\mathcal{O}^{top}_{K(2)}$ on formal affine étale opens

$$f : \mathrm{Spf}(R) \to \mathcal{M}^{ss}_{ell}.$$

Here R is complete with respect to an ideal I. This suffices to construct the presheaf $\mathcal{O}^{top}_{K(2)}$ on $\mathcal{M}^{ss}_{ell}$ by Construction 2.3.

The induced map of special fibers

$$f_0 : \mathrm{Spec}(R/I) \to (\mathcal{M}^{ss}_{ell})_{\mathbb{F}_p}$$

is étale. Since $(\mathcal{M}_{ell}^{ss})_{\mathbb{F}_p}$ is smooth and zero-dimensional, $\mathrm{Spec}(R/I)$ must be étale over $\mathbb{F}_p$. We deduce that there is an isomorphism

$$R/I \cong \prod_i k_i,$$

a finite product of finite fields of characteristic p. Let C be the elliptic curve classified by f, and let C_0 be the elliptic curve classified by f_0. The decomposition of R/I induces a decomposition

$$C_0 \cong \coprod_i C_0^{(i)}.$$

Since f is étale, the elliptic curve C is a universal deformation of the elliptic curve C_0, and hence by Corollary 4.3 there is an isomorphism

$$R \cong \prod_i B(k_i, \widehat{C}_0^{(i)}).$$

We define

$$\mathcal{O}_{K(2)}^{top}(\mathrm{Spf}(R)) := \prod_i E(k_i, \widehat{C}_0^{(i)}).$$

Let $\mathbb{G}$ be the formal group of this even periodic ring spectrum. By Theorem 3.1, since $\mathbb{G}$ is a universal deformation of $\widehat{C}_0$ and C is a universal deformation of C_0, there is an isomorphism

$$\mathbb{G} \cong \widehat{C}.$$

We have therefore verified

Proposition 4.4. *The spectrum of sections* $\mathcal{O}_{K(2)}^{top}(\mathrm{Spf}(R))$ *is an elliptic spectrum associated to the elliptic curve* $C/\mathrm{Spf}(R)$.

5. The Igusa tower

If C is a generalized elliptic curve over a p-complete ring R, let C_{ns} denote the non-singular locus of $C \to \mathrm{Spf}(R)$. Then C_{ns} is a group scheme over R. Given a closed point $x \in \mathrm{Spf}(R)$, the fiber $(C_{ns})_x$ is given by

$$(C_{ns})_x = \begin{cases} C_x & C_x \text{ nonsingular,} \\ \mathbb{G}_m & C_x \text{ singular.} \end{cases}$$

The formal group $\widehat{C}$ is the formal group $\widehat{C}_{ns}$. We still may consider the ind-quasi-finite group-scheme

$$C(p) = \varinjlim_k C_{ns}[p^k].$$

$C(p)$ is technically *not* a p-divisible group, because its height is not uniform. Rather, we have the following table:

C_x	$ht(C(p)_x)$	$ht((C(p)_x)_{et})$	$ht(\widehat{C}_x)$
supersingular	2	0	2
ordinary	2	1	1
singular	1	0	1

If the classifying map

$$C : \mathrm{Spf}(R) \to (\overline{\mathcal{M}}_{ell})_p$$

factors through $\mathcal{M}_{ell}^{ord}$, then C has no supersingular fibers, but may have singular fibers. We shall call such a generalized elliptic curve C *ordinary.*

Let $\mathcal{M}_{ell}^{ord}(p^k)$ be the moduli stack whose R-points (for a p-complete ring R) is the groupoid of pairs (C, η) where

$$C/R = \text{ordinary generalized elliptic curve,}$$
$$(\eta : \mu_{p^k} \xrightarrow{\cong} \widehat{C}[p^k]) = \text{isomorphism of finite group schemes.}$$

The isomorphism η is a p^k-level structure. The stacks $\mathcal{M}_{ell}^{ord}(p^k)$ are representable by Deligne-Mumford stacks.

A p^{k+1}-level structure induces a canonical p^k-level structure, inducing a map

$$\mathcal{M}_{ell}^{ord}(p^{k+1}) \to \mathcal{M}_{ell}^{ord}(p^k). \tag{5.1}$$

Lemma 5.1. *The map $\mathcal{M}_{ell}^{ord}(p^{k+1}) \to \mathcal{M}_{ell}^{ord}(p^k)$ is an étale $\mathbb{Z}/p$-torsor (an étale $(\mathbb{Z}/p)^\times$-torsor if $k = 0$).*

Proof. (This proof is stolen from Paul Goerss.) By Lubin-Tate theory, the p-completed moduli stack $\mathcal{M}_{FG}^{mult}$ of multiplicative formal groups admits a presentation

$$\mathrm{Spf}(\mathbb{Z}_p) \to \mathcal{M}_{FG}^{mult}$$

which is a pro-étale torsor for the group:

$$\mathrm{Aut}(\widehat{\mathbb{G}}_m/\mathbb{Z}_p) = \mathbb{Z}_p^\times.$$

Associated to the closed subgroup $1 + p^k\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ is the étale torsor

$$\mathcal{M}_{FG}^{mult}(p^k) \to \mathcal{M}_{FG}^{mult}$$

for the group $(\mathbb{Z}/p^k)^\times$. The intermediate cover

$$\mathcal{M}_{FG}^{mult}(p^{k+1}) \to \mathcal{M}_{FG}^{mult}(p^k)$$

is an étale $\mathbb{Z}/p$-torsor (it is an étale $(\mathbb{Z}/p)^\times$-torsor if $k = 0$). The R-points of $\mathcal{M}_{FG}^{mult}(p^k)$ is the groupoid whose objects are pairs $(\mathbb{G}, \eta)$ where $\mathbb{G}$ is a formal group over $\mathrm{Spf}(R)$ locally isomorphic to $\widehat{\mathbb{G}}_m$, and η is a level p^k-structure:

$$\eta : \mu_{p^k} \xrightarrow{\cong} \mathbb{G}[p^k].$$

The stacks $\mathcal{M}_{ell}^{ord}(p^k)$ are therefore given by the pullbacks

$$\begin{tikzcd}
\mathcal{M}_{ell}^{ord}(p^{k+1}) \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord}(p^k) \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord} \arrow[d] \\
\mathcal{M}_{FG}^{mult}(p^{k+1}) \arrow[r] & \mathcal{M}_{FG}^{mult}(p^k) \arrow[r] & \mathcal{M}_{FG}^{mult}
\end{tikzcd}$$

where the map $\mathcal{M}_{ell}^{ord} \to \mathcal{M}_{FG}^{mult}$ classifies the formal group of the universal ordinary generalized elliptic curve. The result follows. ☐

Thus we have a tower of étale covers:

$$\begin{tikzcd}
\vdots \arrow[d] \\
\mathcal{M}_{ell}^{ord}(p^{k+1}) \arrow[d] \\
\mathcal{M}_{ell}^{ord}(p^{k}) \arrow[d] \\
\vdots \arrow[d] \\
\mathcal{M}_{ell}^{ord}
\end{tikzcd}$$

This is the *Igusa tower.*

Lemma 5.2. *For $k \geq 1$ ($k \geq 2$ if $p = 2$) the stack $\mathcal{M}_{ell}^{ord}(p^k)$ is formally affine: there is a p-complete ring V_k such that*

$$\mathcal{M}_{ell}^{ord}(p^k) = \mathrm{Spf}(V_k).$$

Proof. This is actually well known — see, for instance, Theorem 2.9.4, and the discussion at the beginning of Section 3.2.2 of [Hid04]. However, the proof of Theorem 2.9.4 in the above cited book only addresses the case where $p > 3$. The idea there is that the ordinary locus of the moduli of elliptic curves, with sufficient level structure, is affine. The result then follows from geometric invariant theory *provided one can show that the moduli problem $\mathcal{M}_{ell}^{ord}(p^k)$ is rigid (i.e. it takes values in sets, not groupoids).* Since p generates the ideal of definition of the formal stack $\mathcal{M}_{ell}^{ord}(p^k)$, it follows from Proposition 3.5.1 of [BL10] that it suffices to show that the mod p reduction $\mathcal{M}_{ell}^{ord}(p^k)_{\mathbb{F}_p}$ is rigid.

Let $(\mathcal{M}_{ell}^{ord})^n_{\mathbb{F}_p}$ denote the moduli stack (over $\mathbb{F}_p$) of ordinary elliptic curves with the structure of an n-jet at the basepoint. (Note that jets on an elliptic curve are the same thing as jets on the formal group of the elliptic curve.) By fixing a coordinate T_0 of $\widehat{\mathbb{G}}_m$, we observe that there is a closed inclusion

$$\mathcal{M}_{ell}^{ord}(p^k)_{\mathbb{F}_p} \hookrightarrow (\mathcal{M}_{ell}^{ord})^{p^k-1}_{\mathbb{F}_p}$$

as a level p^k-structure η gives an elliptic curve the structure of a $(p^k - 1)$-jet $\eta_* T_0$, and this jet uniquely determines the level structure. Thus it suffices to show that $(\mathcal{M}_{ell}^{ord})^{p^k-1}_{\mathbb{F}_p}$ is rigid. In fact, we will establish that it is affine.

Case 1: $p > 3$. Let R be an $\mathbb{F}_p$-algebra. Suppose that (C, T) is an object of $(\mathcal{M}_{ell}^{ord})^{p^k-1}_{\mathbb{F}_p}(R)$ for $k \geq 1$. Zariski-locally over $\mathrm{Spec}(R)$, there is a Weierstrass parameterization

$$C = C_{\mathbf{a}} : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

The Weierstrass curve $C_{\mathbf{a}}$ has a canonical coordinate at infinity given by $T_{\mathbf{a}} = -x/y$. Suppose that T is a $(p^k - 1)$-jet on $C_{\mathbf{a}}$, given by

$$T = m_0 T_{\mathbf{a}} + m_1 T_{\mathbf{a}}^2 + \cdots + m_{p^k-2} T_{\mathbf{a}}^{p^k-1} + O(T_{\mathbf{a}}^{p^k}).$$

According to [Rez, Rmk. 20.3], there are unique values

$$\begin{aligned}
\lambda &= \lambda(m_0) \\
s &= s(m_0, m_1) \\
r &= r(m_0, m_1, m_2) \\
t &= t(m_0, m_1, m_2, m_3)
\end{aligned}$$

such that under the transformation

$$\begin{aligned} f_{\lambda,s,r,t} : C_{\mathbf{a}} &\to C_{\mathbf{a}'} \\ x &\mapsto \lambda^2 x + r \\ y &\mapsto \lambda^3 y + sx + t \end{aligned}$$

the induced level (p^k-1)-jet $T' = (f_{\lambda,s,r,t})_* T$ is of the form

$$T' = T_{\mathbf{a}'} + m'_4 T^5_{\mathbf{a}'} + \cdots + m'_{p^k-2} T^{p^k-1}_{\mathbf{a}'} + O(T^{p^k}_{\mathbf{a}'}).$$

We have shown that the pair (C, η) is (Zariski locally) uniquely representable by a pair $(C_{\mathbf{a}}, T)$ where

$$T = T_{\mathbf{a}} + m_4 T^5_{\mathbf{a}} + \cdots m_{p^k-2} T^{p^k-1}_{\mathbf{a}} + O(T^{p^k}_{\mathbf{a}}).$$

The only morphism $f_{\lambda,s,r,t} : C_{\mathbf{a}} \to C_{\mathbf{a}'}$ which satisfies

$$f^*_{\lambda,s,r,t} T_{\mathbf{a}'} = T_{\mathbf{a}} + O(T^5_{\mathbf{a}})$$

has $\lambda = 1$ and $s = r = t = 0$. Thus (C, T) is determined, Zariski locally, up to unique isomorphism, by the functions

$$a_1, a_2, a_3, a_4, a_6, m_4, \ldots, m_{p^k-2}.$$

The uniqueness of these functions implies that they are compatible on the intersections of a Zariski open cover, and hence patch to give global invariants of (C, T). Expressing the Eisenstein series (Hasse invariant) E_{p-1} of $C_{\mathbf{a}}$ as

$$E_{p-1} = E_{p-1}(a_1, a_2, a_3, a_4, a_6),$$

it follows that we have

$$(\mathcal{M}^{ord}_{ell})^{p^k-1}_{\mathbb{F}_p} \cong \mathrm{Spec}(\mathbb{F}_p[a_1, a_2, a_3, a_4, a_6, m_4, \ldots, m_{p^k-2}])[E^{-1}_{p-1}]).$$

With minor modification, the method for $p > 3$ extends to the cases $p = 2, 3$. The canonical forms for (C, T) just change slightly.

Case 2: $p = 3$. Suppose that (C, T) is an object of $(\mathcal{M}^{ord}_{ell})^{3^k-1}_{\mathbb{F}_3}$ for $k \geq 1$. If $k > 1$, then $3^k - 1 \geq 4$, and thus (C, T) admits a canonical Weierstrass presentation.

If $k = 1$, then this no longer holds. Instead, choosing

$$\begin{aligned} \lambda &= \lambda(m_0) \\ s &= s(m_0, m_1) \end{aligned}$$

there exists (Zariski locally) a Weierstrass curve $(C_{\mathbf{a}}, T) \cong (C, T)$ such that

$$T = T_{\mathbf{a}} + O(T^3_{\mathbf{a}}).$$

Choosing t_0 accordingly, there is a transformation

$$\begin{aligned} f_{t_0} : C_{\mathbf{a}} &\to C_{\mathbf{a}'} \\ (x, y) &\mapsto (x, y + t_0) \end{aligned}$$

such that $a'_3 = 0$. The induced 2-jet $T' = (f_{t_0})_* T$ still satisfies $T' \equiv T_{\mathbf{a}'} \mod T^3_{\mathbf{a}'}$. The automorphisms $f_{\lambda,s,r,t}$ of $(C_{a'}, T')$ preserving the property that $a_3 = 0$, and the trivialization of the 2-jet, satisfy

$$\begin{aligned} \lambda &= 1 \\ s &= 0 \\ t &= -a_1 r/2. \end{aligned}$$

Under such a transformation, we find that

$$a_4 \mapsto a_4 + 2b_2 r + 3r^2$$

where $b_2 = a_2 + a_1^2/4$. Because C is assumed to be ordinary, the element b_2 is a unit. Because R is an $\mathbb{F}_3$-algebra, there is a unique r such that $a_4 \mapsto 0$. Thus we have shown that there is a canonical Weierstrass presentation which trivializes the 2-jet, and for which $a_3 = a_4 = 0$.

Case 3: $p = 2$. Assume that $k = 2$ (for $k > 2$, we have $2^k - 1 \geq 4$, and therefore an elliptic curve with a $2^k - 1$-jet admits a canonical Weierstrass presentation). Let (C, T) be an object of $(\mathcal{M}_{ell}^{ord})^3_{\mathbb{F}_2}$. Choose (Zariski locally) a Weierstrass presentation $(C_{\mathbf{a}}, T) \cong (C, T)$. Choosing

$$\begin{aligned} \lambda &= \lambda(m_0) \\ s &= s(m_0, m_1) \\ r &= r(m_0, m_1, m_2) \end{aligned}$$

we may assume that T satisfies

$$T = T_{\mathbf{a}} + O(T_{\mathbf{a}}^4).$$

The automorphisms $f_{\lambda,s,r,t}$ of (C_a, T) preserving the trivialization of the 3-jet satisfy

$$\begin{aligned} \lambda &= 1, \\ s &= 0, \\ r &= 0. \end{aligned}$$

Under such a transformation, we find that

$$a_4 \mapsto a_4 - a_1 t.$$

Because C is assumed to be ordinary, the element a_1 is a unit. Letting $t = a_4/a_1$, we have $a_4 \mapsto 0$. Thus we have shown that there is a canonical Weierstrass presentation which trivializes the 3-jet, and for which $a_4 = 0$. ☐

Define

$$V_\infty^\wedge := \varprojlim_m \varinjlim_k V_k / p^m V_k.$$

The ring $V_\infty^\wedge$ is the ring of *generalized p-adic modular functions (of level 1)*.

Let $\mathcal{M}_{ell}^{ord}(p^\infty)$ be the formal scheme $\mathrm{Spf}(V_\infty^\wedge)$. There is an isomorphism between the R-points $\mathcal{M}_{ell}^{ord}(p^\infty)(R)$ and isomorphism classes of pairs (C, η) where

$$\begin{aligned} C &= \text{a generalized elliptic curve over } R, \\ (\eta : \widehat{\mathbb{G}}_m \cong \mu_{p^\infty} \xrightarrow{\cong} \widehat{C}) &= \text{an isomorphism of formal groups.} \end{aligned}$$

(Note that the existence of η implies that C has ordinary reduction modulo p.)

The ring $V_\infty^\wedge$ possesses a special structure: it is a θ-algebra (see [GH]). That is, it has actions of operators

$$\begin{aligned} \psi^k, &\quad k \in \mathbb{Z}_p^\times, \\ \psi^p, &\quad \text{lift of the Frobenius}, \\ \theta, &\quad \text{satisfying } \psi^p(x) = x^p + p\theta(x). \end{aligned}$$

The operations ψ^k and ψ^p are ring homomorphisms. The operation θ is determined from ψ^p, since $V_\infty^\wedge$ is torsion-free and we have

$$\psi^p(x) \equiv x^p \mod p.$$

We determine ψ^k and ψ^p on the functors of points of $\mathcal{M}_{ell}^{ord}(p^\infty)$. Suppose that R is a p-complete ring. Note that

$$\mathrm{Aut}_{\mathbb{Z}_p}(\widehat{\mathbb{G}}_m) \cong \mathbb{Z}_p^\times.$$

We may therefore regard $\mathbb{Z}_p^\times$ as acting on $\widehat{\mathbb{G}}_m/R$. Let $[k]$ be the automorphism corresponding to $k \in \mathbb{Z}_p^\times$. Define

$$\begin{aligned} (\psi^k)^* : \mathcal{M}_{ell}^{ord}(p^\infty)(R) &\to \mathcal{M}_{ell}^{ord}(p^\infty)(R) \\ (C, \eta) &\mapsto (C, \eta \circ [k]). \end{aligned}$$

The map $(\psi^k)^*$ is represented by a map

$$\psi^k : V_\infty^\wedge \to V_\infty^\wedge.$$

Suppose that (C, η) is an R-point of $\mathcal{M}_{ell}^{ord}(p^\infty)$. Since C has ordinary reduction mod p, the pth power endomorphism of C_{ns} factors as

(5.2)
$$\begin{tikzcd}
C_{ns} \arrow[rr, "{[p]}"] \arrow[dr, swap, "\Phi_{insep}"] & & C_{ns} \\
& C_{ns}^{(p)} \arrow[ur, swap, "\Phi_{sep}"] &
\end{tikzcd}$$

where Φ_{insep} is purely inseparable. The morphism Φ_{sep} is not, in general, étale, but $\ker \Phi_{sep}$ is an étale group scheme over R. On the non-singular fibers of C, Φ_{sep} has degree p, whereas on the singular fibers it has degree 1.

These morphisms, and their kernels, fit into a 3×3 diagram of short exact sequences of group schemes:

$$\begin{tikzcd}
\widehat{C}[p] \arrow[r] \arrow[d, equal] & C_{ns}[p] \arrow[r] \arrow[d] & C[p]_{et} \arrow[d] \\
\widehat{C}[p] \arrow[r] \arrow[d] & C_{ns} \arrow[r, swap, "\Phi_{insep}"] \arrow[d, swap, "{[p]}"] & C_{ns}^{(p)} \arrow[d, "\Phi_{sep}"] \\
0 \arrow[r] & C_{ns} \arrow[r, equal] & C_{ns}
\end{tikzcd}$$

where $\widehat{C}[p]$ is the p-torsion subgroup of the height 1 formal group $\widehat{C}$ and $C[p]_{et}$ is the p-torsion of the ind-finite group scheme $C(p)_{et}$.

