Ausoni - K-thy of local flds.

Note Title 5/14/2009

Hesselholt - Madsen, On the K-thy of local flds

<u>Annals</u>

Let A be a complete DVR

$K = \text{frac}(A)$, k = Residue fld

mixed char: char $K = 0$

char $k = p \neq 2$, k perfect

<u>Structure</u> <u>thm</u> (Serre's local flds)

$$\begin{tikzcd} W(k) \arrow[rr, "\exists !"] \arrow[dr] & & A \arrow[dl] \\ & k & \end{tikzcd}$$

$$A = \frac{W(k)[\pi]}{\phi(\pi)}$$

ramification index

$$\phi(x) = x^e + p\,\Theta(x)$$

e.g. $A = \mathbb{Z}[\zeta_p]$

$$\phi(x) = \frac{(x-1)^p - 1}{x}$$

Aim: Compute $K_*(A)$, $K_*(K)$

by Trace methods

localization sequence:

$$K(k) \xrightarrow[i']{} K(A) \xrightarrow[j]{} K(K)$$

Problem: don't have localization sequence for

$$THH(-) = T(-)$$

$T(K)$ is an HK-alg

$$\Longrightarrow \quad \overline{\pi}_*(T(K)) = 0 \qquad \left[\begin{array}{c} \overline{\pi}_*(-) \\ \parallel \\ \pi_*(-; \mathbb{Z}/p) \end{array}\right]$$

$$\Longrightarrow \overline{\pi}_* TC(K) = 0$$

Cannot have localization sequence

Def: Let $C^b_{z}(P_A)$

bounded chain cx's of projective A-modules of finite type

W.e.'s $f : X \to Y$ s.t. f is a quasi-iso in $C^b(K\text{-Mod})$

Note: $K(K) \simeq K(C^b_{z}(P_A))$

Thm: 1.5.6 + 1.5.7

We have maps of cofiber sequences

$$\begin{tikzcd}
K(k) \arrow[r] \arrow[d] & K(A) \arrow[r] \arrow[d] & K(K) \arrow[d] \\
TC(k) \arrow[r] \arrow[d] & TC(A) \arrow[r] \arrow[d] & TC(A|K) \arrow[d] \\
TR^n(k) \arrow[r] & TR^n(A) \arrow[r] & TR^n(A|K)
\end{tikzcd}$$

$TR^1(A) = T(A)$

has complicated π_*

but $T(A|K)$ is <u>Nice</u>.

(has good alg description)

The structure of $\pi_* T(A|K)$

(1) Connes' operator

$$d: T_*(A|K) \to T_{*+1}(A|K)$$

from $S^1 \circlearrowleft T(A|K)$

$$d^2 = \eta d = d\eta \qquad \eta \in \pi_1^S$$

at odd primes, $d^2 = 0$

Def: A <u>log</u> <u>ring</u> is a pair (R, M)

- R comm ring
- M comm monoid

$\alpha : M \longrightarrow (R, \times)$ map of monoids

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(2) $M = A \cap K^\times$

$\alpha : M \hookrightarrow A = T_0(A|K)$

So $(T_0(A|K), M)$ is a log ring

(3) $d\log : M \longrightarrow T_1(A|K)$

$M \hookrightarrow K^\times = K_1(K) \longrightarrow T_1(A|K)$

"log derivation"

$(T_0(A|K), M) \to T_1(A|K)$

Set: $E = T_1(A|K)$

$d : A \longrightarrow E$ is a derivation

$d\log : M \longrightarrow E$

Satisfies

Def: a log-derivation from a log-ring (A, M) into an A-mod is given by a pair $(d, d\log)$

$$(A, M) \longrightarrow E$$

d = derivation

$d\log$ = map of monoids

$$d\,\alpha(a) = \alpha(a)\, d\log(a)$$

for $a \in M$

" logarithmic diff'l in calculus "

$$d \log(f) = \frac{df}{f}$$

or $f \, d\log(f) = df$

We have a universal log-derivation, containing the universal derivation

$$A \xrightarrow{d} \Omega^1_A$$

and universal log structure

$$M \longrightarrow A \otimes_{\mathbb{Z}} Gp(M)$$

<u>Def:</u> $$W^1_{(A,M)} = \frac{\left[\Omega^1_A \oplus \left(A \otimes_{\mathbb{Z}} Gp(M)\right)\right]}{\left(d\alpha(a) - a \otimes \alpha(a) \mid a \in M\right)}$$

$$d : M \longrightarrow W^1(A,M)$$
$$a \longmapsto (da, 0)$$

$$d\log : a \longmapsto (0, 1 \otimes a)$$

Computed explicitly.

