Davis: de Rham Witt complex

Note Title 2/26/2009

Outline

1. Witt vectors
2. de Rham - Witt complex
   - is a complex
   - degree zero = Witt vectors
   - Some maps: F, V on full complex

Motivation

$R =$ k-alg k - perfect fld, $\mathrm{char}(k) = p$

e.g. X/k, $X = \mathrm{Spec}(R)$

dR-W cx is a Zariski sheaf of complexes on X

$H^*(\text{dR-W complex}) =$ crystalline cohomology of X

Advertisement:

No cohomology in this talk
but there will be in STAGE
on Mon, March 2, 2-143
at 1:00 pm

Davis will speak about "our convergent subcomplexes, and its cohomology.

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1. Witt vectors

A. Special Cases

$k =$ perfect fld in char p

$F: k \longrightarrow k$ is iso.

$x \longmapsto x^p$

$k \longmapsto W(k)$ p-adically complete DVR
- maximal ideal p
- residue fld k

So totally ramified

Rule: in this talk:

Witt vectors = p-typical Witt vectors

as opposed to "big Witt vectors"

Examples

$W(\mathbb{F}_p) = \mathbb{Z}_p$

$W(\mathbb{F}_{p^n})$ = Ring of integers of the unique deg n unramified extension

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Have a multiplicative map "Teichmüller"

$$[\ ] : k \longrightarrow W(k)$$

$[x]$ is the unique lift of x which has $(p^n)^{th}$ roots $\forall\, n$

Every $w \in W(k)$ can be written uniquely as

$$\sum p^i [x_i]$$

Note: it's not clear how to add

$$\sum p^i [x_i] \quad \text{and} \quad \sum p^j [y_j]$$

B. General case: R = any ring

$$R \rightsquigarrow W(R) \text{ - ring}$$

$$W(R) \equiv \{(r_0, r_1, \ldots) \mid r_i \in R\}$$

as a set

Warning! not as a ring!

How to define ring operators

Define the ghost map:

$$w: W(R) \longrightarrow R^{\mathbb{N}}$$

componentwise operators

$$(r_0, r_1, \ldots) \longmapsto (r_0, r_0^p + p r_1, r_0^{p^2} + p r_1^p + p^2 r_2, \ldots)$$

Want this map to be an algebra homomorphism

<u>Note</u> If R has no p-torsion, this defines it uniquely (since in this case the ghost map is injective)

O/w we need to use functor property:

$$f : R \longrightarrow S$$

$$\Longrightarrow \quad W(f) : W(R) \longrightarrow W(S)$$

$$(r_i)_i \longmapsto (f(r_i))_i$$

<u>Note:</u> if f is injective / surjective

so is $W(f)$

So can define $W(R)$ if R has p-torsion by writing R as a quotient of something w/o p-torsion

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Where does the ghost map come from?

$$\sum p^i [x_i] \longleftrightarrow (x_0, x_1^p, x_2^{p^2}, \ldots\ldots)$$

↑ perfect fld case

Observe: In the 0^{th} coordinate, addition and multiplication are defined as usual

$$W(R) \longrightarrow R$$

Have a multiplicative map

$$[\ ] : R \longrightarrow W(R)$$

$$r \longmapsto (r, 0, 0, \ldots)$$

and

$$w(r, 0, 0, 0, \ldots) = (r, r^p, r^{p^2}, \ldots)$$

Two maps F + V (F: Ring map; V: additive)

Frobenius + Verschiebung "Shove"

F: First define

$$\tilde{F} : R^{\mathbb{N}} \longrightarrow R^{\mathbb{N}}$$

$$(r_0, r_1, \ldots) \longmapsto (r_1, r_2, \ldots)$$

F is defined so that

$$\begin{array}{ccc} W(R) & \xrightarrow{w} & R^{\mathbb{N}} \\ \downarrow F & & \downarrow \tilde{F} \\ W(R) & \xrightarrow[w]{} & R^{\mathbb{N}} \end{array} \qquad \text{commutes}$$

If $R = k$, Frobenius is induced from p^{th} power map.

$$F = W(\text{Frob}) : W(k) \longrightarrow W(k)$$

$$(r_0, r_1, \ldots) \longmapsto (r_0^p, r_1^p, \ldots)$$

$$V : W(R) \longrightarrow W(R)$$

$$(x_0, x_1, \ldots) \longmapsto (0, x_0, x_1, \ldots)$$

- additive
- $V(x)\, y = V(x\, F(y))$

- $FV = p$ (always)
- $VF = p$ if $\mathrm{char}(R) = p$
- $F([x]) = [x^p]$ always