Given a uniformization

$$\eta : \widehat{\mathbb{G}}_m \xrightarrow{\cong} \widehat{C}$$

we get an induced uniformization $\eta^{(p)}$:

(5.3)
$$\begin{tikzcd}
\mu_p \arrow[r] \arrow[d, "\cong"'] & \widehat{\mathbb{G}}_m \arrow[r, "{[p]}"] \arrow[d, "\cong"', "\eta"] & \widehat{\mathbb{G}}_m \arrow[d, dotted, "\eta^{(p)}"] \\
\widehat{C}[p] \arrow[r] & \widehat{C} \arrow[r, swap, "(\Phi_{insep})_*"] & \widehat{C^{(p)}}
\end{tikzcd}$$

Remark 5.3. The uniformization $\eta^{(p)}$ admits a different characterization: it is the unique isomorphism of formal groups making the following diagram commute:

$$\begin{tikzcd}
& \widehat{\mathbb{G}}_m \arrow[dl, dotted, swap, "\eta^{(p)}"] \arrow[d, "\eta"] \\
\widehat{C^{(p)}} \arrow[r, swap, "(\Phi_{sep})_*"] & \widehat{C}
\end{tikzcd}$$

(The isogeny Φ_{sep} induces an isomorphism on formal groups.) The equivalence of this definition of $\eta^{(p)}$ with the previous is proved by the following diagram.

$$\begin{tikzcd}
\widehat{\mathbb{G}}_m \arrow[r, "{[p]}"] \arrow[d, "\eta"'] & \widehat{\mathbb{G}}_m \arrow[r, equal] \arrow[d, "\eta^{(p)}"'] & \widehat{\mathbb{G}}_m \arrow[d, "\eta"] \\
\widehat{C} \arrow[r, "(\Phi_{insep})_*"] \arrow[rr, bend right, "{[p]}"'] & \widehat{C^{(p)}} \arrow[r, "(\Phi_{sep})_*", "\cong"'] & \widehat{C}
\end{tikzcd}$$

We get a map on R-points

$$\begin{aligned}(\psi^p)^* : \mathcal{M}_{ell}^{ord}(p^\infty)(R) &\to \mathcal{M}_{ell}^{ord}(p^\infty)(R) \\ (C, \eta) &\mapsto (C^{(p)}, \eta^{(p)})\end{aligned}$$

which is represented by a ring map

$$\psi^p : V_\infty^\wedge \to V_\infty^\wedge.$$

It is easy to see that ψ^p commutes with ψ^k. To show that ψ^p induces a θ-algebra structure on $V_\infty^\wedge$, it suffices to prove the following:

Lemma 5.4. *We have $\psi^p(x) \equiv x^p$ mod p.*

Proof. It suffices to show that $(\psi^p)^*$ is represented by the Frobenius when restricted to characteristic p. That is, we must show that if R is an $\mathbb{F}_p$-algebra, and (C, η) is an R point of $\mathcal{M}_{ell}^{ord}(p^\infty)$, then the Frobenius

$$\begin{aligned}\sigma : R &\to R \\ x &\mapsto x^p\end{aligned}$$

gives rise to an isomorphism

$$(C^{(p)}, \eta^{(p)}) \cong (\sigma^* C, \sigma^* \eta).$$

We briefly introduce some notation: if X is a scheme over R, then we have the following diagram of morphisms.

$$\begin{tikzcd}
X \arrow[dr, "Fr^{rel}"'] \arrow[drr, bend left, "Fr^{tot}"] \arrow[ddr, bend right] & & \\
& \sigma^* X \arrow[r, "Fr"] \arrow[d] & X \arrow[d] \\
& \mathrm{Spec}(R) \arrow[r, "\sigma^*"'] & \mathrm{Spec}(R)
\end{tikzcd}$$

The square is a pullback square, and Fr is the pullback of σ. The map Fr^{tot} is the total Frobenius, and the universal property of the pullback induces the relative Frobenius Fr^{rel}.

Because the isogeny Fr^{rel} has degree p, we have a factorization

$$\begin{tikzcd}
C_{ns} \arrow[rr, "{[p]}"] \arrow[dr, "Fr^{rel}"'] & & C_{ns} \\
& \sigma^* C_{ns} \arrow[ur, "\widehat{Fr^{rel}}"'] &
\end{tikzcd}$$

Because C has no supersingular fibers, the dual isogeny $\widehat{Fr^{rel}}$ has separable kernel (see, for instance, [Sil86, Thm. V.3.1]).

Therefore, we have

$$\sigma^* C \cong C^{(p)},$$

$$\Phi_{insep} \cong Fr^{rel},$$

$$\Phi_{sep} \cong \widehat{Fr^{rel}}.$$

We just have to show that under these isomorphisms, we have $\sigma^*\eta \cong \eta^{(p)}$. We have the following diagram of formal groups.

$$\begin{tikzcd}
\widehat{\mathbb{G}}_m \arrow[r, "Fr^{rel}"] \arrow[d, "\eta"'] & \widehat{\mathbb{G}}_m \arrow[r, "Fr"] \arrow[d, "\sigma^*\eta"'] & \widehat{\mathbb{G}}_m \arrow[d, "\eta"] \\
\widehat{C} \arrow[r, "Fr^{rel}"'] & \widehat{\sigma^* C} \arrow[r, "Fr"'] & \widehat{C}
\end{tikzcd}$$

On $\mathbb{G}_m$, the relative Frobenius is the pth power map. Therefore, by the definition of $\eta^{(p)}$, we have $\sigma^*\eta \cong \eta^{(p)}$ under the isomorphism $\sigma^* C \cong C^{(p)}$. ☐

More generally, letting ω_∞ denote the canonical line bundle over $\mathrm{Spf}(V_\infty^\wedge)$, then the graded algebra

$$(V_\infty^\wedge)_{2*} := \Gamma \omega_\infty^{\otimes *}$$

inherits the structure of an even periodic graded θ-algebra. The θ-algebra structure may be described by the isomorphism

$$(V_\infty^\wedge)_* \cong (K_p)_* \otimes_{\mathbb{Z}_p} V_\infty^\wedge.$$

Here $(K_p)_*$ is the coefficients of p-adic K-theory, and the θ-algebra structure is induced from the diagonal action of the Adams operations.

Remark 5.5. By defining ψ^p on $V_\infty^\wedge$ using its modular interpretation, I have glossed over several technical issues related to extending the quotient by the canonical subgroup of ordinary elliptic curves to the singular fibers of a generalized elliptic curve. The careful reader could instead choose a different path to defining the operation ψ^p: define it just as I have on the non-singular fibers, and then explicitly define its effect on q-expansions to extend this definition over the cusp. (The effect of ψ^p on q expansions is to raise q to its pth power.)

6. $K(1)$ LOCAL ELLIPTIC SPECTRA

In this section we will investigate the abstract properties satisfied by a $K(1)$-local elliptic spectrum. Throughout this section, suppose that (E, α, C) be an elliptic spectrum. That is to say, E is a $K(1)$-local weakly even periodic ring spectrum, C is a generalized elliptic curve over $R = \pi_0 E$, and α is an isomorphism of formal groups

$$\alpha : \mathbb{G}_E \to \widehat{C}.$$

We shall furthermore assume that R is p-complete, and that the classifying map

$$f : \mathrm{Spf}(R) \to (\mathcal{M}_{ell})_p$$

is flat. This implies:

1. E is Landweber exact (Remark 1.4),
2. C has ordinary reduction modulo p (Lemma 8.1).

There are three distinct subjects we shall address in this section.

1. The p-adic K-theory of $K(1)$-local elliptic spectra.
2. θ-compatible $K(1)$-local elliptic E_∞-ring spectra.
3. The θ-algebra structure of the p-adic K-theory of a supersingular elliptic E_∞-ring spectrum.

The p-adic K-theory of $K(1)$-local elliptic spectra.

Let

$$(K_p^\wedge)_*E := \pi_*((K \wedge E)_p)$$

denote the p-adic K-homology of E. It is geometrically determined by the following standard proposition.

Proposition 6.1. *Let* $\mathrm{Spf}(W)$ *be the pullback of* $\mathrm{Spf}(R)$ *over* $\mathcal{M}_{ell}^{ord}(p^\infty)$. *Then there is an isomorphism*

$$(K_p^\wedge)_0 E \cong W.$$

This isomorphism is $\mathbb{Z}_p^\times$*-equivariant, where the* $\mathbb{Z}_p^\times$*-acts on the left hand side through stable Adams operations, and it acts on the right hand side due to the fact that* $\mathrm{Spf}(W)$ *is an ind-étale* $\mathbb{Z}_p^\times$*-torsor over* $\mathrm{Spf}(R)$.

Proof. By Landweber exactness, choosing complex orientations for K_p and E, we have

$$(K_p^\wedge)_0 E = ((K_p)_0 \otimes_{MUP_0} MUP_0MUP \otimes_{MUP_0} R)_p^\wedge.$$

Using the fact that $\mathrm{Spec}(MUP_0MUP) = \mathrm{Spec}(MUP_0) \times_{\mathcal{M}_{FG}} \mathrm{Spec}(MUP_0)$, it is not hard to deduce from this that we have

$$\mathrm{Spf}((K_p^\wedge)_0 E) \cong \mathrm{Spf}(R) \times_{\mathcal{M}_{FG}} \mathrm{Spf}((K_p)_0).$$

Consider the induced diagram

$$\begin{tikzcd}
\mathrm{Spf}((K_p^\wedge)_0 E) \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[r] \arrow[d] & \mathrm{Spf}((K_p^\wedge)_0) \arrow[d] \\
\mathrm{Spf}(R) \arrow[r, "f"'] & \mathcal{M}_{ell}^{ord} \arrow[r] & \mathcal{M}_{FG}
\end{tikzcd}$$

The right-hand square is a pullback by the proof of Lemma 5.1, and we have established that the composite is a pullback. We deduce that the left-hand side is a pullback, which completes the proof. ☐

θ-compatible $K(1)$-local elliptic E_∞-ring spectra.

If E is an E_∞-ring spectrum, then the completed K_p-homology

$$(K_p^\wedge)_*E := \pi_*((K \wedge E)_p)$$

naturally carries the structure of a θ-algebra: for $k \in \mathbb{Z}_p^\times$, the operations ψ^k are the stable Adams operations in K_p-theory, and the operation θ arises from the action of the $K(1)$-local Dyer-Lashof algebra [GH].

If the classifying map

$$f : \mathrm{Spf}(R) \to \mathcal{M}_{ell}^{ord}$$

is étale, then the pullback W of Proposition 6.1 carries naturally the structure of a θ-algebra which we now explain. Since $\mathrm{Spf}(R)$ is étale over $\mathcal{M}_{ell}^{ord}$, the pullback $\mathrm{Spf}(W)$ is étale over $\mathcal{M}_{ell}^{ord}(p^\infty) = \mathrm{Spf}(V_\infty^\wedge)$. It is in particular formally étale, and therefore there exists a unique lift

$$\begin{tikzcd}
\mathrm{Spec}(W/p) \arrow[r, "\sigma^*"] \arrow[d] & \mathrm{Spec}(W/p) \arrow[r] & \mathrm{Spf}(W) \arrow[d] \\
\mathrm{Spf}(W) \arrow[r] \arrow[urr, dotted, "(\psi^p)^*"'] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[r, "(\psi^p)^*"'] & \mathcal{M}_{ell}^{ord}(p^\infty)
\end{tikzcd}$$

where

$$(\psi^p)^* : \mathcal{M}_{ell}^{ord}(p^\infty) \to \mathcal{M}_{ell}^{ord}(p^\infty)$$

is the lift of the Frobenius coming from θ-algebra structure of $V_\infty^\wedge$ and $\sigma : W/p \to W/p$ is the Frobenius. Note that because $\mathrm{Spf}(R)$ is étale over $\mathcal{M}_{ell}^{ord}$, it is in particular flat, and so W must be torsion-free. Therefore, the induced homomorphism

$$\psi^p : W \to W$$

determines a unique θ-algebra structure on W.

If E is E_∞, and the classifying map f is étale, it is *not* necessarily the case that the isomorphism

$$(K_p^\wedge)_0 E \cong W$$

of Proposition 6.1 preserves the operation ψ^p. This is rather a reflection of the choice of E_∞-structure on E. We therefore make the following definition.

Definition 6.2. Suppose that E is a $K(1)$-local E_∞ elliptic spectrum associated to an elliptic curve C/R, and suppose that the classifying map

$$\mathrm{Spf}(R) \to (\overline{\mathcal{M}}_{ell})_p$$

is étale. If the isomorphism $(K_p^\wedge)_0 E \cong W$ is a map of θ-algebras, then we shall say that (E, C) is a *θ-compatible.*

Remark 6.3. As a *side-effect* of our construction of $\mathcal{O}_{K(1)}^{top}$ it will be the case that the E_∞-structure on the spectrum of sections $E = \mathcal{O}_{K(1)}^{top}(\mathrm{Spf}(R))$ is θ-compatible.

Remark 6.4. In [AHS04], the authors define the notion of an H_∞-elliptic ring spectrum, which is a stronger notion than that of an elliptic H_∞-ring spectrum in that they require a compatibility between the H_∞-structure and the elliptic structure. It is easily seen that every $K(1)$-local H_∞-elliptic spectrum whose classifying map is étale over the p-completion of the moduli stack of elliptic curves is θ-compatible.

The θ-algebra structure of the p-adic K-theory of supersingular elliptic E_∞-ring spectra. In [AHS04, Sec. 3], previous work of Ando and Strickland is condensed into an elegant perspective on Dyer-Lashof operations on an even periodic complex orientable H_∞-ring spectrum T. Namely, suppose that

1. T is *homogeneous* — it is a homotopy commutative algebra spectrum over an even periodic E_∞-ring spectrum (such as MU).
2. $\pi_0 T$ is a complete local ring with residue field of characteristic p.
3. The reduction $\bar{\mathbb{G}}_T$ of $\mathbb{G}_T$ modulo the maximal ideal has finite height.
4. $\mathbb{G}_T$ is *Noetherian* — it is obtained by pullback from a formal group over $\mathrm{Spf}(S)$ where S is Noetherian.

Then, for every morphism

$$i : \mathrm{Spf}(R) \to \mathrm{Spf}(\pi_0 T)$$

and every finite subgroup $H < i^*\mathbb{G}_T$ (i.e. H is an effective relative Cartier divisor of $i^*\mathbb{G}_T$ represented by a subgroup-scheme) there is a new morphism

$$\psi_H : \mathrm{Spf}(R) \to \mathrm{Spf}(\pi_0 T)$$

and an isogeny of formal groups

$$f_H : i^*\mathbb{G}_T \to \psi_H^*\mathbb{G}_T$$

with kernel H. This structure is called *descent data for subgroups.*

Remark 6.5. The authors of [AHS04] actually describe the structure of *descent data for level structures.* However, their treatment carries over to subgroups (see [AHS04, Rmk. 3.12]).

Example 6.6. Suppose that T is a $K(1)$-local E_∞-ring spectrum. Then the formal group $\mathbb{G}_T$ must have height 1 (see the proof of Lemma 8.1), and it follows that $\mathbb{G}_T$ has a unique subgroup of order p, given by the p-torsion subgroup $\mathbb{G}_T[p]$. Taking i to be the identity map, we get an operation

$$\psi_{\mathbb{G}_T[p]} : \pi_0 T \to \pi_0 T.$$

This operation coincides with ψ^p. We shall let f_p denote the associated degree p isogeny

$$f_p = f_{\mathbb{G}_T[p]} : \mathbb{G}_T \to (\psi^p)^* \mathbb{G}_T.$$

Example 6.7. Suppose that $T = E(k, \mathbb{G})$ is the Morava E-theory associated to a height n formal group $\mathbb{G}/k$, with universal deformation $\tilde{\mathbb{G}}/B$, and the E_∞-structure of Goerss and Hopkins [GH04]. Then in [AHS04] it is proven that the associated descent data for subgroups is given as follows. Let

$$i : \mathrm{Spf}(R) \to \mathrm{Spf}(B)$$

be a morphism classifying a deformation $i^*\tilde{\mathbb{G}}/R$ of $i^*\mathbb{G}/k'$ (where $k' := R/\mathfrak{m}_R$). Suppose that $\tilde{H} < i^*\tilde{\mathbb{G}}$ is a finite subgroup, and let H denote the restriction of $\tilde{H}$ to $i^*\mathbb{G}$. Because $\mathbb{G}$ is a formal group of finite height over a field of characteristic p, the only subgroups of $\mathbb{G}$ are of the form

$$H_r = \ker((Fr^{rel})^r : i^*\mathbb{G} \to i^*\mathbb{G})$$

where Fr^{rel} is the relative Frobenius. Therefore, we have $H = H_r$ for some r. The quotient $(i^*\mathbb{G})/H_r$ is the pullback of $\mathbb{G}$ under the composite $i^{(p^r)}$:

$$i^{(p^r)} : \mathrm{Spf}(k') \xrightarrow{(\sigma^r)^*} \mathrm{Spf}(k') \xrightarrow{i} \mathrm{Spf}(k)$$

where σ is the Frobenius. The quotient $i^*\tilde{\mathbb{G}}/\tilde{H}$ is then a deformation of $(i^*\mathbb{G})/H \cong (i^{(p^r)})^*\mathbb{G}$, hence is classified by a morphism

$$\psi_{\tilde{H}} : \mathrm{Spf}(R) \to \mathrm{Spf}(B).$$

This determines the operation $\psi_{\tilde{H}}$. The morphism $f_{\tilde{H}}$ is given by

$$f_{\tilde{H}} : i^*\tilde{\mathbb{G}} \to (i^*\tilde{\mathbb{G}})/\tilde{H} \cong (\psi_{\tilde{H}})^*\tilde{\mathbb{G}}.$$

Suppose now that k is a finite field, and that C/k is a supersingular elliptic curve. Then, by Serre-Tate theory, there is a unique elliptic curve $\tilde{C}$ over the universal deformation ring

$$B := B(k, \widehat{C}) \cong \mathbb{W}(k)[[u_1]]$$

such that the formal group $\tilde{C}^\wedge$ is the universal deformation of the formal group $\widehat{C}$. Furthermore, we have seen that the Goerss-Hopkins-Miller theorem associates to $\widehat{C}/k$ an elliptic E_∞-ring spectrum

$$E := E(k, \widehat{C}) = \mathcal{O}^{top}_{K(2)}(\mathrm{Spf}(B))$$

with associated elliptic curve $\tilde{C}$.

The curve $\tilde{C}$ is to be regarded as an elliptic curve over $\mathrm{Spf}(B)$, but by Remark 1.6, there is a unique elliptic curve $\tilde{C}^{alg}$ over $\mathrm{Spec}(B)$ which restricts to $\tilde{C}/\mathrm{Spf}(B)$. Let B^{ord} be the ring

$$B^{ord} = B[u_1^{-1}]^\wedge_p.$$

We regard B^{ord} as being complete with respect to the ideal (p). Let $\tilde{C}^{ord}$ be the restriction of $\tilde{C}^{alg}$ to $\mathrm{Spf}(B^{ord})$. The following lemma follows from Lemma 8.1.

Lemma 6.8. *The spectrum $E_{K(1)}$ is an elliptic spectrum for the elliptic curve $\tilde{C}^{ord}/B^{ord}$.*

The Goerss-Hopkins E_∞-structure on E induces an E_∞ structure on the $K(1)$-localized spectrum $E_{K(1)}$, and there is an induced operation

$$\psi^p : B^{ord} \to B^{ord}$$

on $B^{ord} = \pi_0 E_{K(1)}$ which lifts the Frobenius in characteristic p. We have the following proposition.

Proposition 6.9. *There is an isomorphism*

$$(\psi^p)^*\tilde{C}^{ord} \cong (\tilde{C}^{ord})^{(p)}$$

(where $(\tilde{C}^{ord})^{(p)}$ is the quotient of $\tilde{C}^{ord}$ of Diagram (5.2)) making the following diagram of isogenies of formal groups commute.

(6.1)
$$\begin{tikzcd}
(\tilde{C}^{ord})^\wedge \arrow[r, "f_p"] \arrow[dr, "(\Phi_{insep})_*"'] & (\psi^p)^*(\tilde{C}^{ord})^\wedge \arrow[d, "\cong"] \\
 & ((\tilde{C}^{ord})^{(p)})^\wedge
\end{tikzcd}$$

Proof. (In some sense, this proposition is one of the most important ingredients to the construction of *tmf*, and I would have gotten it wrong except for the help of Niko Naumann and Charles Rezk.) Let

$$i : \mathrm{Sub}_p(\tilde{C}) \to \mathrm{Spf}(B)$$

be the formal scheme of "subgroups of $\tilde{C}$ of order p" (see, for instance, [Str97]). The formal scheme $\mathrm{Sub}_p(\tilde{C})$ is of the form

$$\mathrm{Spf}(\Gamma_0(p)(\tilde{C})).$$

Observe that because the p-divisible group of $\tilde{C}$ is entirely formal, we have

$$\Gamma_0(p)(\tilde{C}) = \Gamma_0(p)(\tilde{C}^\wedge).$$

Let $\tilde{H}_{can}$ be the universal degree p subgroup of $i^*\tilde{C}$. There is a corresponding operation

$$\psi_{\tilde{H}_{can}} : B = \pi_0 E \to \Gamma_0(p)(\tilde{C})$$

and an isomorphism

$$\psi^*_{\tilde{H}_{can}} \tilde{C}^\wedge \cong (i^*\tilde{C}^\wedge)/\tilde{H}_{can}. \tag{6.2}$$

This operation arises topologically from the total power operation

$$\psi_{\tilde{H}_{can}} : B = \pi_0 E \xrightarrow{\mathcal{P}_E} \pi_0 E^{B\Sigma_{p+}} \twoheadrightarrow \Gamma_0(p)(\tilde{C}^\wedge)$$

where the surjection is the quotient by the image of the transfer morphisms [AHS04, Rmk. 3.12]).