Relation to THH

Def: A <u>log</u>-<u>diff'l</u> <u>graded</u> ring is (E^*, d) (graded diff'l ring)

w/ monoid map

$$\alpha : M \longrightarrow E^0$$

a monoid map

$$d\log : M \longrightarrow E^1$$

s.t.

$$(d, d\log) : (E^0, M) \longrightarrow E^1$$

is a log-derivation

and s.t. $d \cdot d\log : M \longrightarrow E^2$ is zero.

Universal example!

$$\bigwedge_A W'(A,M) =: W^*_{(A,M)}$$

"Kähler diff'ls w/ log poles"

Rmk 2.2.5

The p-torsion submod:

$${}_p(W'(A,M)) \subset W'(A,M)$$

is a cyclic A/p - module

on $d\log(-p)$

i.e. $\quad {}_p(W(A,M)) = A/p \{d\log(-p)\}$

Rmk: A. Lindenstrauss - Madsen compute

$$T_*(A)$$

$\Rightarrow$ $T_2(A|K)$ is a uniquely divisible gp

$$\Rightarrow \quad \pi_2 T(A|K) \xrightarrow[\cong]{\gamma} {}_p\pi_1 T(A|K)$$

$$K \longmapsto d\log(-p)$$

Thm B The canonical map

$$\omega^*_{(A,M)} \otimes \mathbb{F}_p[K] \longrightarrow \pi_* T(A|K)$$

is an iso!

Rmk:

$\omega^i_{(A,M)}$ is uniquely divisible for $i \geq 2$

$$\Longrightarrow \quad \omega^1_{(A,M)} \cong D \oplus {}_p\omega^1_{(A,M)}$$

$$\Longrightarrow \quad \omega^*_{(A,M)} \otimes \mathbb{F}_p[K] \cong$$

$$\left(A \oplus A/p\,\{d\log(-p)\}\right) \otimes \mathbb{F}_p[\mathcal{K}]$$

Note! This formula is much simpler than Kindenstrauss-Madsen (but relies on it!)

e.g.

For $A = \mathbb{Z}_p[\zeta_p]$

we obtain

$$\pi_* T(\mathbb{Z}_p[\zeta_p] \mid \mathbb{Q}_p(\zeta_p))$$

$$= P_{p-1}(\pi) \otimes E[d] \otimes P[\mathcal{K}]$$

$$P_{p-1}(\pi) = \mathbb{F}_p[\pi] / (\pi^{p-1})$$

But

$\pi_{0\ast} T(\mathbb{Z}_p[S_p])$ is MUCH WORSE!

<u>Moral</u> relative TC is easier to compute than absolute TC.

"~~ramified~~ tamely ramified extensions are log-étale"

$\Longrightarrow$ Descent.

<u>Thm</u> L tamely ramified
$\mid G$
K (i.e. $p \nmid e_{L|K}$)

Then we have a iso.

$$T(A|K) \cong T(B|L)^{hG}$$

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uses that $B \otimes_A \omega_{(A,M_A)}/\omega(k) \xrightarrow{\cong} \omega_{(B,M_B)}/\omega(l)$

Now for:

$$TR^{\bullet}(A|K)$$

The $TR^n_*(A|K)$ is also a log-diff'l graded rly

for $M = A \cap K^\times$

$$\alpha_n : M \xrightarrow{\alpha} A \xrightarrow{[-]_n} W_n(A) = T^n_0(A|K)$$

$d : TR^n_m \longrightarrow TR^n_{m+1}$ comes from S^1-action

$d\log_n : M \hookrightarrow K_1(K) \xrightarrow[\text{trc}]{} TR^n_1(A|K)$

This makes (TR^n_*, M) into a log-diff'l graded ring.

$$
\begin{tikzcd}
M \arrow[r, "\alpha_3"'] & W_3(A) \arrow[r, "\lambda", "\cong"'] \arrow[d, shift right] \arrow[d, shift left] & TR^3_0(A|K) \arrow[r] \arrow[d, "F"', shift right=2] \arrow[d, "R", shift left] & \cdots & & \\
M \arrow[r, "\alpha_2"'] & W_2(A) \arrow[r, "\lambda", "\cong"'] \arrow[u, bend right] \arrow[d, shift right] \arrow[d, shift left] & TR^2_0(A|K) \arrow[r, "d"'] \arrow[u, bend right, "V"'] \arrow[d, "F"', shift right=2] \arrow[d, "R", shift left] & TR^2_1(A|K) \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & TR^2_2 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & \cdots \\
 & & M \arrow[ur, "d\log_2"] & & & \\
M \arrow[r, "\alpha_1"'] & W_1(A) \arrow[r, "\lambda", "\cong"'] \arrow[uu, bend right] & TR^1_0(A|K) \arrow[r, "d"'] \arrow[uu, bend right, "V"'] & TR^1_1 \arrow[r, "d"'] \arrow[uu, bend right] & TR^1_2 \arrow[r] \arrow[uu, bend right] & \cdots \\
 & & M \arrow[ur, "d\log_1"] & & &
\end{tikzcd}
$$