Truncated Witt vectors:

$$W_n(R) = \{(r_0, r_1, \ldots, r_{n-1})\}$$

Ring ops defined as before.

$$W(R) \longrightarrow W_n(R)$$

$$W(R) = \varprojlim W_n(R)$$

Examples: (1) Let R be a $\mathbb{Z}_{(p)}$-algebra

Then: $W(R)$ is a $\mathbb{Z}_{(p)}$-algebra

(2) Let R be a $\mathbb{Z}_{(p)}$-algebra

Then $V(W(R)) \subseteq W(R)$ is an ideal with divided powers.

There is a SES

$$0 \longrightarrow V(W(R)) \to W(R) \longrightarrow R \longrightarrow 0$$

$\Rightarrow$ $V(W(R))$ is an ideal

Functorial Divided powers

Need: $\dfrac{V(w)^n}{n!}$

$$V(w)V(w) = V(w\, FV(w))$$

$$= V(w\, p\, w)$$

$$= p\, V(w^2)$$

$$\Longrightarrow \quad p^{n-1} \mid (V(w))^n$$

But **Warning** $W(R)$ might have p-torsion

$$R = k[x]/(x^p)$$

$$p[x] = FV[x] = (0, x^p, 0, 0, \cdots)$$

$$= 0$$

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2. The dR-W complex

Let A be a $\mathbb{Z}_{(p)}$-algebra

Defi a **Witt complex over A** is

(i) A pro-differential graded alg $E^{*}_{\bullet}$

degree in complex

inverse limit

and a strict map of pro-rings

$$\lambda : W_{\bullet}(A) \longrightarrow E^{0}_{\bullet}$$

(ii) A strict map of pro-graded algs

$$F : E^{0}_{\bullet} \longrightarrow E^{0}_{\bullet-1}$$

compatible w/ λ

$$F d[a] = [a]^{p-1} d[a]$$

(iii) A strict map of graded $E_\bullet^*$-modules

$$V : F_* E^*_{\bullet-1} \longrightarrow E^*_\bullet$$

module structure

i.e. $V(F(x)y) = x V(y)$

s.t.

- $V\lambda = \lambda V$
- $FdV = d$
- $FV = p$

Note: $W(R)$ is a Witt complex (concentrated in deg 0)

Def: The dR-W cx over A is the initial object in the cty of Witt cx's:

$$W_\bullet \Omega_A^*$$

existence: Freyd adjoint functor thm

Lots of properties

- $V(x\,dy) = V(x)\,d\,V(y)$
- $Vd = p\,dV$

Prop 1: There is a surjective map:

$$\Omega^*_{W.(A)} \longrightarrow W.\Omega^*_A$$

Comes from extending

$$\begin{array}{ccc} W.(A) & \longrightarrow & W.\Omega^*_A \\ \downarrow & \nearrow & \\ \Omega^*_{W.(A)} & & \end{array}$$

(Sketch of pf) Image is a Witt complex, call it $E^*_\bullet$.

$$W\Omega_A \longrightarrow E \longrightarrow W.\Omega_A$$

composite is identity.

Prop 2: $W_1(A) = A$

$$\Omega^*_{W_1(A)} = \Omega^*_A \longrightarrow W_1\Omega^*_A$$

is an iso.

Prop 3: The map

$$\Omega^0_{W_0(A)} = W_0(A) \longrightarrow W_0\Omega^0_A$$

is an isomorphism.

Example

(1) $A = k$ perfect field

$W\Omega^i_k = 0$ for $i > 0$

(pf) Show that

$$\Omega^i_{W_n(k)} = 0 \quad \text{for } i > 0$$

(2.) $A = \mathbb{Z}_{(p)}$

From the surjective map

$$\Omega_{W_n(\mathbb{Z}_{(p)})} \twoheadrightarrow W_n \Omega_{\mathbb{Z}_{(p)}}$$

Want to study terms like

$$V^i[1]\, d\, V^j[1]$$

↑ suffices to use 1 because of additivity

Using Verschiebung formulas

get $W\Omega^i_{\mathbb{Z}_{(p)}} = 0$ for $i \geq 2$

Have p-torsion in degree 1.