Forgetting the topology on the ring B, we can regard $\tilde{C}^\wedge$ simply as a formal group over the ring B, and we get an induced formal group $(\tilde{C}^{ord})^\wedge/B^{ord}$ and degree p subgroup

$$\tilde{H}^{ord}_{can} < i^*(\tilde{C}^{ord})^\wedge$$

over

$$\Gamma_0(p)(\tilde{C})^{ord} := \Gamma_0(p)(\tilde{C})[u_1^{-1}]^\wedge_p \cong \Gamma_0(p)(\tilde{C}^{ord}).$$

The last isomorphism follows from the fact that *forgetting* about formal schemes, there is an isomorphism

$$\mathrm{Sub}_p(\tilde{C}^{alg}) \times_{\mathrm{Spec}(B)} \mathrm{Spec}(B^{ord}) \cong \mathrm{Sub}_p(\tilde{C}^{alg} \times_{\mathrm{Spec}(B)} \mathrm{Spec}(B^{ord})).$$

Let

$$c : \Gamma_0(p)(\tilde{C}^{ord}) \to \Gamma_0(p)((\tilde{C}^{ord})^\wedge) \cong B^{ord}$$

be the map classifying the subgroup $\tilde{H}^{ord}_{can}$ of order p of $\Gamma_0(p)((\tilde{C}^{ord})^\wedge)$, regarded as a subgroup of $\tilde{C}^{ord}$. The isomorphism $\Gamma_0(p)((\tilde{C}^{ord})^\wedge) \cong B^{ord}$ reflects the fact that there is one and only one degree p-subgroup of a deformation of a height 1 formal group. Thus

$$\tilde{H}^{ord}_{can} = i^*(\tilde{C}^{ord})^\wedge[p]. \tag{6.3}$$

By Example 6.6, the corresponding operation

$$\psi_{c^*\tilde{H}^{ord}_{can}} : B^{ord} \to \Gamma_0(p)((\tilde{C}^{ord})^\wedge) \cong B^{ord}$$

is nothing more than the operation ψ^p on the $K(1)$-local E_∞ ring spectrum $E_{K(1)}$. Since localization $E \to E_{K(1)}$ is a map of E_∞-ring spectra, E and $E_{K(1)}$ have compatible descent data for subgroups, and we deduce that there is a commutative diagram:

(6.4)
$$\begin{tikzcd}
B \arrow[r] \arrow[d, equal] \arrow[ddd, bend right=60, "\psi_{\tilde{H}_{can}}"'] & B^{ord} \arrow[d, equal] \arrow[dddrr, bend left, "\psi^p"] & & \\
\pi_0 E \arrow[r] \arrow[d, "\mathcal{P}_E"] & \pi_0 E_{K(1)} \arrow[d, "(\mathcal{P}_E)_{K(1)}"'] \arrow[dr, "\mathcal{P}_{E_{K(1)}}"] & & \\
\pi_0 E^{B\Sigma_{p+}} \arrow[r] \arrow[d, two heads] & \pi_0(E^{B\Sigma_{p+}})_{K(1)} \arrow[r] \arrow[d, two heads] & \pi_0(E_{K(1)})^{B\Sigma_{p+}} \arrow[d, two heads] & \\
\Gamma_0(p)(\tilde{C}) \arrow[r, "j"'] & \Gamma_0(p)(\tilde{C}^{ord}) \arrow[r, "c"'] & \Gamma_0(p)((\tilde{C}^{ord})^\wedge) \arrow[r, equal] & B^{ord}
\end{tikzcd}$$

Using Serre-Tate theory, the isomorphism (6.2) lifts to an isomorphism

$$(i^* \tilde{C})/\tilde{H}_{can} \cong (\psi_{\tilde{H}_{can}})^* \tilde{C}.$$

Using the isomorphism above, equality (6.3), and the commutativity of Diagram 6.4, we deduce that there are isomorphisms

$$\begin{aligned}
(\tilde{C}^{ord})^{(p)} &= \tilde{C}^{ord}/c^* \tilde{H}^{ord}_{can} \\
&= c^* j^* ((i^* \tilde{C})/\tilde{H}_{can}) \\
&\cong c^* j^* (\psi_{\tilde{H}_{can}})^* \tilde{C} \\
&= (\psi^p)^* \tilde{C}^{ord}.
\end{aligned}$$

Diagram (6.1) commutes because both f_p and $(\Phi_{insep})_*$ are lifts of the relative Frobenius with the same kernel. ☐

Consider the pullback diagram

(6.5)
$$\begin{tikzcd}
\mathrm{Spf}((K_p^\wedge)_0 E_{K(1)}) \arrow[r, "g"] \arrow[d, "q"'] & \mathrm{Spf}(V_\infty^\wedge) \arrow[d, "q'"] \\
\mathrm{Spf}(\pi_0 E_{K(1)}) \arrow[r, "g'"'] & \mathcal{M}_{ell}^{ord}
\end{tikzcd}$$

of Proposition 6.1. The following theorem will be essential in our construction of the map α_{chrom}.

Theorem 6.10. *The map*

$$g : V_\infty^\wedge \to (K_p^\wedge)_0 E_{K(1)}$$

of Diagram (6.5) is a map of θ-algebras.

Proof. We just need to check that g commutes with ψ^p (we already know from Proposition 6.1 that g commutes with the action of $\mathbb{Z}_p^\times$). The map g classifies a level structure

$$\eta : \widehat{\mathbb{G}}_m \xrightarrow{\cong} q^* (\tilde{C}^{ord})^\wedge.$$

We need to verify that there is an isomorphism

$$((\psi^p)^* q^* \tilde{C}^{ord}, (\psi^p)^* \eta) \cong ((q^* \tilde{C}^{ord})^{(p)}, \eta^{(p)}).$$

The descent data for level structures arising from E_∞-structures is natural with respect to maps of E_∞-ring spectra (see [AHS04]). It follows that the maps of E_∞-ring spectra:

$$K_p \xrightarrow{r} (K_p \wedge E_{K(1)})_p \xleftarrow{q} E_{K(1)}.$$

induce a diagram

$$\begin{tikzcd}
r^*\widehat{\mathbb{G}}_m \arrow[r, "r^*f_p"] \arrow[d, "\eta"'] & r^*(\psi^p)^*\widehat{\mathbb{G}}_m \arrow[d, "(\psi^p)^*\eta"] \\
(q^*\tilde{C}^{ord})^\wedge \arrow[r, "q^*f_p"'] & q^*(\psi^p)^*(\tilde{C}^{ord})^\wedge
\end{tikzcd}$$

The E_∞-structure on p-adic K-theory associates to the formal subgroup $\mu_p < \widehat{\mathbb{G}}_m$ over $\mathbb{Z}_p$ the pth power isogeny

$$f_p = [p] : \widehat{\mathbb{G}}_m \to \widehat{\mathbb{G}}_m$$

(this is a special case of Example 6.7). Combined with Diagram (6.1), we have a diagram

$$\begin{tikzcd}
\widehat{\mathbb{G}}_m \arrow[r, "{[p]}"] \arrow[d, "\eta"'] & \widehat{\mathbb{G}}_m \arrow[d, "(\psi^p)^*\eta"] \\
q^*(\tilde{C}^{ord})^\wedge \arrow[r, "(\Phi_{insep})_*"'] & q^*((\tilde{C}^{ord})^{(p)})^\wedge
\end{tikzcd}$$

We deduce from Diagram (5.3) that with respect to the isomorphism $(\psi^p)^*\tilde{C}^{ord} \cong (\tilde{C}^{ord})^{(p)}$ we have $(\psi^p)^*\eta \cong \eta^{(p)}$. ☐

7. CONSTRUCTION OF $\mathcal{O}^{top}_{K(1)}$

For a θ-algebra A/k and a θ-A-module M, let

$$H^*_{Alg_\theta}(A/k, M)$$

denote the θ-algebra Andre-Quillen cohomology of A with coefficients in M. In [GH] (see also [GH04]), an obstruction theory for $K(1)$-local E_∞ ring spectra is developed. We summarize their main results:

Theorem 7.1 (Goerss-Hopkins)**.**

(1) *Given a graded θ-algebra A_*, the obstructions to the existence of a $K(1)$-local E_∞-ring spectrum E, for which there is an isomorphism*

$$(K_p^\wedge)_*E \cong A_*$$

of θ-algebras, lie in

$$H^s_{Alg_\theta}(A_*/(K_p)_*, A_*[-s+2]), \qquad s \geq 3.$$

The obstructions to uniqueness lie in

$$H^s_{Alg_\theta}(A_*/(K_p)_*, A_*[-s+1]), \qquad s \geq 2.$$

(2) *Given $K(1)$-local E_∞-ring spectra E_1, E_2 such that $K_*^\wedge E_i$ is p-complete, and a map of graded θ-algebras*

$$f_* : (K_p^\wedge)_*E_1 \to (K_p^\wedge)_*E_2,$$

the obstructions to the existence of a map $f : E_1 \to E_2$ of E_∞-ring spectra which induces f_* on p-adic K-homology lie in

$$H^s_{Alg_\theta}((K_p^\wedge)_*E_1/(K_p)_*, (K_p^\wedge)_*E_2[-s+1]), \qquad s \geq 2.$$

(Here, the θ-$(K_p^\wedge)_*E_1$-module structure on $(K_p^\wedge)_*E_2$ arises from the map f_*.) The obstructions to uniqueness lie in

$$H^s_{Alg_\theta}((K_p^\wedge)_*E_1/(K_p)_*, (K_p^\wedge)_*E_2[-s]), \qquad s \geq 1.$$

(3) *Given such a map f above, there is a spectral sequence which computes the higher homotopy groups of the space $E_\infty(E_1, E_2)$ of E_∞ maps:*

$$H^s_{Alg_\theta}((K_p^\wedge)_* E_1/(K_p)_*, (K_p^\wedge)_* E_2[t]) \Rightarrow \pi_{-t-s}(E_\infty(E_1, E_2), f).$$

Remark 7.2. The notation $A_*[u]$ corresponds to the notation $\Omega^{-u} A_*$ in [GH04], [GH].

Remark 7.3. To simplify notation in the remainder of this paper, we will write

$$H^*_{Alg_\theta}(A_*, M_*) := H^*_{Alg_\theta}(A_*/(K_p)_*, M_*).$$

(That is, we will always be taking our Andre-Quillen cohomology groups in the category of graded θ-$(K_p)_*$-algebras unless we specify a different base explicitly.)

Remark 7.4. The homotopy groups of a $K(1)$-local E_∞-ring spectrum E are recovered from its p-adic K-homology by an Adams-Novikov spectral sequence. Assuming that the pro-system

$$\{K_t(E \wedge M(p^i))\}_i$$

is Mittag-Leffler (see [Dav06, Thm. 10.2]) this spectral sequence takes the form

$$H^s_c(\mathbb{Z}_p^\times, (K_p^\wedge)_t E) \Rightarrow \pi_{t-s} E. \tag{7.1}$$

Let A_* be a graded even periodic θ-algebra, and M_* be a graded θ-A_*-module. In [GH, Sec. 2.4.3], it is explained how the cohomology of the cotangent complex $\mathbb{L}(A_0/\mathbb{Z}_p)$ inherits a canonical θ-A_0-module structure from that of A_0, and that there is a spectral sequence

$$\mathrm{Ext}^s_{Mod^\theta_{A_0}}(H_t(\mathbb{L}(A_0/\mathbb{Z}_p)), M_*) \Rightarrow H^{s+t}_{Alg_\theta}(A_*, M_*). \tag{7.2}$$

The following lemma simplifies the computation of these Andre-Quillen cohomology groups.

Lemma 7.5. *Suppose that $A_*/(K_p)_*$ is a torsion-free graded θ-algebra, and that M_* is a torsion-free graded θ-A_*-module. Let A_*^k denote the fixed points $A_*^{1+p^k\mathbb{Z}_p}$ ($A_*^0 = A_*^{\mathbb{Z}_p^\times}$). Note that we have*

$$A_* = \varprojlim_m \varinjlim_k A_*^k / p^m A_*^k.$$

Let $\bar{A}_*$ (respectively $\bar{A}_*^0$ and $\bar{M}_*$) denote A_*/pA_* (respectively A_*^0/pA_*^0 and M_*/pM_*). Assume that:

(1) *A_* and M_* are even periodic,*
(2) *$\bar{A}_0^0$ is formally smooth over $\mathbb{F}_p$,*
(3) *$H^s_c(\mathbb{Z}_p^\times, \bar{M}_0) = 0$ for $s > 0$,*
(4) *$\bar{A}_0$ is ind-étale over $\bar{A}_0^0$.*

Then we have:

$$H^s_{Alg_\theta}(A_*, M_*[t]) = 0$$

if $s > 1$ or t is odd.

Proof. By [GH04, Prop. 6.8], there is a spectral sequence

$$H^s_{Alg^\theta_{\mathbb{F}_p}}(\bar{A}_*, p^m M_*/p^{m+1} M_*[t]) \Rightarrow H^s_{Alg_\theta}(A_*, M_*[t]).$$

Thus it suffices to prove the mod p result. Note that because M is torsion-free, there is an isomorphism

$$\bar{M}_* \cong p^m M_*/p^{m+1} M_*.$$

Since $\bar{A}_0^0$ is formally smooth over $\mathbb{F}_p$, and since $\bar{A}_0$ is ind-étale over $\bar{A}_0^0$, we deduce that $\bar{A}_0$ is formally smooth over $\mathbb{F}_p$. Therefore, the spectral sequence

$$\mathrm{Ext}^s_{Mod^\theta_{\bar{A}_0}}(H_t(\mathbb{L}(\bar{A}_0/\mathbb{F}_p)), \bar{M}_*[t]) \Rightarrow H^{s+t}_{Alg^\theta_{\mathbb{F}_p}}(\bar{A}_*, \bar{M}_*)$$

collapses to give an isomorphism

$$\mathrm{Ext}^s_{Mod^\theta_{\bar{A}_0}}(\Omega_{\bar{A}_0/\mathbb{F}_p}, \bar{M}_{-t}) \cong H^s_{Alg^\theta_{\mathbb{F}_p}}(\bar{A}_*, \bar{M}_*[t]).$$

Since $\bar{A}_0$ is ind-étale over $\bar{A}_0^0$, there is an isomorphism

$$\Omega_{\bar{A}_0/\mathbb{F}_p} \cong \bar{A}_0 \otimes_{\bar{A}_0^0} \Omega_{\bar{A}_0^0/\mathbb{F}_p}$$

of θ-$\bar{A}_0$-modules. Because $\bar{A}_0$ is flat over $\bar{A}_0^0$, this induces a change of rings isomorphism

$$\mathrm{Ext}^s_{Mod^\theta_{\bar{A}_0}}(\Omega_{\bar{A}_0/\mathbb{F}_p}, \bar{M}_{-t}) \cong \mathrm{Ext}^s_{Mod^\theta_{\bar{A}_0^0}}(\Omega_{\bar{A}_0^0/\mathbb{F}_p}, \bar{M}_{-t}).$$

There is a composite functors spectral sequence

$$\mathrm{Ext}^s_{\bar{A}_0^0[\theta]}(\Omega_{\bar{A}_0^0/\mathbb{F}_p}, H^t_c(\mathbb{Z}_p^\times, \bar{M}_u)) \Rightarrow \mathrm{Ext}^{s+t}_{Mod^\theta_{\bar{A}_0^0}}(\Omega_{\bar{A}_0^0/\mathbb{F}_p}, \bar{M}_u)$$

which, by our hypotheses, collapses to an isomorphism

$$\mathrm{Ext}^s_{\bar{A}_0^0[\theta]}(\Omega_{\bar{A}_0^0/\mathbb{F}_p}, \bar{M}_u^{\mathbb{Z}_p^\times}) \cong \mathrm{Ext}^s_{Mod^\theta_{\bar{A}_0^0}}(\Omega_{\bar{A}_0^0/\mathbb{F}_p}, \bar{M}_u). \tag{7.3}$$

Because $\bar{A}_0^0$ is formally smooth over $\mathbb{F}_p$, the module of Kähler differentials $\Omega_{\bar{A}_0^0/\mathbb{F}_p}$ is projective as an $\bar{A}_0^0$-module. The Ext groups in the left hand side of (7.3) therefore vanish for $s > 1$, and, since M_* is concentrated in even degrees, for u odd. □

There is a relative form of Theorem 7.1. Fix a $K(1)$-local E_∞-ring spectrum E. The entire statement of Theorem 7.1 is valid if you work in the category of $K(1)$-local commutative E-algebras instead of $K(1)$-local E_∞-ring spectra. The obstructions live in the Andre-Quillen cohomology groups for graded θ-W_*-algebras:

$$H^s_{Alg^\theta_{W_*}}(A_*, M_*)$$

where $W_* = (K_p^\wedge)_*E$.

Lemma 7.6. *Suppose that W_* and A_* are even periodic, and that A_0 is étale over W_0. Then for all s,*

$$H^s_{Alg^\theta_{W_*}}(A_*, M_*) = 0.$$

Proof. Consider the spectral sequence

$$\mathrm{Ext}^s_{Mod^\theta_{A_*}}(H_t(\mathbb{L}(A_*/W_*)), M_*) \Rightarrow H^{s+t}_{Alg^\theta_{W_*}}(A_*, M_*).$$

Because A_* is étale over W_*, the cotangent complex is contractible, and the spectral sequence collapses to zero. □

We outline our construction of $\mathcal{O}^{top}_{K(1)}$:

Step 1: We will construct a $K(1)$-local E_∞-ring spectrum $tmf(p)^{ord}$. This will be our candidate for the spectrum of sections of $\mathcal{O}^{top}_{K(1)}$ over the étale cover

$$\begin{tikzcd} \mathcal{M}^{ord}_{ell}(p) \arrow[d, "\gamma"] \\ \mathcal{M}^{ord}_{ell} \end{tikzcd}$$

This cover is Galois, with Galois group $(\mathbb{Z}/p)^\times$. We will show that there is a corresponding action of $(\mathbb{Z}/p)^\times$ on the spectrum $tmf(p)^{ord}$ by E_∞-ring maps. We will define $tmf_{K(1)}$ to be homotopy fixed points

$$tmf_{K(1)} := (tmf(p)^{ord})^{h(\mathbb{Z}/p)^\times}.$$

Step 2: We will construct the sheaf $\mathcal{O}^{top}_{K(1)}$ in the category of commutative $tmf_{K(1)}$-algebras.

We now give the details of our constructions.

Step 1: construction of $tmf_{K(1)}$.

Case 1: assume that p is odd.

Let $\mathcal{X}$ be the formal pullback

(7.4)
$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[d] \\
\mathcal{M}_{ell}^{ord}(p) \arrow[r] & \mathcal{M}_{ell}^{ord}
\end{tikzcd}$$

For a p-complete ring R, the R-points of $\mathcal{X}$ are given by

$$\mathcal{X} = \{(C, \eta, \eta')\}$$

where the data is given by:

C a generalized elliptic curve over R,

$\eta : \widehat{\mathbb{G}}_m \xrightarrow{\cong} \widehat{C}$ an isomorphism of formal groups,

$\eta' : \mu_p \xrightarrow{\cong} \widehat{C}[p]$ an isomorphism of finite group schemes.

Since $\mathcal{M}_{ell}^{ord}(p) = \mathrm{Spf}(V_1)$ is formally affine, we deduce that $\mathcal{X} = \mathrm{Spf}(W)$ for some ring W. Since $\mathcal{M}_{ell}^{ord}(p)$ is étale over $\mathcal{M}_{ell}^{ord}$, the ring W possesses a canonical θ-algebra structure extending that of $V_\infty^\wedge$. For $k \in \mathbb{Z}_p^\times$, the operations ψ^k are induced by the natural transformation on R-points:

$$(\psi^k)^* \mathcal{X}(R) \to \mathcal{X}(R)$$
$$(C, \eta, \eta') \mapsto (C, \eta \circ [k], \eta')$$

The operation ψ^p is induced by the natural transformation

$$(\psi^p)^* \mathcal{X}(R) \to \mathcal{X}(R)$$
$$(C, \eta, \eta') \mapsto (C^{(p)}, \eta^{(p)}, (\eta')^{(p)})$$

Here, given η', the level structure $(\eta')^{(p)}$ is the one making the following diagram commute (see Remark 5.3).

$$\begin{tikzcd}
& \mu_p \arrow[d, "\eta'"] \arrow[dl, dotted, "(\eta')^{(p)}"'] \\
\widehat{C^{(p)}}[p] \arrow[r, "(\Phi_{sep})_*"'] & \widehat{C}[p]
\end{tikzcd}$$

Taking $\omega_{\infty,1}$ to be the canonical line bundle over $\mathcal{X}$, we can construct an evenly graded θ-algebra W_* as

$$W_{2*} := \Gamma \omega_{\infty,1}^{\otimes *}.$$

Theorem 7.7. *There is a $(\mathbb{Z}/p)^\times$-equivariant, even periodic, $K(1)$-local E_∞-ring spectrum $tmf(p)^{ord}$ such that*

1. $\pi_0 tmf(p)^{ord} \cong V_1$,
2. *Letting $(\mathbf{C}_1, \boldsymbol{\eta}_1)$ be the universal tuple over $\mathcal{M}_{ell}^{ord}(p)$, there is an isomorphism of formal groups $\mathbb{G}_{tmf(p)^{ord}} \cong \widehat{\mathbf{C}}_1$.*
3. *There is an isomorphism of θ-algebras*

$$(K_p^\wedge)_* tmf(p)^{ord} \cong W_*.$$

Proof. Observe the following.

1. W_* is concentrated in even degrees.

(2) W is ind-etale over $W^{\mathbb{Z}_p^\times} = V_1$, and V_1 is smooth over $\mathbb{Z}_p$. This is because in the following pullback

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[d] \\
\mathcal{M}_{ell}^{ord}(p) \arrow[r] & \mathcal{M}_{ell}^{ord}
\end{tikzcd}$$

we have $\mathcal{M}_{ell}^{ord}(p^\infty)$ ind-etale over $\mathcal{M}_{ell}^{ord}$, thus $\mathcal{X} = \mathrm{Spf}(W)$ is ind-etale over $\mathcal{M}_{ell}^{ord}(p) = \mathrm{Spf}(V_1)$, and $\mathcal{M}_{ell}^{ord}(p)$ is smooth over $\mathrm{Spf}(\mathbb{Z}_p)$.