These satisfy:

$$E^n_m = TR^n_m(A|K)$$

(1) $(E^\bullet_*, M)$ is a pro-log diff'l graded ring (w.r.t. R)

i.e. R commutes w/ everything

(2) $\lambda : (W_\bullet(A), M) \longrightarrow (E^\bullet_0, M)$

is a map of pro-log rings

(3) $F : E^{n+1}_* \longrightarrow E^n_*$

is a map of pro-log rings.

$\lambda F = F \lambda$

$$F\,d\log_n = d\log_{n-1},$$

$$F d[a]_n = [a]_{n-1}^{p-1} d([a]_{n-1})$$

(4) $V : E_*^n \longrightarrow E_*^{n+1}$ is

a map of pro-graded

E_*^{n+1}-modules

(via $E_*^{n+1} \xrightarrow[F]{} E_*^n$)

$$\begin{cases} \lambda F = F\lambda \\ FV = p \\ FdV = d \end{cases}$$

(sec 3.2.1)

This structure is a log-Witt complex

over (A, M)

there is a ctgy of such

Thm The is an initial object,
the de-Rham-Witt cx w/
log poles:

$$W_\bullet \omega^*_{(A,M)}$$

$$W_\bullet \omega^*_{(A,M)} \longrightarrow TR^\bullet_*$$

Gives names to all elts
in spectral sequences.

Assume: $\mu_p \subset A \subset K$

$\Longrightarrow$ $K(K)$ is 2-periodic

$$
\begin{tikzcd}
\mu_p = \pi_1 B\mu_p & \pi_2 B\mu_p \arrow[l, "\partial", "\cong"'] \arrow[r] & K_2(k) \arrow[d] \\
 & & TR_2(A|k)
\end{tikzcd}
$$

Take:

$\mathrm{Sym}(\mu_p)$, $\quad F, V, R = 1$, $d = 0$

Thm 6.14

the canonical map

$$
W_\bullet \omega^*_{(A,M)} \otimes \overbrace{\mathrm{Sym}(\mu_p)}^{\mathbb{F}_p[S]} \longrightarrow TR^*_\bullet(A|K)
$$

is a pro-isomorphism!

iso of pro-objects

Rmk: $W_n \omega^i_{(A,M)}$ is uniquely divisible for $n \geq 1$ and $i \geq 2$

[so will not contribute]

$\Rightarrow$ $\overline{TR}_*(A|k)$ is 2-periodic

$$\overline{TC} \longrightarrow \overline{TR} \xrightarrow[F-1]{} \overline{TR}$$

$\Rightarrow$ $\overline{TC}_*(A|k)$ is 2-periodic

and also have

$\overline{K}_*(k)$ 2-periodic starting from $* \geq 1$

$$\begin{tikzcd}
\overline{K}_0(k) \arrow[r] \arrow[d] & \overline{K}_0(A) \arrow[r] \arrow[d] & \overline{K}_0(k) \arrow[r] \arrow[d] & 0 \\
\overline{TC}_0(k) \arrow[r] & \overline{TC}_0(A) \arrow[r] & \overline{TC}_0(A|k) \arrow[r] & \overline{TC}_{-1}(k) = 0
\end{tikzcd}$$

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know K_0, TC_0

$\Rightarrow$ $\overline{K}_1(k) = k^\times / (k^\times)^p$

$$TC_0(A|k) = \mathbb{Z}/p \oplus \mathbb{Z}/p$$

If you don't have root of unity, adjoin it, and use log-étale descent.

$$TC(A|k) \simeq TC(A[\zeta_p]|k(\zeta_p))^{h\Delta}$$

Identify homotopy type using étale K-theory.

<u>example:</u>

$$K(\mathbb{Q}(\zeta_p)) \simeq_p BF\psi^{g^{p-1}} \times F\psi^{g^{p-1}} \times U^{p-1}$$

$$\overline{T}(\mathbb{Z}_p|\mathbb{Q}_p) = E(1) \otimes P(\kappa)$$