(3) $H_c^s(\mathbb{Z}_p^\times, W) = 0$ for $s > 0$. This is because W is is an ind-étale $\mathbb{Z}_p^\times$-torsor over V_1.

We deduce, from Lemma 7.5, that there exists a $K(1)$-local E_∞-ring spectrum $tmf(p)^{ord}$ such that we have an isomorphism

$$(K_p^\wedge)_* tmf(p)^{ord} \cong W_*$$

of graded θ-algebras. As a consequence of (3) above, we deduce that the spectral sequence (7.1) collapses to give an isomorphism

$$\pi_* tmf(p)^{ord} \cong (V_1)_*$$

where, if ω_1 is the canonical line bundle over $\mathcal{M}_{ell}^{ord}(p)$, then

$$(V_1)_{2*} = \Gamma \omega_1^{\otimes *}.$$

Let $(\mathbf{C}_1, \boldsymbol{\eta}_1)$ be the universal tuple over $\mathcal{M}_{ell}^{ord}(p)$. The existence of the isomorphism

$$\boldsymbol{\eta}_1 : \mu_p \xrightarrow{\cong} \widehat{\mathbf{C}}_1[p]$$

implies that ω_1 admits a trivialization. In particular, $tmf(p)^{ord}$ is even periodic.

We now show that the formal group of $\mathbb{G}_{tmf(p)^{ord}}$ is isomorphic to the formal group $\widehat{\mathbf{C}}_1$. Choose complex orientations Φ_K, $\Phi_{tmf(p)^{ord}}$ of K and $tmf(p)^{ord}$. Consider the following diagram.

$$\begin{tikzcd}
MUP_0 \arrow[r, "\Phi_{tmf(p)^{ord}}"] \arrow[d, "\eta_R"'] & \pi_0 tmf(p)^{ord} \arrow[r, equal] \arrow[d, "\eta_R"] & V_1 \arrow[d, hook] \\
MUP_0 MUP \arrow[r, "\Phi_K \wedge \Phi_{tmf(p)^{ord}}"'] & (K_p^\wedge)_0 tmf(p)^{ord} \arrow[r, equal] & W
\end{tikzcd}$$

The map $\Phi_K \wedge \Phi_{tmf(p)^{ord}}$ classifies an isomorphism of formal groups

$$\alpha : \eta_L^* \widehat{\mathbb{G}}_m \xrightarrow{\cong} \eta_R^* \mathbb{G}_{tmf(p)^{ord}}.$$

over W. At the same time, the universal tuple $(\mathbf{C}, \boldsymbol{\eta}, \boldsymbol{\eta}')$ over W has as part of its data an isomorphism of formal groups

$$\boldsymbol{\eta} : \widehat{\mathbb{G}}_m \xrightarrow{\cong} \widehat{\mathbf{C}}.$$

The generalized elliptic curve $\mathbf{C}$ over W is a pullback of the elliptic curve $\mathbf{C}_1$ over V_1 — thus it is invariant under the action of $\mathbb{Z}_p^\times$. The same holds for the formal group $\eta_R^* \mathbb{G}_{tmf(p)^{ord}}$ — it is tautologically the pullback of $\mathbb{G}_{tmf(p)^{ord}}$. Under the action of an element $k \in \mathbb{Z}_p^\times$, the isomorphisms α and $\boldsymbol{\eta}$ transform as

$$[k]^* \alpha = \alpha \circ [k],$$
$$[k]^* \boldsymbol{\eta} = \boldsymbol{\eta} \circ [k].$$

The isomorphism

$$\boldsymbol{\eta} \circ \alpha^{-1} : \eta_R^* \mathbb{G}_{tmf(p)^{ord}} \xrightarrow{\cong} \widehat{\mathbf{C}}$$

is therefore invariant under the action of $\mathbb{Z}_p^\times$. Thus it descends to an isomorphism

$$\alpha_1 : \mathbb{G}_{tmf(p)^{ord}} \xrightarrow{\cong} \widehat{\mathbf{C}}_1.$$

The Galois group $(\mathbb{Z}/p)^\times$ of $\mathcal{M}_{ell}^{ord}(p)$ over $\mathcal{M}_{ell}^{ord}$ acts on V_1. The last thing we need to show is that this action lifts to a point-set level action of $(\mathbb{Z}/p)^\times$ by E_∞-ring maps. Because W_* satisfies the hypotheses of Lemma 7.5, we may deduce from Theorem 7.1 that the K_p-Hurewicz map

$$[tmf(p)^{ord}, tmf(p)^{ord}]_{E_\infty} \to \mathrm{Hom}_{Alg_\theta}(W_*, W_*)$$

is an isomorphism. The action of $(\mathbb{Z}/p)^\times$ on V_1 lifts to W in an obvious way: on the R-points of $\mathrm{Spf}(W) = \mathcal{X}$, an element $[k] \in (\mathbb{Z}/p)^\times$ acts by

$$\begin{aligned} [k]^* : \mathcal{X}(R) &\to \mathcal{X}(R) \\ (C, \eta, \eta') &\mapsto (C, \eta, \eta' \circ [k]) \end{aligned}$$

This action is easily seen to commute with the action of ψ^l for $l \in \mathbb{Z}_p^\times$, and ψ^p. Thus $(\mathbb{Z}/p)^\times$ acts on W through maps of θ-algebras. We deduce that there is a map of groups

$$(\mathbb{Z}/p)^\times \to [tmf(p)^{ord}, tmf(p)^{ord}]_{E_\infty}^\times.$$

The obstructions to lifting this homotopy action to a point-set action may be identified using the obstruction theory of Cooke [Coo78] (adapted to the topological category of E_∞-ring spectra). Namely, the obstructions lie in the group cohomology

$$H^s((\mathbb{Z}/p)^\times, \pi_{s-2}(E_\infty(tmf(p)^{ord}, tmf(p)^{ord}), \mathrm{Id})), \qquad s \geq 3.$$

Since the space $E_\infty(tmf(p)^{ord}, tmf(p)^{ord})$ is p-complete, and the order of the group $(\mathbb{Z}/p)^\times$ is prime to p, these obstructions must vanish. □

Define

$$tmf_{K(1)} := (tmf(p)^{ord})^{h(\mathbb{Z}/p)^\times}.$$

The following lemma is a useful corollary of a theorem of N. Kuhn.

Lemma 7.8. *Suppose that G is a finite group which acts on a $K(n)$-local E_∞-ring spectrum E through E_∞-ring maps. Then the Tate spectrum E^{tG} is $K(n)$-acyclic, and the norm map*

$$N : E_{hG} \to E^{hG}$$

is a $K(n)$-local equivalence.

Proof. Kuhn proves that the localized Tate spectrum $((S_{T(n)})^{tG})_{T(n)}$ is acyclic [Kuh04, Thm. 1.5], where $T(n)$ is the telescope of a v_n-periodic map on a type n complex. The Tate spectrum $(E^{tG})_{K(n)}$ is an algebra spectrum over $((S_{T(n)})^{tG})_{T(n)}$. In particular, it is a module spectrum over an acyclic ring spectrum, and hence must be acyclic. □

Lemma 7.9. *There is an isomorphism of θ-algebras $(K_p^\wedge)_* tmf_{K(1)} \cong (V_\infty^\wedge)_*$.*

Proof. By Lemma 7.8, the natural map

$$(K_p \wedge (tmf(p)^{ord})^{h(\mathbb{Z}/p)^\times})_{K(1)} \to (K_p \wedge tmf(p)^{ord})_{K(1)}^{h(\mathbb{Z}/p)^\times}$$

is an equivalence (the homotopy fixed points are commuted past the smash product by changing them to homotopy orbits). The homotopy fixed point spectral sequence computing the homotopy groups of the latter collapses to give an isomorphism:

$$(V_\infty^\wedge)_* \cong (W_*)^{(\mathbb{Z}/p)^\times} \cong (K_p^\wedge)_* tmf_{K(1)}.$$

(The first isomorphism above comes from the fact that $\mathcal{X}$ is an étale $(\mathbb{Z}/p)^\times$-torsor over $\mathcal{M}_{ell}^{ord}(p^\infty)$.)

□

Case 2: $p = 2$.

If one were to try to duplicate the odd-primary argument, one would do the following: the first stack in the 2-primary Igusa tower which is formally affine is

$$\mathcal{M}_{ell}^{ord}(4) = \mathrm{Spf}(V_2).$$

The cover $\mathcal{M}_{ell}^{ord}(4) \xrightarrow{\gamma} \mathcal{M}_{ell}^{ord}$ is Galois with Galois group $(\mathbb{Z}/4)^\times$. One must begin by constructing the $K(1)$-local E_∞-ring spectrum $tmf(4)^{ord}$. One would like to use the obstruction theory of Cooke to make this spectrum $(\mathbb{Z}/4)^\times$-equivariant, but the order of the group is 2, so we cannot conclude that the obstructions vanish.

We instead replace K with KO. Define a graded *reduced θ-algebra* to be a graded θ-algebra over KO_* where the action of $\mathbb{Z}_2^\times$ is replaced with an action of $\mathbb{Z}_2^\times/\{\pm 1\}$.

Suppose that V is a θ-algebra, and that the subgroup $\{\pm 1\} \subset \mathbb{Z}_2^\times$ acts trivially on V. Then V may be regarded as a reduced θ-algebra. One may form a corresponding graded reduced θ-algebra by taking

$$W_* = KO_* \otimes V. \tag{7.5}$$

Definition 7.10. We shall say that a graded reduced θ-algebra W_* is *Bott periodic* if it takes the form (7.5). We shall say that a $K(1)$-local E_∞ ring spectrum is *Bott periodic* if

1. $(K_2^\wedge)_* E$ is torsion-free and concentrated in even degrees.
2. The map $(KO_2^\wedge)_0 E \to (K_2^\wedge)_0 E$ is an isomorphism.

The relevance of this definition is given by the following lemma.

Lemma 7.11. *Suppose that E is a Bott periodic $K(1)$-local E_∞-ring spectrum. Then we have*

$$(KO_2^\wedge)_* E \cong KO_* \otimes (K_2^\wedge)_0 E$$

In particular, the graded reduced θ-algebra $(KO_2^\wedge)_* E$ is Bott periodic. Conversely, if E is an E_∞ ring spectrum with

$$(KO_2^\wedge)_* E \cong KO_* \otimes V$$

as KO_*-modules, then E is Bott periodic, and $(K_2^\wedge)_0 E \cong V$.

Proof. The first part uses the homotopy fixed point spectral sequence

$$H^s(\mathbb{Z}/2, (K_2^\wedge)_t E) \Rightarrow (KO_2^\wedge)_{t-s} E.$$

The second part follows easily from the Künneth spectral sequence. □

Remark 7.12. Both KO_2 and $tmf_{K(1)}$ (once we construct it) are Bott periodic.

Unfortunately the homology theory $KO_2^\wedge$ does not seem to satisfy all of the hypotheses required for the Goerss-Hopkins obstruction theory to apply. Nevertheless, when restricted to Bott periodic spectra with vanishing positive cohomology as a $\mathbb{Z}_2^\times/\{\pm 1\}$-module, it can be made to work. This is discussed in Appendix A. There it is shown that given a Bott periodic graded reduced θ-algebra W_* satisfying

$$H_c^s(\mathbb{Z}_2^\times/\{\pm 1\}, W_0) = 0 \text{ for } s > 0,$$

the obstructions to the existence of a $K(1)$-local E_∞-ring spectrum E with $(KO_2^\wedge)_* E \cong W_*$ lie in the cohomology groups

$$H^s_{Alg_\theta^{red}}(W_*, W_*[-s+2]), \quad s \geq 3.$$

Given Bott periodic $K(1)$-local E_∞-ring spectra E_1 and E_2, the obstructions to realizing a map of graded reduced θ-algebras

$$(KO_2^\wedge)_* E_1 \to (KO_2^\wedge)_* E_2$$

lie in

$$H^s_{Alg_\theta^{red}}((KO_2^\wedge)_* E_1, (KO_2^\wedge)_* E_2[-s+1]), \quad s \geq 2.$$

We have the following analog of Lemma 7.5.

Lemma 7.13. *Suppose that $A_*/(KO_2)_*$ is a graded 2-complete reduced θ-algebra, and that M_* is a graded 2-complete reduced θ-A_*-module. Let A_*^k denote the fixed points $A_*^{1+2^k\mathbb{Z}_2}$ ($A_*^0 = A_*^{\mathbb{Z}_2^\times}$). Note that we have*

$$A_* = \varprojlim_m \varinjlim_k A_*^k/p^m A_*^k.$$

Let $\bar{A}_*$ (respectively $\bar{A}_*^0$ and $\bar{M}_*$) denote the mod 2 reduction. Assume that:

1. *A_* and M_* are Bott periodic,*
2. *$\bar{A}_0^0$ is formally smooth over $\mathbb{F}_2$,*
3. *$H_c^s(\mathbb{Z}_2^\times/\{\pm 1\}, \bar{M}_0) = 0$ for $s > 0$,*
4. *$\bar{A}_0$ is ind-étale over $\bar{A}_0^0$.*

Then we have:

$$H^s_{Alg_\theta^{red}}(A_*, M_*[t]) = 0$$

if either $s > 1$ or $-t \equiv 3, 5, 6, 7 \mod 8$.

The following lemma is of crucial importance.

Lemma 7.14. *Let $V_\infty^\wedge$ be the representing ring for $\mathcal{M}_{ell}^{ord}(2^\infty)$ (a.k.a. the θ-algebra of generalized 2-adic modular functions).*

1. *The element $[-1] \in \mathbb{Z}_2^\times$ acts trivially on $V_\infty^\wedge$.*
2. *The subring $V_2 \subset V_\infty$ is isomorphic to the fixed points under the induced action of the group $\mathbb{Z}_2^\times/\{\pm 1\}$.*
3. *We have $H_c^s(\mathbb{Z}_2^\times/\{\pm 1\}, V_\infty^\wedge/2V_\infty^\wedge) = 0$ for $s > 0$.*

Proof. The stack $\mathcal{M}_{ell}^{ord}(2^\infty)$ represents pairs (η, C) where

$$\eta : \widehat{\mathbb{G}}_m \to \widehat{C}$$

is an isomorphism. However, we have $([-1]^*\eta, C) \cong (\eta, C)$:

$$\begin{tikzcd}
\widehat{\mathbb{G}}_m \arrow[r, "{[-1]}"] \arrow[rr, bend left, "{[-1]^*\eta}"] \arrow[rrd, "\eta"'] & \widehat{\mathbb{G}}_m \arrow[r, "\eta"] & \widehat{C} \arrow[d, "{[-1]}"] \\
& & \widehat{C}
\end{tikzcd}$$

This proves (1). Under the isomorphism given by the composite

$$1 + 4\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2^\times \to \mathbb{Z}_2^\times/\{\pm 1\}$$

the action of the subgroup $1 + 4\mathbb{Z}_2$ agrees with the induced action of $\mathbb{Z}_2^\times/\{\pm 1\}$ on $V_\infty^\wedge$. But $V_\infty^\wedge/2V_\infty^\wedge$ is ind-Galois over $V_2/2V_2$ (the representing ring for $\mathcal{M}_{ell}^{ord}(4) \otimes \mathbb{F}_2$) with Galois group $1 + 4\mathbb{Z}_2$. This proves (2) and (3). □

The algebra $V_\infty^\wedge/2V_\infty^\wedge$ is ind-etale over $V_2/2V_2$, and $\mathcal{M}_{ell}^{ord}(4) \otimes \mathbb{F}_2$ is smooth. Lemma 7.13 implies that the groups

$$H^s_{Alg_\theta^{red}}(KO_* \otimes V_\infty^\wedge, KO_* \otimes V_\infty^\wedge[u])$$

vanish for $s > 1$ and $-u \equiv 3, 5, 6, 7 \mod 8$. This is enough to deduce that there exists a $K(1)$-local E_∞-ring spectrum $tmf_{K(1)}$ such that there is an isomorphism of graded reduced θ-algebras

$$(KO_2^\wedge)_* tmf_{K(1)} \cong KO_* \otimes V_\infty^\wedge.$$

Remark 7.15. There is another construction of $tmf_{K(1)}$ at $p = 2$ which is described in [Lau04] (see also [Hop]). The spectrum is explicitly constructed by attaching two $K(1)$-local E_∞-cells to the $K(1)$-local sphere. Unfortunately, it seems that this approach does not generalize to primes $p \geq 5$, though it does work at $p = 3$ as well [Hop].

Step 2: construction of the presheaf $\mathcal{O}^{top}_{K(1)}$. We shall now construct the sections of a presheaf $\mathcal{O}^{top}_{K(1)}$ on $(\mathcal{M}^{ord}_{ell})_{et}$. By Remark 2.5, it suffices to produce the values of $\mathcal{O}^{top}_{K(1)}$ on étale formal affine opens of $\mathcal{M}^{ord}_{ell}$.

Let $\mathrm{Spf}(R) \xrightarrow{f} \mathcal{M}^{ord}_{ell}$ be an étale formal affine open. Consider the pullback:

$$\begin{tikzcd}
\mathrm{Spf}(W) \arrow[r, "f'"] \arrow[d] & \mathcal{M}^{ord}_{ell}(p^\infty) \arrow[d] \\
\mathrm{Spf}(R) \arrow[r, "f"'] & \mathcal{M}^{ord}_{ell}
\end{tikzcd}$$

Since f is étale, W is an étale $V^\wedge_\infty$-algebra, and W carries a canonical θ-algebra structure (Section 6). We have an associated even periodic graded θ-$(V^\wedge_\infty)_*$-algebra W_*.

The relative form of Theorem 7.1 indicates that the obstructions to the existence and uniqueness of a $K(1)$-local commutative $tmf_{K(1)}$-algebra E such that there is an isomorphism

$$(K^\wedge_p)_* E \cong W_*$$

of θ-$(V^\wedge_\infty)_*$-algebras lie in the Andre-Quillen cohomology groups

$$H^s_{Alg^\theta_{(V^\wedge_\infty)_*}}(W_*, W_*[u]).$$

These cohomology groups vanish by Lemma 7.6.

Given a map

$$g : \mathrm{Spf}(R_2) \to \mathrm{Spf}(R_1)$$

in $(\mathcal{M}^{ord}_{ell})_{et}$, we get an induced map

$$g^* : (W_1)_* \to (W_2)_*$$

of the corresponding θ-$(V^\wedge_\infty)_*$-algebras. Let E_1, E_2 be the corresponding $K(1)$-local commutative $tmf_{K(1)}$-algebras. The obstructions for existence and uniqueness of a map of $tmf_{K(1)}$-algebras

$$\tilde{g}^* : E_1 \to E_2$$

realizing the map g^* on K_p-homology lie in the groups

$$H^s_{Alg^\theta_{(V^\wedge_\infty)_*}}((W_1)_*, (W_2)_*[u]).$$

Furthermore, given the existence of $\tilde{g}^*$, there is a spectral sequence

$$H^s_{Alg^\theta_{(V^\wedge_\infty)_*}}((W_1)_*, (W_2)_*[u]) \Rightarrow \pi_{-u-s}(\mathrm{Alg}_{tmf_{K(1)}}(E_1, E_2), \tilde{g}^*).$$

Again, these cohomology groups all vanish by Lemma 7.6. We deduce that:

(1) The K_p-Hurewicz map

$$[E_1, E_2]_{Alg_{tmf_{K(1)}}} \to \mathrm{Hom}_{Alg^\theta_{(V^\wedge_\infty)_*}}((W_1)_*, (W_2)_*)$$

is an isomorphism.

(2) The mapping spaces $\mathrm{Alg}_{tmf_{K(1)}}(E_1, E_2)$ have contractible components.

We have constructed a functor

$$\bar{\mathcal{O}}^{top}_{K(1)} : ((\mathcal{M}^{ord}_{ell})_{et,\mathit{aff}})^{op} \to Ho(\text{Commutative } tmf_{K(1)}\text{-algebras}).$$

Since the mapping spaces are contractible, this functor lifts to give a presheaf (see [DKS89])

$$\mathcal{O}^{top}_{K(1)} : ((\mathcal{M}^{ord}_{ell})_{et,\mathit{aff}})^{op} \to \text{Commutative } tmf_{K(1)}\text{-algebras}.$$

The same argument used to prove part (2) of Theorem 7.7 proves the following.

Proposition 7.16. *Suppose that* $\mathrm{Spf}(R) \to \mathcal{M}_{ell}^{ord}$ *is an étale open classifying a generalized elliptic curve* C/R*. Then the associated spectrum of sections* $\mathcal{O}_{K(1)}^{top}$ *is an elliptic spectrum for the curve* C/R*.*

8. Construction of $\mathcal{O}_p^{top}$

To construct $\mathcal{O}_p^{top}$ it suffices to construct the map

$$\alpha_{chrom} : (i_{ord})_* \mathcal{O}_{K(1)}^{top} \to ((i_{ss})_* \mathcal{O}_{K(2)}^{top})_{K(1)}.$$

Our strategy will be to do this in two steps:

Step 1: We will construct

$$\alpha_{chrom} : tmf_{K(1)} \to (tmf_{K(2)})_{K(1)}$$

where

$$tmf_{K(2)} := \mathcal{O}_{K(2)}^{top}(\mathcal{M}_{ell}^{ss}).$$

Step 2: We will use the $K(1)$-local obstruction theory in the category of $tmf_{K(1)}$-algebra spectra to show that this map can be extended to a map of presheaves of spectra:

$$(\iota_{ord})_* \mathcal{O}_{K(1)}^{top} \to ((\iota_{ss})_* \mathcal{O}_{K(2)}^{top})_{K(1)}.$$

We will need the following lemma.

Lemma 8.1. *Suppose that* C *is a generalized elliptic curve over a ring* R*, and that* E *is an elliptic spectrum associated with* C*. Then*

1. E *is* $E(2)$*-local.*
2. *Suppose that* R *is* p*-complete, and that the classifying map*

$$\mathrm{Spf}(R) \to (\overline{\mathcal{M}}_{ell})_p$$

for C *is flat. Then there is an equivalence*

$$E_{K(1)} \simeq E[v_1^{-1}]_p.$$

Proof. Greenlees and May [GM95] proved that there is an equivalence

$$E_{E(n)} \simeq E[I_{n+1}^{-1}].$$

They also showed there is a spectral sequence

$$H^s(\mathrm{Spec}(R) - X_n, \omega^{\otimes t}) \Rightarrow \pi_{2t-s} E[I_{n+1}^{-1}] \tag{8.1}$$

where $X_n = \mathrm{Spec}(R/I_{n+1})$ is the locus of $\mathrm{Spec}(R/p)$ where the formal group of E has height greater than n. (1) therefore follows from the fact that $\widehat{C}$ never has height greater than 2. For (2), since R is assumed to be p-complete, there is an isomorphism

$$\pi_0(E[v_1^{-1}]_p) \cong R[v_1^{-1}]_p.$$

Over $R[v_1^{-1}]/pR[v_1^{-1}]$, the generalized elliptic curve C is ordinary, hence X_1 is empty and the spectral sequence (8.1) collapses to show that $E[v_1^{-1}]_p$ is $E(1)$-local. It is also p-complete by construction, and since $K(1)$-localization is the p-completion of $E(1)$-localization, we deduce that $E[v_1^{-1}]_p$ is $K(1)$-local. It therefore suffices to show that the map

$$E \to E[v_1^{-1}]_p$$

is a $K(1)$-equivalence. It suffices to show that it yields an equivalence on p-adic K-theory. However, by Proposition 6.1, both $(K_p^\wedge)_0 E$ and $(K_p^\wedge)_0(E[v_1^{-1}]_p)$ are given by W, where we have pullback squares:

$$\begin{tikzcd}
\mathrm{Spf}(W) \arrow[r] \arrow[d] & \mathrm{Spf}(R[v_1^{-1}]_p^\wedge) \arrow[r] \arrow[d] & \mathrm{Spf}(R) \arrow[d] \\
\mathcal{M}_{ell}^{ord}(p^\infty) \arrow[r] & \mathcal{M}_{ell}^{ord} \arrow[r] & (\overline{\mathcal{M}}_{ell})_p
\end{tikzcd}$$

□

Step 1: construction of $\alpha_{chrom} : tmf_{K(1)} \to (tmf_{K(2)})_{K(1)}$.

We shall temporarily assume that p is odd. After we complete Step 1 for odd primes, we shall address the changes necessary for the prime 2.

Fix N to be a positive integer greater than or equal to 3 and coprime to p. Let $\mathcal{M}_{ell}(N)/\mathbb{Z}[1/N]$ denote the moduli stack of pairs (C, ρ) where C is an elliptic curve and ρ is a "full level N structure":

$$\rho : (\mathbb{Z}/N)^2 \xrightarrow{\cong} C[N].$$

Since N is greater than 3, this stack is a scheme [DR73, Cor. 2.9]. The cover

$$\mathcal{M}_{ell}(N) \to \mathcal{M}_{ell} \otimes \mathbb{Z}[1/N]$$

given by forgetting the level structure is an étale $GL_2(\mathbb{Z}/N)$-torsor. Let $\mathcal{M}_{ell}(N)_p$ denote the completion of $\mathcal{M}_{ell}(N)$ at p, and let $\mathcal{M}_{ell}^{ss}(N)$ denote the pullback

$$\begin{tikzcd}
\mathcal{M}_{ell}^{ss}(N) \arrow[r] \arrow[d] & \mathcal{M}_{ell}(N)_p \arrow[d] \\
\mathcal{M}_{ell}^{ss} \arrow[r] & (\mathcal{M}_{ell})_p
\end{tikzcd}$$

Since $\mathcal{M}_{ell}(N)_p$ is a formal scheme, $\mathcal{M}_{ell}^{ss}(N)$ is also a formal scheme. By Serre-Tate theory, the formal scheme $\mathcal{M}_{ell}^{ss}(N)$ is given by

$$\mathcal{M}_{ell}^{ss}(N) = \coprod_i \mathrm{Spf}(\mathbb{W}(k_i)[[u_1]])$$

for a finite set of finite fields $\{k_i\}$ (this set of finite fields depends on N). Let A_N denote the representing ring

$$A_N := \prod_i \mathbb{W}(k_i)[[u_1]]$$

and let B_N be the ring

$$B_N := A_N[u_1^{-1}]_p^\wedge = \prod_i \mathbb{W}(k_i)((u_1))_p^\wedge.$$

(Elements in the ring $\mathbb{W}(k_i)((u_1))_p^\wedge$ are bi-infinite Laurent series

$$\sum_{j \in \mathbb{Z}} a_j u_1^j$$

where we require that $a_j \to 0$ as $j \to -\infty$.) We shall use the notation

$$\mathcal{M}_{ell}^{ss}(N) = \mathrm{Spf}_{(p,u_1)}(A_N)$$

to indicate that Spf is taken with respect to the ideal of definition (p, u_1). Define $\mathcal{M}_{ell}^{ss}(N)^{ord}$ to be the formal scheme given by

$$\mathcal{M}_{ell}^{ss}(N)^{ord} = \mathrm{Spf}_{(p)}(B_N).$$

Let $(C_N^{ss}, \eta_N^{ss})/\mathcal{M}_{ell}^{ss}(N)$ be the elliptic curve with full level structure classified by the map

$$\mathcal{M}_{ell}^{ss}(N) \to \mathcal{M}_{ell}(N).$$

We regard $\mathcal{M}_{ell}^{ss}(N)^{ord}$ as the "ordinary locus" of C_N^{ss}. This does not actually make sense in the context of formal schemes — $\mathcal{M}_{ell}^{ss}(N)^{ord}$ is *not* a formal subscheme of $\mathcal{M}_{ell}^{ss}(N)$. Nevertheless, by Remark 1.6, there is a canonical elliptic curve (with level structure) $((C_N^{ss})^{alg}, \eta_N^{ss})$ which lies over $\mathcal{M}_{ell}^{ss}(N)^{alg} := \mathrm{Spec}(A_N)$, and restricts to $C_N^{ss}/\mathrm{Spf}_{(p,u_1)}(A_N)$. The formal scheme $\mathcal{M}_{ell}^{ss}(N)^{ord}$ is given by the pullback

$$\begin{tikzcd}
\mathcal{M}_{ell}^{ss}(N)^{ord} \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord} \arrow[d] \\
\mathcal{M}_{ell}^{ss}(N)^{alg} \arrow[r] & \overline{\mathcal{M}}_{ell}
\end{tikzcd}$$

We let $((C_N^{ss})^{ord}, \eta_N^{ss})$ denote the restriction of the pair $((C_N^{ss})^{alg}, \eta_N^{ss})$ to $\mathcal{M}_{ell}^{ss}(N)^{ord}$. We define $\mathcal{M}_{ell}^{ss}(N,p)^{ord}$ to be the pullback

$$\begin{tikzcd}
\mathcal{M}_{ell}^{ss}(N,p)^{ord} \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord}(p) \arrow[d] \\
\mathcal{M}_{ell}^{ss}(N)^{ord} \arrow[r] & \mathcal{M}_{ell}^{ord}
\end{tikzcd}$$

and denote the pullback of $(C_N^{ss})^{ord}$ to $\mathcal{M}_{ell}^{ss}(N,p)^{ord}$ by $(C_{N,1}^{ss})^{ord}$. Since $\mathcal{M}_{ell}^{ss}(N)^{ord}$ and $\mathcal{M}_{ell}^{ord}(p)$ are formally affine, we deduce that $\mathcal{M}_{ell}^{ss}(N,p)^{ord}$ is formally affine, and is of the form $\mathrm{Spf}_{(p)}(B_{N,1})$.

Let $\mathcal{M}_{ell}^{ord}(p)^{ns}$ denote the locus of the formal affine scheme $\mathcal{M}_{ell}^{ord}(p)$ where the universal curve is nonsingular; it is covered by an étale $GL_2(\mathbb{Z}/N)$-torsor given by the pullback

$$\begin{tikzcd}
\mathcal{M}_{ell}^{ord}(N,p)^{ns} \arrow[r] \arrow[d] & \mathcal{M}_{ell}(N)_p \arrow[d] \\
\mathcal{M}_{ell}^{ord}(p)^{ns} \arrow[r] & (\mathcal{M}_{ell})_p
\end{tikzcd}$$

The action of $GL_2(\mathbb{Z}/N)$ on the formal affine scheme $\mathcal{M}_{ell}^{ss}(N,p)^{ord} = \mathrm{Spf}_{(p)}(B_{N,1})$ over $\mathcal{M}_{ell}^{ord}(N,p)^{ns}$, gives descent data which, by faithfully flat descent (see, for instance, [Hid00, Sec. 1.11.3]), yields a new formal affine scheme

$$\mathcal{M}_{ell}^{ss}(p)^{ord} = \mathrm{Spf}_{(p)}(B_1)$$

over $\mathcal{M}_{ell}^{ord}(p)^{ns}$ (where $B_1 = B_{N,1}^{GL_2(\mathbb{Z}/N)}$) together with a pullback diagram

$$\begin{tikzcd}
\mathcal{M}_{ell}^{ss}(N,p)^{ord} \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ss}(p)^{ord} \arrow[d] \\
\mathcal{M}_{ell}^{ord}(N,p)^{ns} \arrow[r] & \mathcal{M}_{ell}^{ord}(p)^{ns}
\end{tikzcd}$$

Define $(V_\infty^\wedge)^{ss}$ to be the pullback

$$\begin{tikzcd}
\mathrm{Spf}_{(p)}((V_\infty^\wedge)^{ss}) \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[d] \\
\mathcal{M}_{ell}^{ss}(p)^{ord} \arrow[r] & \mathcal{M}_{ell}^{ord}(p)
\end{tikzcd}$$

and define W^{ss} and $\tilde{W}^{ss}$ to be the pullbacks

$$\begin{tikzcd} \mathrm{Spf}_{(p)}(\tilde{W}^{ss}) \arrow[r] \arrow[d] & \mathrm{Spf}_{(p)}(W^{ss}) \arrow[r] \arrow[d] & \mathrm{Spf}_{(p)}((V_\infty^\wedge)^{ss}) \arrow[d] \\ \mathcal{M}_{ell}^{ord}(N,p)^{ns} \arrow[r] & \mathcal{M}_{ell}^{ord}(p)^{ns} \arrow[r] & (\mathcal{M}_{ell}^{ord})^{ns} \end{tikzcd} \tag{8.2}$$

By faithfully flat descent, we have

$$W^{ss} = (\tilde{W}^{ss})^{GL_2(\mathbb{Z}/N)},$$
$$(V_\infty^\wedge)^{ss} = (W^{ss})^{(\mathbb{Z}/p)^\times}.$$

Remark 8.2. Both $\tilde{W}^{ss}$ and W^{ss} possess alternative descriptions. They are given by pullbacks

$$\begin{tikzcd} \mathrm{Spf}(\tilde{W}^{ss}) \arrow[r] \arrow[d] & \mathrm{Spf}(W^{ss}) \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[d] \\ \mathcal{M}_{ell}^{ss}(N,p)^{ord} \arrow[r] & \mathcal{M}_{ell}^{ss}(p)^{ord} \arrow[r] & \mathcal{M}_{ell}^{ord} \end{tikzcd}$$

Let $tmf(N)_{K(2)}$ be the spectrum of sections

$$tmf(N)_{K(2)} := \mathcal{O}^{top}_{K(2)}(\mathcal{M}_{ell}^{ss}(N)).$$

The action of $GL_2(\mathbb{Z}/N)$ on the torsor $\mathcal{M}_{ell}^{ss}(N)$ induces an action of $GL_2(\mathbb{Z}/N)$ on $tmf(N)_{K(2)}$. Since the sheaf $\mathcal{O}^{top}_{K(2)}$ satisfies homotopy decent, we have

$$(tmf(N)_{K(2)})^{hGL_2(\mathbb{Z}/N)} \simeq tmf_{K(2)}.$$

Lemma 8.3. *There is an equivalence*

$$((tmf(N)_{K(2)})_{K(1)})^{hGL_2(\mathbb{Z}/N)} \simeq (tmf_{K(2)})_{K(1)}.$$

Proof. Using Lemma 7.8, and descent, we may deduce that there are equivalences

$$\begin{aligned} ((tmf(N)_{K(2)})_{K(1)})^{hGL_2(\mathbb{Z}/N)} &\simeq ((tmf_{K(2)})^{hGL_2(\mathbb{Z}/N)})_{K(1)} \\ &\simeq (tmf_{K(2)})_{K(1)}. \end{aligned}$$

□

Consider the finite $(\mathbb{Z}/p)^\times$ Galois extension $E_1^{h(1+p\mathbb{Z}_p)}$ of $S_{K(1)}$ given by the homotopy fixed points of E_1-theory with respect to the open subgroup $1+p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ (see [DH04], [Rog08]). Note that we have

$$(E_1^{h(1+p\mathbb{Z}_p)})^{h(\mathbb{Z}/p)^\times} \simeq S_{K(1)}. \tag{8.3}$$

Define spectra

$$\begin{aligned} (tmf(N,p)_{K(2)})_{K(1)} &:= (tmf(N)_{K(2)})_{K(1)} \wedge_{S_{K(1)}} E_1^{h(1+p\mathbb{Z}_p)} \\ (tmf(p)_{K(2)})_{K(1)} &:= (tmf_{K(2)})_{K(1)} \wedge_{S_{K(1)}} E_1^{h(1+p\mathbb{Z}_p)} \end{aligned}$$

These spectra inherit an action by the group $(\mathbb{Z}/p)^\times = \mathbb{Z}_p^\times/1+p\mathbb{Z}_p$.

Using Lemma 7.8, Lemma 8.3 and Equation (8.3), we have the following.

Lemma 8.4. *There are equivalences of E_∞-ring spectra*

$$((tmf(N,p)_{K(2)})_{K(1)})^{hGL_2(\mathbb{Z}/N)} \simeq (tmf(p)_{K(2)})_{K(1)}$$
$$((tmf(p)_{K(2)})_{K(1)})^{h(\mathbb{Z}/p)^\times} \simeq (tmf_{K(2)})_{K(1)}$$

We now link up some homotopy calculations with our previous algebro-geometric constructions.

Lemma 8.5. *There is an $GL_2(\mathbb{Z}/N) \times (\mathbb{Z}/p)^\times$-equivariant isomorphism*

$$\pi_0(tmf(N,p)_{K(2)})_{K(1)} \cong B_{N,1}$$

making $(tmf(N,p)_{K(2)})_{K(1)}$ an elliptic spectrum with associated elliptic curve $(C^{ss}_{N,1})^{ord}$.

Proof. By construction, there is a $GL_2(\mathbb{Z}/N)$-equivariant isomorphism

$$\pi_0 tmf(N)_{K(2)} \cong A_N$$

making $tmf(N)_{K(2)}$ an elliptic spectrum with associated elliptic curve C^{ss}_N. By Lemma 8.1, this gives rise to an isomorphism

$$\pi_0(tmf(N)_{K(2)})_{K(1)} \cong B_N$$

making the pair $((tmf(N)_{K(2)})_{K(1)}, (C^{ss}_N)^{ord})$ an elliptic spectrum. For any $K(1)$-local even periodic Landweber exact cohomology theory E, the homotopy groups of

$$E' = E \wedge_{S_{K(1)}} E_1^{h(1+p\mathbb{Z}_p)}$$

are given by the pullback

$$\begin{tikzcd}
\mathrm{Spf}(\pi_0 E') \arrow[r] \arrow[d] & \mathcal{M}^{mult}_{FG}(p) \arrow[d] \\
\mathrm{Spf}(\pi_0 E) \arrow[r] & \mathcal{M}^{mult}_{FG}
\end{tikzcd}$$

(where the notation here is the same as in the proof of Lemma 5.1). This is easily deduced from the cofiber sequence

$$E' \to (K_p \wedge E)_p \xrightarrow{\psi^k - 1} (K_p \wedge E)_p$$

where k is chosen to be a topological generator of the subgroup $1 + \mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$. In particular, we have the desired isomorphism

$$\pi_0(tmf(N,p)_{K(2)})_{K(1)} \simeq B_{N,1}.$$

The formal group of E' is the pullback of the formal group of E along the map $\pi_0 E \to \pi_0 E'$. The elliptic curve $(C^{ss}_{N,1})^{ord}$ is the pullback of $(C^{ss}_N)^{ord}$ under the same homomorphism. The canonical isomorphism between the formal group of E and the formal group of $(C^{ss}_N)^{ord}$ thus pulls back to give the required isomorphism between the formal group of E' and the formal group of $(C^{ss}_{N,1})^{ord}$. □

Lemma 8.6. *There are isomorphisms*

$$\begin{aligned}
(K_p^\wedge)_*(tmf(N,p)_{K(2)})_{K(1)} &\cong (K_p)_* \otimes_{\mathbb{Z}_p} \tilde{W}^{ss} \\
(K_p^\wedge)_*(tmf(p)_{K(2)})_{K(1)} &\cong (K_p)_* \otimes_{\mathbb{Z}_p} W^{ss} \\
(K_p^\wedge)_*(tmf_{K(2)})_{K(1)} &\cong (K_p)_* \otimes_{\mathbb{Z}_p} (V_\infty^\wedge)^{ss}
\end{aligned}$$

(We shall denote these graded objects as $\tilde{W}^{ss}_*$, W^{ss}_*, and $(V^\wedge_\infty)^{ss}_*$, respectively.)

Proof. We deduce the first isomorphism by combining Proposition 6.1 with Remark 8.2. Using Lemma 8.4, and Lemma 7.8, we have equivalences

$$\begin{aligned}
((K_p \wedge (tmf(N,p)_{K(2)})_{K(1)})_p)^{hGL_2(\mathbb{Z}/N)} &\simeq (K_p \wedge (tmf(p)_{K(2)})_{K(1)})_p \\
((K_p \wedge (tmf(p)_{K(2)})_{K(1)})_p)^{h(\mathbb{Z}/p)^\times} &\simeq (K_p \wedge (tmf(p)_{K(2)})_{K(1)})_p
\end{aligned}$$

The pullback diagram (8.2) implies that $\tilde{W}^{ss}$ is an étale $GL_2(\mathbb{Z}/N)$-torsor over W^{ss}, and W^{ss} is an étale $(\mathbb{Z}/p)^\times$-torsor over $(V^\wedge_\infty)^{ss}$. The resulting homotopy fixed point spectral sequence

$$H^*(GL_2(\mathbb{Z}/N), (K_p^\wedge)_*(tmf(N,p)_{K(2)})_{K(1)}) \Rightarrow (K_p^\wedge)_*(tmf(p)_{K(2)})_{K(1)}$$

therefore collapses to give the required isomorphism

$$(K_p^\wedge)_*(tmf(p)_{K(2)})_{K(1)} \cong (\tilde{W}^{ss})_*^{GL_2(\mathbb{Z}/N)} = W^{ss}_*.$$

This in turn allows us to conclude that the homotopy fixed point spectral sequence

$$H^*((\mathbb{Z}/p)^\times, (K_p^\wedge)_*(tmf(p)_{K(2)})_{K(1)}) \Rightarrow (K_p^\wedge)_*(tmf_{K(2)})_{K(1)}$$

collapses to give the isomorphism

$$(K_p^\wedge)_*(tmf_{K(2)})_{K(1)} \cong (W_*^{ss})^{(\mathbb{Z}/p)^\times} = (V_\infty^\wedge)_*^{ss}.$$

□

The universal property of the pullback, together with the diagram of Remark 8.2, gives a $(\mathbb{Z}/p)^\times$-equivariant map $\tilde{\alpha}^*$:

$$\begin{tikzcd}
\mathrm{Spf}(W^{ss}) \arrow[d] \arrow[dr, dotted, "\tilde{\alpha}^*"] \arrow[drr, bend left] & & \\
\mathcal{M}_{ell}^{ss}(p)^{ord} \arrow[dr] & \mathrm{Spf}(W) \arrow[r] \arrow[d] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[d] \\
& \mathcal{M}_{ell}^{ord}(p) \arrow[r] & \mathcal{M}_{ell}^{ord}
\end{tikzcd}$$

Here, $\mathrm{Spf}(W) = \mathcal{X}$ is the pro-Galois cover of $\mathcal{M}_{ell}^{ord}(p)$ given by Diagram (7.4).

To construct our desired map

$$\alpha_{chrom} : tmf_{K(1)} \to (tmf_{K(2)})_{K(1)}$$

it suffices to construct a $(\mathbb{Z}/p)^\times$-equivariant map

$$\alpha'_{chrom} : tmf(p)_{K(1)} \to (tmf(p)_{K(2)})_{K(1)}.$$

The map α_{chrom} is then recovered by taking homotopy fixed point spectra.

The map $\tilde{\alpha}^*$ induces a map

$$\tilde{\alpha} : W_* \to W_*^{ss}$$

of graded θ-algebras. The obstructions to the existence of a map of $K(1)$-local E_∞-ring spectra

$$\alpha'_{chrom} : tmf(p)_{K(1)} \to (tmf(p)_{K(2)})_{K(1)}$$

inducing the map $\tilde{\alpha}$ on p-adic K-theory lie in:

$$H^s_{Alg_\theta}(W_*, W_*^{ss}[-s+1]) \qquad s \geq 2.$$

These groups are seen to vanish using Lemma 7.5. The obstructions to uniqueness (that is, uniqueness up to homotopy) lie in

$$H^s_{Alg_\theta}(W_*, W_*^{ss}[-s]) \qquad s \geq 1,$$

and these groups are also zero. Because $\tilde{\alpha}$ is $(\mathbb{Z}/p)^\times$-equivariant, we deduce that the map α'_{chrom} commutes with the action of $(\mathbb{Z}/p)^\times$ *in the homotopy category of E_∞-ring spectra*. Because we are working in an injective diagram model category structure, after performing a suitable fibrant replacement of $(tmf(p)_{K(2)})_{K(1)}$, there is an equivalence of (derived) mapping spaces

$$E_\infty(tmf(p)_{K(1)}, (tmf(p)_{K(2)})_{K(1)})_{(\mathbb{Z}/p)^\times\text{-equivariant}} \simeq E_\infty(tmf(p)_{K(1)}, (tmf(p)_{K(2)})_{K(1)})^{h(\mathbb{Z}/p)^\times}.$$

Because the order of $(\mathbb{Z}/p)^\times$ is prime to p, the spectral sequence

$$H^s((\mathbb{Z}/p)^\times, \pi_t E_\infty(tmf(p)_{K(1)}, (tmf(p)_{K(2)})_{K(1)})) \Rightarrow \pi_{t-s} E_\infty(tmf(p)_{K(1)}, (tmf(p)_{K(2)})_{K(1)})^{h(\mathbb{Z}/p)^\times}$$

collapses to show that the natural map

$$[tmf(p)_{K(1)}, (tmf(p)_{K(2)})_{K(1)}]_{\substack{E_\infty \\ (\mathbb{Z}/p)^\times-equivariant}} \to [tmf(p)_{K(1)}, (tmf(p)_{K(2)})_{K(1)}]_{E_\infty}^{(\mathbb{Z}/p)^\times}$$

is an isomorphism. In particular, we may choose α'_{chrom} to be a $(\mathbb{Z}/p)^\times$-equivariant map of E_∞-ring spectra.

Modifications for the prime 2.

At the prime 2, the first stage of the Igusa tower which is a formal affine scheme is $\mathcal{M}_{ell}^{ord}(4)$. All of the algebro-geometric constructions such as $\mathcal{M}_{ell}^{ss}(N,p)^{ord}$, $\mathcal{M}_{ell}^{ss}(p)^{ord}$, etc for p an odd prime go through for the prime 2 with $\mathcal{M}_{ell}^{ord}(p)$ replaced by $\mathcal{M}_{ell}^{ord}(4)$ to produce formal affine schemes $\mathcal{M}_{ell}^{ss}(N,4)^{ord}$ and $\mathcal{M}_{ell}^{ss}(4)^{ord}$. One then defines $(V_\infty^\wedge)^{ss}$ as the pullback

$$\begin{tikzcd}
\mathrm{Spf}((V_\infty^\wedge)^{ss}) \arrow[r, "\alpha^*"] \arrow[d] & \mathcal{M}_{ell}^{ord}(2^\infty) \arrow[d] \\
\mathcal{M}_{ell}^{ss}(4)^{ord} \arrow[r] & \mathcal{M}_{ell}^{ord}(4)
\end{tikzcd} \tag{8.4}$$

Define

$$\begin{aligned}
(\mathit{tmf}(N)_{K(2)})_{K(1)} &:= (\mathcal{O}^{top}_{K(2)}(\mathcal{M}_{ell}^{ss}(N)) \\
(\mathit{tmf}(N,4)_{K(2)})_{K(1)} &:= (\mathit{tmf}(N)_{K(2)})_{K(1)} \wedge_{S_{K(1)}} E_1^{h(1+4\mathbb{Z}_2)} \\
(\mathit{tmf}(4)_{K(2)})_{K(1)} &:= (\mathit{tmf}_{K(2)})_{K(1)} \wedge_{S_{K(1)}} E_1^{h(1+4\mathbb{Z}_2)}
\end{aligned}$$

Just as in the odd primary case, argue (in this order) that we have

$$\begin{aligned}
(K_2^\wedge)_0(\mathit{tmf}(N,4)_{K(2)})_{K(1)} &\cong \tilde{W}^{ss} \\
(K_2^\wedge)_0(\mathit{tmf}(4)_{K(2)})_{K(1)} &\cong W^{ss} \\
(K_2^\wedge)_0(\mathit{tmf}_{K(2)})_{K(1)} &\cong (V_\infty^\wedge)^{ss}
\end{aligned}$$

where $\tilde{W}^{ss}$ and W^{ss} are given as the pullbacks

$$\begin{tikzcd}
\mathrm{Spf}(\tilde{W}^{ss}) \arrow[r] \arrow[d] & \mathrm{Spf}(W^{ss}) \arrow[r] \arrow[d] & \mathrm{Spf}((V_\infty^\wedge)^{ss}) \arrow[d] \\
\mathcal{M}_{ell}^{ord}(N,4)^{ns} \arrow[r] & \mathcal{M}_{ell}^{ord}(4)^{ns} \arrow[r] & (\mathcal{M}_{ell}^{ord})^{ns}
\end{tikzcd}$$

Note that the homotopy groups of $(\mathit{tmf}_{K(2)})_{K(1)}$ are easily computed by inverting c_4 in the homotopy fixed point spectral sequence for EO_2:

$$\pi_*(\mathit{tmf}_{K(2)})_{K(1)} = KO_*((j^{-1}))_2^\wedge.$$

It follows that the hypotheses of Lemma 7.11 are satisfied, and we have an isomorphism

$$(KO_2^\wedge)_*(\mathit{tmf}_{K(2)})_{K(1)} \cong KO_2 \otimes_{\mathbb{Z}_2} (V_\infty^\wedge)^{ss}.$$

The map α^* of Equation (8.4) induces a map

$$\alpha : KO_* \otimes V_\infty^\wedge \to KO_* \otimes (V_\infty^\wedge)^{ss}$$

of graded reduced Bott periodic θ-algebras. The obstructions to the existence of a map of $K(1)$-local E_∞-ring spectra

$$\alpha_{chrom} : \mathit{tmf}_{K(1)} \to (\mathit{tmf}_{K(2)})_{K(1)}$$

inducing the map α on 2-adic KO-theory lie in:

$$H^s_{Alg_\theta^{red}}(KO_* \otimes V_\infty^\wedge, KO_* \otimes (V_\infty^\wedge)^{ss}[-s+1]) \qquad s \geq 2.$$

These groups are seen to vanish using Lemmas 7.11 and 7.13.

Step 2: construction of α_{chrom} as a map of presheaves over $\overline{\mathcal{M}}_{ell}$.

We will now construct a map of presheaves

$$\alpha_{chrom} : (\iota_{ord})_* \mathcal{O}^{top}_{K(1)} \to ((\iota_{ss})_* \mathcal{O}^{top}_{K(2)})_{K(1)}.$$

By the results of Section 2, it suffices to construct this map on the sections of formal affine étale opens of $\overline{\mathcal{M}}_{ell}$.

Let R be a p-complete ring, and let

$$\mathrm{Spf}_{(p)}(R) \to (\overline{\mathcal{M}}_{ell})_p$$

be a formal affine étale open, classifying a generalized elliptic curve C/R. Let ω_R be the pullback of the line bundle ω over $\overline{\mathcal{M}}_{ell}$. The invertible sheaf corresponds to an invertible R-module I. Let R_* denote the evenly graded ring where

$$R_{2t} = I^{\otimes_R t}.$$

Consider the pullbacks:

$$\begin{tikzcd}
\mathrm{Spf}(R^{ord}) \arrow[r] \arrow[d] & \mathrm{Spf}(R) \arrow[d] & \mathrm{Spf}(R^{ss}) \arrow[r] \arrow[d] & \mathrm{Spf}(R) \arrow[d] \\
\mathcal{M}_{ell}^{ord} \arrow[r] & (\overline{\mathcal{M}}_{ell})_p & \mathcal{M}_{ell}^{ss} \arrow[r] & (\overline{\mathcal{M}}_{ell})_p
\end{tikzcd} \tag{8.5}$$

Remark 8.7. It is not immediately clear why these pullbacks are formal affine schemes.

(1) The pullback of $\mathrm{Spf}(R)$ over $\mathcal{M}_{ell}^{ord}$ is a formal affine scheme because the Hasse invariant can be regarded as a section of the restriction of the line bundle $\omega_R^{\otimes p-1}$ to $\mathrm{Spec}(R/p)$. Indeed, if $v_1 \in I^{\otimes_R (p-1)}$ is a lift of the Hasse invariant, then R^{ord} is the zeroth graded piece of the graded ring

$$R_*^{ord} := (R_*)[v_1^{-1}]_p^\wedge.$$

(2) The pullback of $\mathrm{Spf}(R)$ over $\mathcal{M}_{ell}^{ss}$ is formally affine because, by Serre-Tate theory, and the fact that the classifying map is étale, we know that

$$R^{ss} \cong \prod_i W(k_i)[[u_1]],$$

where $\{k_i\}$ is a finite set of finite fields. In Diagram (8.5), $\mathrm{Spf}(R^{ss})$ is taken with respect to the ideal $(p, u_1) \subset R^{ss}$, while $\mathrm{Spf}(R)$ is taken with respect to the ideal $(p) \subset R$. The ring R^{ss} has an alternative characterization: it is the zeroth graded piece of the completion

$$R_*^{ss} := (R_*)_{(v_1)}^\wedge.$$

Define

$$(R^{ss})_*^{ord} := (R_*^{ss}[v_1^{-1}])_p^\wedge$$

and let $(R^{ss})^{ord} \cong R^{ss}[u_1^{-1}]_p^\wedge$ be the zeroth graded piece. Define generalized elliptic curves:

$$\begin{aligned}
C^{ord} &= C \otimes_R R^{ord} \\
C^{ss} &= C \otimes_R R^{ss} \\
(C^{ss})^{ord} &= C^{ss} \otimes_{R^{ss}} (R^{ss})^{ord}
\end{aligned}$$

Since the image of v_1 is invertible in $(R^{ss})_*^{ord}$, the curve $(C_R^{ss})^{ord}$ has ordinary reduction modulo p, and there exists a factorization

$$\begin{tikzcd}
R_* \arrow[r] \arrow[d] & R_*^{ss} \arrow[r] & (R^{ss})_*^{ord} \\
R_*^{ord} \arrow[urr, dotted, "\bar{g}"'] & &
\end{tikzcd} \tag{8.6}$$

We have $K(1)$-local E_∞-ring spectra:

$$\begin{aligned} E^{ord} &:= (\iota_{ord})_* \mathcal{O}^{top}_{K(1)}(\mathrm{Spf}(R)), \\ E^{ss} &:= (\iota_{ss})_* \mathcal{O}^{top}_{K(2)}(\mathrm{Spf}(R)), \\ (E^{ss})^{ord} &:= E^{ss}_{K(1)}. \end{aligned}$$

Combining Propositions 4.4 and 7.16 with Lemma 8.1, we have the following.

Lemma 8.8. *The spectra E^{ord}, E^{ss}, and $(E^{ss})^{ord}$ are elliptic with respect to the generalized elliptic curves C^{ord}/R^{ord}, C^{ss}/R^{ss}, and $(C^{ss})^{ord}/(R^{ss})^{ord}$, respectively.*

Consider the pullbacks

$$\begin{tikzcd} \mathrm{Spf}((W_{ss})^{ord}) \arrow[r, "g"] \arrow[d] & \mathrm{Spf}(W^{ord}) \arrow[r] \arrow[d] & \mathcal{M}^{ord}_{ell}(p^\infty) \arrow[d] \\ \mathrm{Spf}((R_{ss})^{ord}) \arrow[r, "\bar{g}"'] & \mathrm{Spf}(R^{ord}) \arrow[r] & \mathcal{M}^{ord}_{ell} \end{tikzcd}$$

We have, by Proposition 6.1, the following isomorphisms of graded θ-$(V^\wedge_\infty)_*$-algebras:

$$\begin{aligned} (K^\wedge_p)_* E^{ord} &\cong W^{ord}_* \\ (K^\wedge_p)_* (E_{ss})^{ord} &\cong (W_{ss})^{ord}_* \end{aligned}$$

where W^{ord}_* and $(W_{ss})^{ord}_*$ are the even periodic graded θ-algebras associated to the θ-algebras W^{ord} and $(W_{ss})^{ord}$.

We wish to construct a map:

$$\begin{tikzcd} tmf_{K(1)} \arrow[r, "\alpha_{chrom}"] \arrow[d] & (tmf_{K(2)})_{K(1)} \arrow[d] \\ E^{ord} \arrow[r, dotted, "\alpha_{chrom}"'] & (E_{ss})^{ord} \end{tikzcd}$$

The map g induces a map of graded θ-$(V^\wedge_\infty)_*$-algebras

$$g : W^{ord}_* \to (W_{ss})^{ord}_*.$$

The obstructions to realizing this map to the desired map

$$\alpha_{chrom} : E^{ord} \to (E_{ss})^{ord}$$

of $K(1)$-local commutative $tmf_{K(1)}$-algebras lie in

$$H^s_{Alg^\theta_{(V^\wedge_\infty)_*}}(W^{ord}_*, (W_{ss})^{ord}_*[-s+1]), \qquad s > 1.$$

Because W^{ord} is étale over $V^\wedge_\infty$, Lemma 7.6 implies that these obstruction groups all vanish. Thus the realization α_{chrom} exists.

Suppose that we are given a pair of étale formal affine opens

$$\mathrm{Spf}(R_i) \to \overline{\mathcal{M}}_{ell}, \qquad i = 1, 2.$$

Associated to these are $K(1)$-local commutative $tmf_{K(1)}$-algebras

$$\begin{aligned} E^{ord}_i &:= (\iota_{ord})_* \mathcal{O}^{top}_{K(1)}(\mathrm{Spf}(R_i)), \\ (E_{i,ss})^{ord} &:= (\iota_{ss})_* \mathcal{O}^{top}_{K(2)}(\mathrm{Spf}(R_i))_{K(1)}. \end{aligned}$$

and graded θ-$(V^\wedge_\infty)_*$-algebras

$$(K^\wedge_p)_* E_i^{ord} \cong (W_i)_*^{ord},$$
$$(K^\wedge_p)_* (E_{i,ss})^{ord} \cong (W_{i,ss})_*^{ord}.$$

Again, Lemma 7.6 implies that

$$H^s_{Alg^\theta_{(V^\wedge_\infty)_*}}((W_1)_*^{ord}, (W_{2,ss})_*^{ord}[u]) = 0.$$

We deduce that

(1) the Hurewicz map

$$[E_1^{ord}, (E_{2,ss})^{ord}]_{\mathrm{Alg}_{tmf_{K(1)}}} \to \mathrm{Hom}_{Alg^\theta_{(V^\wedge_\infty)_*}}((W_1)_*^{ord}, (W_{2,ss})_*^{ord})$$

is an isomorphism.

(2) The mapping spaces $\mathrm{Alg}_{tmf_{K(1)}}(E_1^{ord}, (E_{2,ss})^{ord})$ have contractible components.

We conclude that:

(1) The maps α_{chrom} assemble to give a natural transformation

$$\alpha_{chrom} : (\iota_{ord})_* \bar{\mathcal{O}}^{top}_{K(1)} \to ((\iota_{ss})_* \bar{\mathcal{O}}^{top}_{K(2)})_{K(1)}.$$

of the associated homotopy functors

$$(\iota_{ord})_* \bar{\mathcal{O}}^{top}_{K(1)} : ((\overline{\mathcal{M}}_{ell})_{p,et,aff})^{op} \to Ho(\text{Comm } tmf_{K(1)}\text{-algebras}),$$
$$((\iota_{ss})_* \bar{\mathcal{O}}^{top}_{K(2)})_{K(1)} : ((\overline{\mathcal{M}}_{ell})_{p,et,aff})^{op} \to Ho(\text{Comm } tmf_{K(1)}\text{-algebras}).$$

(2) The contractibility of the mapping spaces implies that the maps α_{chrom} may be chosen to induce a strict natural transformation of functors:

$$\alpha_{chrom} : (\iota_{ord})_* \mathcal{O}^{top}_{K(1)} \to ((\iota_{ss})_* \mathcal{O}^{top}_{K(2)})_{K(1)}.$$

Putting the pieces together.

Define $\mathcal{O}^{top}_p$ to be the presheaf of E_∞ ring spectra given by the pullback

$$\begin{tikzcd}
\mathcal{O}^{top}_p \arrow[r] \arrow[d] & (\iota_{ss})_* \mathcal{O}^{top}_{K(2)} \arrow[d] \\
(\iota_{ord})_* \mathcal{O}^{top}_{K(1)} \arrow[r, "\alpha_{chrom}"'] & ((\iota_{ss})_* \mathcal{O}^{top}_{K(2)})_{K(1)}
\end{tikzcd}$$

Let R be a p-complete ring and suppose that

$$\mathrm{Spf}(R) \to (\overline{\mathcal{M}}_{ell})_p$$

is an étale open classifying a generalized elliptic curve C/R. Using the same notation as we have been using, there are associated elliptic spectra E^{ord}, E^{ss}, and $(E^{ss})^{ord}$. The spectrum of sections $E := \mathcal{O}^{top}_p(\mathrm{Spf}(R))$ is given by the homotopy pullback

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & E^{ss} \arrow[d] \\
E^{ord} \arrow[r, "\alpha_{chrom}"'] & (E^{ss})^{ord}
\end{tikzcd}$$

We then have the following.

Proposition 8.9. *The spectrum E is elliptic for the curve C/R.*

We first need the following lemma.

Lemma 8.10. *Suppose that A is a ring and that $x \in A$ is not a zero-divisor. Then the following square is a pullback.*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A^{\wedge}_{(x)} \arrow[d] \\
A[x^{-1}] \arrow[r] & A^{\wedge}_{(x)}[x^{-1}]
\end{tikzcd}$$

Proof. Because of our assumption, the map $A \to A[x^{-1}]$ is an injection. The result then follows from the fact that the induced map of the cokernels of the vertical maps

$$A/x^{\infty} \to A^{\wedge}_{(x)}/x^{\infty}$$

is an isomorphism. ☐

Remark 8.11. Lemma 8.10 is true in greater generality, at least provided that A is Noetherian, but this is the only case we need.

Proof of Proposition 8.9. The proposition reduces to verifying that the diagram

$$\begin{tikzcd}
R_* \arrow[r] \arrow[d] & R_*^{ss} \arrow[d] \\
R_*^{ord} \arrow[r, "g"'] & (R^{ss})_*^{ord}
\end{tikzcd} \tag{8.7}$$

is a pullback. Since $\mathrm{Spf}(R) \to (\overline{\mathcal{M}}_{ell})_p$ is étale, and the map $(\overline{\mathcal{M}}_{ell})_p \to (\mathcal{M}_{FG})_p$ is flat (Remark 1.4), the composite

$$\mathrm{Spf}(R) \to (\overline{\mathcal{M}}_{ell})_p \to (\mathcal{M}_{FG})_p$$

is flat. In particular, by Landweber's criterion, the sequence $(p, v_1) \subset R_*$ is regular. Therefore R_* is p-torsion-free, and v_1 is not a zero divisor in R_*/pR_*. Using the facts that R_* is p-complete and p-torsion-free, it may be deduced that v_1 is not a zero divisor in R_*. Therefore, by Lemma 8.10, the following square is a pullback.

$$\begin{tikzcd}
R_* \arrow[r] \arrow[d] & (R_*)^{\wedge}_{(v_1)} \arrow[d] \\
R_*[v_1^{-1}] \arrow[r] & (R_*)^{\wedge}_{(v_1)}[v_1^{-1}]
\end{tikzcd}$$

The square (8.7) is the p-completion of the above square. Since p-completion is exact on p-torsion-free modules, we deduce that (8.7) is a pullback diagram, as desired. ☐

9. CONSTRUCTION OF $\mathcal{O}^{top}_{\mathbb{Q}}$ AND $\mathcal{O}^{top}$

In this section we will construct the presheaf $\mathcal{O}^{top}_{\mathbb{Q}}$, and the map

$$\alpha_{arith} : (\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}} \to \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p \right)_{\mathbb{Q}}.$$

By the results of Section 2, it suffices to restrict our attention to affine étale opens.

The Eilenberg-MacLane functor associates to a graded $\mathbb{Q}$-algebra A_* a commutative $H\mathbb{Q}$-algebra $H(A_*)$. Suppose that

$$f : \mathrm{Spec}(R) \to (\overline{\mathcal{M}}_{ell})_{\mathbb{Q}}$$

is an affine étale open. Define an evenly graded ring R_* by

$$R_{2t} := \Gamma f^* \omega^{\otimes t}.$$

We define

$$\mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}(R)) = H(R_*).$$

The functoriality of $H(-)$ makes this a presheaf of commutative $H\mathbb{Q}$-algebras.

Proposition 9.1. *Let C/R be the generalized elliptic curve classified by f. Then the spectrum $H(R_*)$ uniquely admits the structure of an elliptic spectrum for the curve C.*

Proof. We just need to show that there is a unique isomorphism of formal groups

$$\widehat{C} \xrightarrow{\cong} \mathbb{G}_{H(R_*)}.$$

It suffices to show that there is a uniue isomorphism Zariski locally on $\operatorname{Spec} R$. Thus it suffices to consider the case where the line bundle $f^*\omega$ is trivial. In this case, the formal group $\mathbb{G}_{H(R_*)}$ is just the additive formal group. Since we are working over $\mathbb{Q}$, there is a unique isomorphism given by the logarithm. ☐

Because its sections are rational, the presheaf $(\prod_p (\iota_p)_* \mathcal{O}^{top}_p)_{\mathbb{Q}}$ is a presheaf of commutative $H\mathbb{Q}$-algebras.

There is an alternative perspective to the homotopy groups of an elliptic spectrum that we shall employ. Let $(\overline{\mathcal{M}}_{ell})^1$ denote the moduli stack of pairs (C, v) where C is a generalized elliptic curve and v is a tangent vector to the identity. Then the forgetful map

$$f : (\overline{\mathcal{M}}_{ell})^1 \to \overline{\mathcal{M}}_{ell}$$

is a $\mathbb{G}_m$-torsor. There is a canonical isomorphism

$$f_* \mathcal{O}_{(\overline{\mathcal{M}}_{ell})^1} \cong \bigoplus_{t \in \mathbb{Z}} \omega^{\otimes t} \tag{9.1}$$

which gives the weight decomposition of $\mathcal{O}_{(\overline{\mathcal{M}}_{ell})^1}$ induced by the $\mathbb{G}_m$-action. We deduce the following lemma.

Lemma 9.2. *For any étale open*

$$U \to (\overline{\mathcal{M}}_{ell})_{\mathbb{Q}}$$

for which the pullback

$$f^*U \to (\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}$$

is an affine scheme, there is a natural isomorphism

$$\pi_* \mathcal{O}^{top}_{\mathbb{Q}}(U) \cong \mathcal{O}_{(\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}}(f^*U).$$

Consider the substacks:

$$\overline{\mathcal{M}}_{ell}[c_4^{-1}] \subset \overline{\mathcal{M}}_{ell},$$
$$\overline{\mathcal{M}}_{ell}[\Delta^{-1}] \subset \overline{\mathcal{M}}_{ell}.$$

A Weierstrass curve is non-singular if and only if Δ is invertible, whereas a singular Weierstrass curve ($\Delta = 0$) has no cuspidal singularities if and only if c_4 is invertible. Thus the pair $\overline{\mathcal{M}}_{ell}[c_4^{-1}]$, $\overline{\mathcal{M}}_{ell}[\Delta^{-1}]$ form an open cover of $\overline{\mathcal{M}}_{ell}$. Consider the induced cover

$$\{(\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}[c_4^{-1}], (\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}[\Delta^{-1}]\}.$$

The following lemma is a corollary of the computation of the ring of modular forms of level 1 over $\mathbb{Q}$.

Lemma 9.3. *The stack* $(\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}$ *is the open subscheme of*

$$\mathrm{Spec}(\mathbb{Q}[c_4, c_6])$$

given by the union of the affine subschemes

$$(\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}[c_4^{-1}] = \mathrm{Spec}(\mathbb{Q}[c_4^{\pm 1}, c_6]),$$
$$(\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}[\Delta^{-1}] = \mathrm{Spec}(\mathbb{Q}[c_4, c_6, \Delta^{-1}]).$$

where $\Delta = (c_4^3 - c_6^2)/1728$.

Let $\overline{\mathcal{M}}_{ell}[c_4^{-1}, \Delta^{-1}]$ denote the intersection (pullback)

$$\overline{\mathcal{M}}_{ell}[c_4^{-1}] \cap \overline{\mathcal{M}}_{ell}[\Delta^{-1}] \hookrightarrow \overline{\mathcal{M}}_{ell}.$$

For a presheaf $\mathcal{F}$ on $\overline{\mathcal{M}}_{ell}$, let

$$\mathcal{F}[c_4^{-1}], \quad \mathcal{F}[\Delta^{-1}], \quad \mathcal{F}[c_4^{-1}, \Delta^{-1}]$$

denote the presheaves on $\overline{\mathcal{M}}_{ell}$ obtained by taking the pushforwards of the restrictions of $\mathcal{F}$ to the open substacks

$$\overline{\mathcal{M}}_{ell}[c_4^{-1}], \quad \overline{\mathcal{M}}_{ell}[\Delta^{-1}], \quad \overline{\mathcal{M}}_{ell}[c_4^{-1}, \Delta^{-1}],$$

respectively. By descent, to construct α_{arith}, it suffices to construct a diagram of presheaves of $H\mathbb{Q}$-algebras:

(9.2)
$$\begin{tikzcd}
(\iota_{\mathbb{Q}})_*\mathcal{O}^{top}_{\mathbb{Q}}[c_4^{-1}] \arrow[r, "\alpha_{arith}"] \arrow[d] & \left(\prod_p (\iota_p)_*\mathcal{O}^{top}_p\right)_{\mathbb{Q}}[c_4^{-1}] \arrow[d] \\
(\iota_{\mathbb{Q}})_*\mathcal{O}^{top}_{\mathbb{Q}}[c_4^{-1}, \Delta^{-1}] \arrow[r, "\alpha_{arith}"] & \left(\prod_p (\iota_p)_*\mathcal{O}^{top}_p\right)_{\mathbb{Q}}[c_4^{-1}, \Delta^{-1}] \\
(\iota_{\mathbb{Q}})_*\mathcal{O}^{top}_{\mathbb{Q}}[\Delta^{-1}] \arrow[r, "\alpha_{arith}"'] \arrow[u] & \left(\prod_p (\iota_p)_*\mathcal{O}^{top}_p\right)_{\mathbb{Q}}[\Delta^{-1}] \arrow[u]
\end{tikzcd}$$

We accomplish this in two steps:

Step 1: Construct compatible maps on the sections over $\overline{\mathcal{M}}_{ell}[\Delta^{-1}]$, $\overline{\mathcal{M}}_{ell}[c_4^{-1}]$, and $\overline{\mathcal{M}}_{ell}[c_4^{-1}, \Delta^{-1}]$.

Step 2: Construct corresponding maps of presheaves.

Step 1: Construction of the α_{arith} on certain sections.

Define commutative $H\mathbb{Q}$-algebras

$$tmf_{\mathbb{Q}}[c_4^{-1}] := (\iota_{\mathbb{Q}})_*\mathcal{O}^{top}_{\mathbb{Q}}(\overline{\mathcal{M}}_{ell}[c_4^{-1}])$$
$$tmf_{\mathbb{Q}}[c_4^{-1}, \Delta^{-1}] := (\iota_{\mathbb{Q}})_*\mathcal{O}^{top}_{\mathbb{Q}}(\overline{\mathcal{M}}_{ell}[c_4^{-1}, \Delta^{-1}])$$
$$tmf_{\mathbb{Q}}[\Delta^{-1}] := (\iota_{\mathbb{Q}})_*\mathcal{O}^{top}_{\mathbb{Q}}(\overline{\mathcal{M}}_{ell}[\Delta^{-1}])$$
$$tmf_{\mathbb{A}_f}[c_4^{-1}] := \left(\prod_p (\iota_p)_*\mathcal{O}^{top}_p(\overline{\mathcal{M}}_{ell}[c_4^{-1}])\right)_{\mathbb{Q}}$$
$$tmf_{\mathbb{A}_f}[c_4^{-1}, \Delta^{-1}] := \left(\prod_p (\iota_p)_*\mathcal{O}^{top}_p(\overline{\mathcal{M}}_{ell}[c_4^{-1}, \Delta^{-1}])\right)_{\mathbb{Q}}$$
$$tmf_{\mathbb{A}_f}[\Delta^{-1}] := \left(\prod_p (\iota_p)_*\mathcal{O}^{top}_p(\overline{\mathcal{M}}_{ell}[\Delta^{-1}])\right)_{\mathbb{Q}}$$

Observe that we have

$$\pi_* tmf_{\mathbb{A}_f}[-] \cong \pi_* tmf_{\mathbb{Q}}[-] \otimes_{\mathbb{Q}} \mathbb{A}_f$$

where $\mathbb{A}_f = \left(\prod_p \mathbb{Z}_p\right) \otimes \mathbb{Q}$ is the ring of finite adeles. Therefore there are natural maps of commutative $\mathbb{Q}$-algebras

$$\bar{\alpha}_{arith} : \pi_* tmf_{\mathbb{Q}}[-] \to \pi_* tmf_{\mathbb{A}_f}[-].$$

The Goerss-Hopkins obstructions to existence and uniqueness of maps

$$\alpha_{arith} : tmf_{\mathbb{Q}}[-] \to tmf_{\mathbb{A}_f}[-]$$

of commutative $H\mathbb{Q}$-algebras realizing the maps $\bar{\alpha}_{arith}$ lie in the Andre-Quillen cohomology of commutative $\mathbb{Q}$-algebras:

$$H^s_{comm_{\mathbb{Q}}}(\pi_* tmf_{\mathbb{Q}}[-], \pi_* tmf_{\mathbb{A}_f}[-][-s+1]), \qquad s > 1.$$

Because

$$\pi_* tmf_{\mathbb{Q}}[-] = \mathbb{Q}[c_4, c_6][-]$$

is a smooth $\mathbb{Q}$-algebra, we have

$$H^s_{comm_{\mathbb{Q}}}(\pi_* tmf_{\mathbb{Q}}[-], \pi_* tmf_{\mathbb{A}_f}[-][u]) = 0, \qquad s > 0.$$

We deduce that the Hurewicz map

$$[tmf_{\mathbb{Q}}[-], tmf_{\mathbb{A}_f}[-]]_{Alg_{H\mathbb{Q}}} \to \mathrm{Hom}_{comm_{\mathbb{Q}}}(\pi_* tmf_{\mathbb{Q}}[-], \pi_* tmf_{\mathbb{A}_f}[-])$$

is an isomorphism. In particular, the maps α_{arith} exist.

We similarly find that we have

$$H^s_{comm_{\mathbb{Q}}}(\pi_* tmf_{\mathbb{Q}}[c_4^{-1}], \pi_* tmf_{\mathbb{A}_f}[c_4^{-1}, \Delta^{-1}][u]) = 0, \qquad s > 0,$$
$$H^s_{comm_{\mathbb{Q}}}(\pi_* tmf_{\mathbb{Q}}[\Delta^{-1}], \pi_* tmf_{\mathbb{A}_f}[c_4^{-1}, \Delta^{-1}][u]) = 0, \qquad s > 0.$$

This implies that the diagram

(9.3)
$$\begin{tikzcd}
tmf_{\mathbb{Q}}[c_4^{-1}] \arrow[r, "\alpha_{arith}"] \arrow[d] & tmf_{\mathbb{A}_f}[c_4^{-1}] \arrow[d, "r_1"] \\
tmf_{\mathbb{Q}}[c_4^{-1}, \Delta^{-1}] \arrow[r, "\alpha_{arith}"] & tmf_{\mathbb{A}_f}[c_4^{-1}, \Delta^{-1}] \\
tmf_{\mathbb{Q}}[\Delta^{-1}] \arrow[r, "\alpha_{arith}"] \arrow[u] & tmf_{\mathbb{A}_f}[\Delta^{-1}] \arrow[u, "r_2"']
\end{tikzcd}$$

commutes *up to homotopy* in the category of commutative $H\mathbb{Q}$-algebras.

Because the presheaves $\mathcal{O}_p^{top}$ are fibrant in the Jardine model structure, the maps r_1 and r_2 in Diagram 9.3 are fibrations of commutative $H\mathbb{Q}$-algebras. The following lemma implies that we can rectify Diagram (9.3) to a point-set level commutative diagram of commutative $H\mathbb{Q}$-algebras.

Lemma 9.4. *Suppose that $\mathcal{C}$ is a simplicial model category, and that*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p"'] & X \arrow[d, "q"] \\
B \arrow[r, "g"'] & Y
\end{tikzcd}$$

is a homotopy commutative diagram with A cofibrant and q a fibration. Then there exists a map f', homotopic to f, such that the diagram

$$\begin{tikzcd}
A \arrow[r, "f'"] \arrow[d, "p"'] & X \arrow[d, "q"] \\
B \arrow[r, "g"'] & Y
\end{tikzcd}$$

strictly commutes.

Proof. Let H be a homotopy that makes the diagram commute, and take a lift

$$\begin{tikzcd}
A \otimes 0 \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, "q"] \\
A \otimes \Delta^1 \arrow[r, "H"'] \arrow[ur, dotted, "\tilde{H}"'] & Y
\end{tikzcd}$$

Take $f' = \tilde{H}_1$. ☐

Step 2: construction of Diagram 9.2.

It suffices to construct the diagram on affine opens. Suppose that

$$\mathrm{Spec}(R) \to \overline{\mathcal{M}}_{ell}$$

is an affine étale open. Define commutative $H\mathbb{Q}$-algebras

$$\begin{aligned}
T[c_4^{-1}] &:= (\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}(R[c_4^{-1}])) \\
T[c_4^{-1}, \Delta^{-1}] &:= (\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}(R[c_4^{-1}, \Delta^{-1}])) \\
T[\Delta^{-1}] &:= (\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}(R[\Delta^{-1}])) \\
T'[c_4^{-1}] &:= \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p(\mathrm{Spec}(R[c_4^{-1}])) \right)_{\mathbb{Q}} \\
T'[c_4^{-1}, \Delta^{-1}] &:= \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p(\mathrm{Spec}(R[c_4^{-1}, \Delta^{-1}])) \right)_{\mathbb{Q}} \\
T'[\Delta^{-1}] &:= \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p(\mathrm{Spec}(R[\Delta^{-1}])) \right)_{\mathbb{Q}}
\end{aligned}$$

Let T' be any commutative $tmf_{\mathbb{A}_f}[-]$-algebra, and let

$$\pi_* tmf_{\mathbb{Q}}[-] \to \pi_* T'$$

be a map of $\pi_* tmf_{\mathbb{Q}}[-]$-algebras. We have the following pullback diagram.

$$\begin{tikzcd}
\mathrm{Spec}(\pi_* T[-]) \arrow[rr] \arrow[d] & & \mathrm{Spec}(R \otimes \mathbb{Q}) \arrow[d] \\
\mathrm{Spec}(\mathbb{Q}[c_4, c_6][-]) \arrow[r, equal] & (\overline{\mathcal{M}}_{ell})^1_{\mathbb{Q}}[-] \arrow[r] & (\overline{\mathcal{M}}_{ell})_{\mathbb{Q}}
\end{tikzcd}$$

In particular, we deduce that $\pi_* T[-]$ is étale over

$$\pi_* tmf_{\mathbb{Q}}[-] = \mathbb{Q}[c_4, c_6][-].$$

Therefore, the spectral sequence

$$\mathrm{Ext}^s_{\pi_* T[-]}(H_t(\mathbb{L}(\pi_* T[-]/\pi_* tmf_{\mathbb{Q}}[-])), \pi_* T'[u]) \Rightarrow H^{s+t}_{comm_{\pi_* tmf_{\mathbb{Q}}[-]}}(\pi_* T[-], \pi_* T'[u])$$

collapses to give

$$H^s_{comm_{\pi_* tmf_{\mathbb{Q}}[-]}}(\pi_* T[-], \pi_* T'[u]) = 0.$$

We deduce that

(1) The Hurewicz maps

$$[T[-], T']_{Alg_{tmf_{\mathbb{Q}}[-]}} \to \mathrm{Hom}_{comm_{\pi_* tmf_{\mathbb{Q}}[-]}}(\pi_* T[-], \pi_* T')$$

are isomorphisms.

(2) The mapping spaces $\mathrm{Alg}_{tmf_{\mathbb{Q}}[-]}(T[-], T')$ have contractible components.

This is enough to conclude that there exist maps α_{arith}, functorial in R, making the following diagrams commute

$$\begin{tikzcd}
T[c_4^{-1}] \arrow[r, "\alpha_{arith}"] \arrow[d] & T'[c_4^{-1}] \arrow[d] \\
T[c_4^{-1}, \Delta^{-1}] \arrow[r, "\alpha_{arith}"] & T'[c_4^{-1}, \Delta^{-1}] \\
T[\Delta^{-1}] \arrow[r, "\alpha_{arith}"] \arrow[u] & T'[\Delta^{-1}] \arrow[u]
\end{tikzcd}$$

Since, by homotopy descent, there are homotopy pullbacks

$$\begin{tikzcd}
(\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}(R)) \arrow[r] \arrow[d] & T[c_4^{-1}] \arrow[d] \\
T[\Delta^{-1}] \arrow[r] & T[c_4^{-1}, \Delta^{-1}]
\end{tikzcd}
\qquad
\begin{tikzcd}
\left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p(\mathrm{Spec}(R))\right)_{\mathbb{Q}} \arrow[r] \arrow[d] & T'[c_4^{-1}] \arrow[d] \\
T'[\Delta^{-1}] \arrow[r] & T'[c_4^{-1}, \Delta^{-1}]
\end{tikzcd}$$

We get an induced map on pullbacks

$$\alpha_{arith} : (\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}(R)) \to \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p(\mathrm{Spec}(R)) \right)_{\mathbb{Q}}.$$

which is natural in $\mathrm{Spec}(R)$.

We define $\mathcal{O}^{top}$ to be the presheaf on $\overline{\mathcal{M}}_{ell}$ whose sections over $\mathrm{Spec}(R)$ are given by the pullback

$$\begin{tikzcd}
\mathcal{O}^{top}(\mathrm{Spec}(R)) \arrow[r] \arrow[d] & \prod_p (\iota_p)_* \mathcal{O}^{top}_p(\mathrm{Spec}(R)) \arrow[d] \\
(\iota_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}(R)) \arrow[r, "\alpha_{arith}"'] & \left(\prod_p (\iota_p)_* \mathcal{O}^{top}_p(\mathrm{Spec}(R))\right)_{\mathbb{Q}}
\end{tikzcd}$$

The following proposition concludes our proof of Theorem 1.1.

Proposition 9.5. *The spectrum $\mathcal{O}^{top}(\mathrm{Spec}(R))$ is elliptic with respect to the elliptic curve C/R.*

Proof. The proposition follows from Propositions 8.9 and 9.1, and the pullback

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & \prod_p R^\wedge_p \arrow[d] \\
R \otimes \mathbb{Q} \arrow[r] & \left(\prod_p R^\wedge_p\right) \otimes \mathbb{Q}
\end{tikzcd}$$

□

APPENDIX A. $K(1)$-LOCAL GOERSS-HOPKINS OBSTRUCTION THEORY FOR THE PRIME 2

Theorem 7.1 provides an obstruction theory for producing $K(1)$-local E_∞-ring spectra, and maps between them, at all primes. These obstructions lie in the Andre-Quillen cohomology groups based on p-adic K-homology. Unfortunately, as indicated in Section 7, the K-theoretic obstruction theory is insufficient to produce the sheaf $\mathcal{O}^{top}_{K(1)}$ at the prime 2. At the prime 2 we instead must use a variant of the theory based on 2-adic *real* K-theory. The material in this Appendix is the product of some enlightening discussions with Tyler Lawson.

For a spectrum E, the E-based obstruction theory of [GH] requires the homology theory to be "adapted" to the E_∞ operad. Unfortunately, $KO_2^\wedge$ does not seem to be adapted to the E_∞-operad. While the $KO_2^\wedge$-homology of a free E_∞ algebra generated by the 0-sphere is the free graded reduced θ-algebra on one generator, this fails to occur for spheres of every dimension. Nevertheless, we will show that the obstruction theory can be manually implemented when the spaces and spectra involved are Bott periodic (Definition 7.10).

Theorem A.1.

(1) *Given a Bott-periodic graded reduced θ-algebra A_* satisfying*

$$H^s_c(\mathbb{Z}_2^\times/\{\pm 1\}, A_*) = 0, \text{ for } s > 0, \tag{A.1}$$

the obstructions to the existence of a $K(1)$-local E_∞-ring spectrum E, for which there is an isomorphism

$$(KO_2^\wedge)_* E \cong A_*$$

of graded reduced θ-algebras, lie in

$$H^s_{Alg^{red}_\theta}(A_*/(KO_2)_*, A_*[-s+2]), \qquad s \geq 3.$$

(2) *Given Bott periodic $K(1)$-local E_∞-ring spectra E_1, E_2, and a map of graded θ-algebras*

$$f_* : (KO_2^\wedge)_* E_1 \to (KO_2^\wedge)_* E_2,$$

the obstructions to the existence of a map $f : E_1 \to E_2$ of E_∞-ring spectra which induces f_* on 2-adic KO-homology lie in

$$H^s_{Alg^{red}_\theta}((KO_2^\wedge)_* E_1/(KO_2)_*, (KO_2^\wedge)_* E_2[-s+1]), \qquad s \geq 2.$$

(Here, the θ-$(KO_2^\wedge)_*E_1$-module structure on $(KO_2^\wedge)_*E_2$ arises from the map f_*.) The obstructions to uniqueness lie in

$$H^s_{Alg^{red}_\theta}((KO_2^\wedge)_* E_1/(KO_2)_*, (KO_2^\wedge)_* E_2[-s]), \qquad s \geq 1.$$

(3) *Given such a map f above, there is a spectral sequence which computes the higher homotopy groups of the space $E_\infty(E_1, E_2)$ of E_∞ maps:*

$$H^s_{Alg^{red}_\theta}((KO_2^\wedge)_* E_1/(KO_2)_*, (KO_2^\wedge)_* E_2[t]) \Rightarrow \pi_{-t-s}(E_\infty(E_1, E_2), f).$$

Remark A.2. The author believes that Condition (A.1) is unnecessary, but it makes the proof of the theorem much easier to write down, and is satisfied by in the cases needed in this paper.

The remainder of this section will be devoted to proving the theorem above. Most of the work is in proving (1). As in [GH], consider the category $s\mathrm{Alg}^{K(1)}_{E_\infty}$ of simplicial objects in the $K(1)$-local category of E_∞-ring spectra. Endow this category with a $\mathcal{P}$-resolution model structure[1] with projectives given by

$$\mathcal{P} = \{\Sigma^i T_j\}_{i \in \mathbb{Z}, j > 1}$$

where the spectra T_j are the finite Galois extensions of $S_{K(1)}$ given by

$$T_j = KO_2^{hG_j}$$

---

[1] To be precise, we are endowing the category of simplicial *spectra* with the $\mathcal{P}$-resolution model structure associated to the $K(1)$-local model structure on spectra, and then lifting this to a model structure on simplicial commutative ring spectra.

for

$$G_j = 1 + 2^j \mathbb{Z}_2 \subset \mathbb{Z}_2^\times/\{\pm 1\} =: \Gamma.$$

Note that T_j is $K(1)$-locally dualizible (in fact, it is self-dual), and we have

$$KO_2 \simeq_{K(1)} \varinjlim_j T_j.$$

The forgetful functor $\mathrm{Alg}_\theta^{red} \to \mathrm{Mod}_{\mathbb{Z}_2[[\Gamma]]}$ has a left adjoint — call it $\mathbb{P}_\theta$. Let $\mathbb{P}$ denote the free $K(1)$-local E_∞-algebra functor. Then the natural map is an isomorphism:

$$KO_* \otimes \mathbb{P}_\theta(KO_2^\wedge)_0(S^0) \to (KO_2^\wedge)_*(\mathbb{P}S^0).$$

In fact, the same holds when S^0 is replaced by the spectrum T_j.

As in [GH], an object $X_\bullet$ of $s\mathrm{Alg}_{E_\infty}^{K(1)}$ has two kinds of homotopy groups associated to an object $P \in \mathcal{P}$: the E_2-homotopy groups

$$\pi_{s,t}(X_\bullet; P) := \pi_s[\Sigma^t P, X_\bullet]_{\mathrm{Sp}_{K(1)}}$$

given as the homotopy groups of the simplicial abelian group, and the natural homotopy groups

$$\pi^\natural_{s,t}(X_\bullet; P) := [\Sigma^t P \otimes \Delta^s/\partial\Delta^s, X_\bullet]_{s\mathrm{Sp}_{K(1)}}$$

given as the homotopy classes of maps computed in the homotopy category $h(s\mathrm{Sp}_{K(1)})$. These homotopy groups are related by the spiral exact sequence

$$\cdots \to \pi^\natural_{s-1,t+1}(X_\bullet; P) \to \pi^\natural_{s,t}(X_\bullet; P) \to \pi_{s,t}(X_\bullet; P) \to \pi^\natural_{s-2,t+1}(X_\bullet; P) \to \cdots.$$

We shall omit P from the notation when $P = S^0$.

We will closely follow the explicit treatment of obstruction theory given by Blanc-Johnson-Turner [BJT], adapted to our setting. Namely, we will produce a free simplicial resolution $W_\bullet$ of the reduced theta algebra A_0, and then analyze the obstructions to inductively producing an explicit object $X_\bullet \in s\mathrm{Alg}_{E_\infty}^{K(1)}$ with

$$(KO_2^\wedge)_* X_\bullet \cong KO_* \otimes W_\bullet.$$

The desired E_∞ ring spectrum will then be given by $E := |X_\bullet|$.

Both of the resolutions $W_\bullet$ and $X_\bullet$ will be *CW-objects* in the sense of [BJT, Defn. 1.20] — the spaces of n-simplices take the form:

$$W_n = \bar{W}_n \widehat{\otimes} L_n W_\bullet,$$
$$X_n = (\bar{X}_n \wedge L_n X_\bullet)_{K(1)}.$$

(where $L_n(-)$ denotes the nth latching object). The 'cells' $\bar{W}_n$ (resp. $\bar{X}_n$) will be free reduced θ-algebras (respectively free $K(1)$-local E_∞ rings) and are thus augmented.

For $Y_\bullet$ denoting either $W_\bullet$ or $X_\bullet$, we require that for $i > 0$, the map d_i is the augmentation when restricted to $\bar{Y}_n$. The simplicial structure is then completely determined by the 'attaching maps'

$$\bar{d}_0^{Y_n} : \bar{Y}_n \to Y_{n-1}.$$

and the simplicial identities. Saying that an attaching map $\bar{d}_0^{Y_n}$ satisfies the simplicial identities is equivalent to requiring that the composites $d_i \bar{d}_0^{Y_n}$ factor through the augmentation.

Given such a simplicial free θ-algebra resolution $W_\bullet$ of A_0, and a θ-A_0-module M, the André-Quillen cohomology of A_0 with coefficients in M may be computed as follows. Let QW_n denote the indecomposibles of the augmented free θ-algebra W_n. Then $QW_\bullet$ is a simplicial reduced Morava module, and the Moore chains $(C_*QW_\bullet, d_0)$ form a chain complex of Morava modules. The André-Quillen cohomology is given by the hypercohomology

$$H^n_{\mathrm{Alg}_\theta^{red}}(A_0, M) = \mathbb{H}^n(\mathrm{Hom}^c_{\mathbb{Z}_2[[\Gamma]]}(C_*QW_\bullet), I^*)$$

where I^* is an injective resolution of M in the category of reduced Morava modules. However, if M satisfies

$$H^s_c(\Gamma; M) = 0, \quad s > 0$$

then one can dispense with the injective resolution I^*, and we simply have

$$H^n_{\mathrm{Alg}^{red}_\theta}(A_0, M) = H^n(\mathrm{Hom}^c_{\mathbb{Z}_2[[\Gamma]]}(C_*QW_\bullet, M).$$

We produce $W_\bullet$ and $X_\bullet$ simultaneously and inductively so that $KO_0X_\bullet = W_\bullet$, so that $W_\bullet$ is a resolution of A_0. Start by taking a set of topological generators $\{\alpha^i_0\}$ of A_0 as a θ-algebra. We may take these generators to have open isotropy subgroups in Γ: then there exist j_i so that the isotropy of α^i_0 is contained in the image of $1 + 2^j\mathbb{Z}_2$ in Γ. Note that since there are isomorphisms of Morava modules

$$(KO^\wedge_2)_0T_j \cong \mathbb{Z}_2[(\mathbb{Z}/2^j)^\times/\{\pm 1\}],$$

the generators $\{\alpha^i_0\}$ may be viewed as giving a surjection of θ-algebras

$$\{\alpha^i_0\} : \mathbb{P}_\theta(KO^\wedge_2)_0\bar{Y}_0 \to A$$

for $\bar{Y}_0 = \bigvee_{\alpha^i_0} T_{j_i}$. Define

$$W_0 = \mathbb{P}_\theta(KO^\wedge_2)_0\bar{Y}_0, \qquad X_0 = \mathbb{P}\bar{Y}_0.$$

Then take a collection of open isotropy topological generators $\{\alpha^i_1\}$ (as a Morava module) of the kernel of the map

$$\{\alpha^i_0\} : W_0 \to A_0.$$

Realize these as maps

$$\bar{\alpha}^i_1 : S^0 \to (KO_2 \wedge X_0)_{K(1)}.$$

Suppose that α^i_1 factors through $T_{j_i} \wedge X_0$. Then, since T_{j_i} is $K(1)$-locally Spanier-Whitehead self-dual, there will be resulting maps

$$\tilde{\alpha}^i_1 : T_{j_i} \to X_0.$$

Take

$$\bar{Y}_1 = \bigvee_{\tilde{\alpha}^i_1} T_{j_i}, \quad \bar{W}_1 = \mathbb{P}_\theta(KO^\wedge_2)_0\bar{Y}_1, \quad \bar{X}_1 = \mathbb{P}\bar{Y}_1.$$

and let $\bar{d}^{X_1}_0$ be the map induced from $\{\tilde{\alpha}^i_1\}$. Suppose inductively that we have defined the skeleta $W^{[n-1]}_\bullet$ and $X^{[n-1]}_\bullet$. Note that since

$$\pi_{s,*}(KO \wedge X^{[n-1]}_\bullet) = \begin{cases} A, & s = 0, \\ 0, & 0 < s < n-1 \end{cases}$$

we can deduce from the spiral exact sequence that

$$\pi^\natural_{s,*}(KO \wedge X^{[n-1]}_\bullet) \cong A[-s] \quad 0 \le s \le n-3.$$

Consider the portion of the spiral exact sequence

$$\pi^\natural_{n-1,0}(KO \wedge X^{[n-1]}_\bullet) \to \pi_{n-1,0}(KO \wedge X^{[n-1]}_\bullet) \xrightarrow{\beta_n} \pi^\natural_{n-3,1}(KO \wedge X^{[n-1]}_\bullet) \cong A[-n+2]_0.$$

The map of Morava modules β_n will represent our nth obstruction. Indeed, β_n may be regarded as a map of graded Morava modules

$$\beta_n : \pi_{n-1,*}(KO \wedge X^{[n-1]}_\bullet) \to A[-n+2].$$

Since A satisfies Hypothesis (A.1), there is a short exact sequence

$$\text{(A.2)} \quad \mathrm{Hom}^c_{\mathbb{Z}_2[[\Gamma]]}(C_{n-1}QW^{[n-1]}_\bullet, A[-n+2]_0) \xrightarrow{u} \mathrm{Hom}^c_{\mathbb{Z}_2[[\Gamma]]}(\pi_{n-1,0}(KO \wedge X), A[-n+2]_0) \to H^n_{\mathrm{Alg}^{red}_\theta}(A; A[-n+2]) \to 0$$

and this gives a corresponding class $[\beta_n] \in H^n_{\mathrm{Alg}^{red}_\theta}(A; A[-n+2])$.

Suppose that β_n was zero on the nose. Take a collection $\{\alpha^i_n\}$ of open isotropy topological generators of the Morava module $\pi_{n-1,0}(KO \wedge X^{[n-1]}_\bullet)$. Since β_n is zero, these lift to elements

$$\bar{\alpha}^i_n \in \pi^\natural_{n-1,0}(KO \wedge X^{[n-1]}_\bullet).$$

Assume the lifts also have open isotropy. Then for j_i sufficiently large, the maps

$$\alpha_n^i : S^0 \otimes \Delta^{n-1}/\partial\Delta^{n-1} \to KO \wedge X_\bullet^{[n-1]}$$

come from maps

$$\tilde{\alpha}_n^i : T_{j_i} \otimes \Delta^{n-1}/\partial\Delta^{n-1} \to X_\bullet^{[n-1]}.$$

Define

$$\bar{Y}_n = \bigvee_{\tilde{\alpha}_n^i} T_{j_i}, \quad \bar{W}_n = \mathbb{P}_\theta(KO_2^\wedge)_0\bar{Y}_n, \quad \bar{X}_n = \mathbb{P}\bar{Y}_n.$$

We define a map of simplicial E_∞-algebras

$$\phi_n : \bar{X}_n \otimes \partial\Delta^n \to X_\bullet^{[n-1]}$$

where the restriction

$$\phi_n|_{\Lambda_0^n} : \bar{X}_n \otimes \Lambda_0^n \to X_\bullet^{[n-1]}$$

is taken to be the map which is given by the augmentation on each of the faces of Λ_0^n. The map ϕ_n is then determined by specifying a candidate for the restriction on the 0-face

$$\bar{d}_0^{X_n} = \phi_n|_{\Delta^{n-1}} : \bar{X}_n \otimes \Delta^{n-1} \to X_\bullet^{[n-1]}$$

which restricts to the augmentation on each of the faces of $\partial\Delta^{n-1}$. Thus we just need to produce an appropriate class

$$[\bar{d}_0^{X_n}] \in \pi_{n-1,0}^\natural(X_\bullet^{[n-1]}; \bar{Y}_n).$$

We take $[\bar{d}_0^{X_n}]$ to be the map given by $\{\tilde{\alpha}_n^i\}$. Then we define $X_\bullet^{[n]}$ to be the pushout

$$\begin{tikzcd}
\bar{X}_n \otimes \partial\Delta^n \arrow[r, "\phi_n"] \arrow[d] & X_\bullet^{[n-1]} \arrow[d] \\
\bar{X}_n \otimes \Delta^n \arrow[r] & X_\bullet^{[n]}
\end{tikzcd}$$

in $s\mathrm{Alg}_{E_\infty}^{K(1)}$, and define $W_\bullet^{[n]} := (KO_2^\wedge)_0 X_\bullet^{[n]}$.

However, we claim that if the *cohomology class* $[\beta_n]$ vanishes, then there exists a different choice of ϕ_{n-1} one level down, which will yield a different $(n-1)$-skeleton $X_\bullet^{[n-1]'}$, whose associated obstruction β_n' vanishes on the nose. Backing up a level, different choices ϕ_{n-1}, ϕ_{n-1}' correspond to different lifts of $\{\alpha_{n-1}^i\}$. By the spiral exact sequence, any two lifts differ by an element δ_{n-1}, as depicted in the following diagram in the category of Morava modules:

$$\begin{tikzcd}
& Q\bar{W}_{n-1} \arrow[dl, "\{\alpha_{n-1}^i\}"'] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, dotted, "\delta_{n-1}"] & \\
\pi_{n-2,0}(KO \wedge X_\bullet^{[n-2]}) & \pi_{n-2,0}^\natural(KO \wedge X_\bullet^{[n-2]}) \arrow[l] & \pi_{n-3,1}^\natural(KO \wedge X_\bullet^{[n-2]}) \arrow[l]
\end{tikzcd}$$

The fact that $\beta_{n-1} = 0$, together with the spiral exact sequence

$$\pi_{n-2,*}(KO \wedge X_\bullet^{[n-2]}) \xrightarrow{\beta_{n-1}} \pi_{n-4,*+1}^\natural(KO \wedge X_\bullet^{[n-2]}) \to \pi_{n-3,*}^\natural(KO \wedge X_\bullet^{[n-2]}) \to 0$$

tells us that there is an isomorphism

$$\pi_{n-3,*}^\natural(KO \wedge X_\bullet^{[n-2]}) \xleftarrow{\cong} \pi_{n-4,*+1}^\natural(KO \wedge X_\bullet^{[n-2]}) \cong A[-n+3]$$

and in particular that we can regard δ_{n-1} to lie in (compare [BJT, Lem. 2.11]):

$$\mathrm{Hom}_{\mathbb{Z}_2[[\Gamma]]}^c(Q\bar{W}_{n-1}, A[-n+2]) \cong \mathrm{Hom}_{\mathbb{Z}_2[[\Gamma]]}^c(C_{n-1}QW_\bullet^{[n-1]}, A[-n+2]_0).$$

Let $X_\bullet^{[n-1]'}$ denote the $(n-1)$-skeleton obtained by using the attaching map ϕ'_{n-1}, with associated obstruction β'_n. The difference $\beta_n - \beta'_n$ is the image of δ_{n-1} under the map u of (A.2). Therefore, if the cohomology class $[\beta_n]$ vanishes, then there exists δ_{n-1} such that $u(\delta_{n-1}) = \beta_n$, and a corresponding ϕ'_n, whose associated obstruction $\beta'_n = 0$. This completes the inductive step.

The spectral sequence (3) is the Bousfield-Kan spectral sequence associated to the (diagonal) cosimplicial space

$$E_\infty(B(\mathbb{P}, \mathbb{P}, E_1), KO_2^{\bullet+1} \wedge E_2).$$

The identification of the E_2-term relies on the fact that since E_1 is Bott-periodic,

$$(KO_2^\wedge)_* \mathbb{P}^{\bullet+1} E_1 \cong \mathbb{P}_\theta^{\bullet+1}(KO_2^\wedge)_* E_1.$$

The obstruction theory (2) is just the usual Bousfield obstruction theory [Bou89] specialized to this cosimplicial space.

REFERENCES

[AHS04] Matthew Ando, Michael J. Hopkins, and Neil P. Strickland, *The sigma orientation is an H_∞ map*, Amer. J. Math. **126** (2004), no. 2, 247–334.

[BJT] David Blanc, Mark W. Johnson, and James M. Turner, *Higher homotopy operations and André-Quillen cohomology*, arXiv:1107.4117v1.

[BL10] Mark Behrens and Tyler Lawson, *Topological automorphic forms*, Mem. Amer. Math. Soc. **204** (2010), no. 958, xxiv+141.

[Bla00] David Blanc, *CW simplicial resolutions of spaces with an application to loop spaces*, Topology Appl. **100** (2000), no. 2-3, 151–175.

[Bou89] A. K. Bousfield, *Homotopy spectral sequences and obstructions*, Israel J. Math. **66** (1989), no. 1-3, 54–104.

[Con07] Brian Conrad, *Arithmetic moduli of generalized elliptic curves*, J. Inst. Math. Jussieu **6** (2007), no. 2, 209–278.

[Coo78] George Cooke, *Replacing homotopy actions by topological actions*, Trans. Amer. Math. Soc. **237** (1978), 391–406.

[Dav06] Daniel G. Davis, *Homotopy fixed points for $L_{K(n)}(E_n \wedge X)$ using the continuous action*, J. Pure Appl. Algebra **206** (2006), no. 3, 322–354.

[DH04] Ethan S. Devinatz and Michael J. Hopkins, *Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups*, Topology **43** (2004), no. 1, 1–47.

[DHI04] Daniel Dugger, Sharon Hollander, and Daniel C. Isaksen, *Hypercovers and simplicial presheaves*, Math. Proc. Cambridge Philos. Soc. **136** (2004), no. 1, 9–51.

[DKS89] W. G. Dwyer, D. M. Kan, and J. H. Smith, *Homotopy commutative diagrams and their realizations*, J. Pure Appl. Algebra **57** (1989), no. 1, 5–24.

[DR73] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1973, pp. 143–316. Lecture Notes in Math., Vol. 349.

[GH] Paul G. Goerss and Michael J. Hopkins, *Moduli problems for structured ring spectra*, Preprint, available at www.math.northwestern.edu/∼pgoerss.

[GH04] P. G. Goerss and M. J. Hopkins, *Moduli spaces of commutative ring spectra*, Structured ring spectra, London Math. Soc. Lecture Note Ser., vol. 315, Cambridge Univ. Press, Cambridge, 2004, pp. 151–200.

[GM95] J. P. C. Greenlees and J. P. May, *Completions in algebra and topology*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 255–276.

[GS96] J. P. C. Greenlees and Hal Sadofsky, *The Tate spectrum of v_n-periodic complex oriented theories*, Math. Z. **222** (1996), no. 3, 391–405.

[Har05] Urs Hartl, *Uniformizing the stacks of abelian sheaves*, Number fields and function fields—two parallel worlds, Progr. Math., vol. 239, Birkhäuser Boston, Boston, MA, 2005, pp. 167–222.

[Hid00] Haruzo Hida, *Geometric modular forms and elliptic curves*, World Scientific Publishing Co. Inc., River Edge, NJ, 2000.

[Hid04] ———, *p-adic automorphic forms on Shimura varieties*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2004.

[Hop] Michael J. Hopkins, *$K(1)$-local E_∞-ring spectra*, available from Doug Ravenel's web-page.

[Hop95] ———, *Topological modular forms, the Witten genus, and the theorem of the cube*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 554–565.

[Hov99] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR 1650134 (99h:55031)

[Jar00] J. F. Jardine, *Presheaves of symmetric spectra*, J. Pure Appl. Algebra **150** (2000), no. 2, 137–154.

[KM85] Nicholas M. Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, vol. 108, Princeton University Press, Princeton, NJ, 1985.

[Kuh04] Nicholas J. Kuhn, *Tate cohomology and periodic localization of polynomial functors*, Invent. Math. **157** (2004), no. 2, 345–370.

[Lau04] Gerd Laures, $K(1)$*-local topological modular forms*, Invent. Math. **157** (2004), no. 2, 371–403.

[Nau07] Niko Naumann, *The stack of formal groups in stable homotopy theory*, Adv. Math. **215** (2007), no. 2, 569–600.

[Rez] Charles Rezk, *Supplementary notes for math 512*, Available at http://www.math.uiuc.edu/∼rezk/.

[Rog08] John Rognes, *Galois extensions of structured ring spectra*, Mem. Amer. Math. Soc. **192** (2008), no. 898, 1–97.

[Sil86] Joseph H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 106, Springer-Verlag, New York, 1986.

[Str97] Neil P. Strickland, *Finite subgroups of formal groups*, J. Pure Appl. Algebra **121** (1997), 161–208.

[Tat67] J. T. Tate, $p -$ *divisible groups.*, Proc. Conf. Local Fields (Driebergen, 1966), Springer, Berlin, 1967, pp. 158–183.