Blumberg - THH + TC

Note Title 4/23/2009

Cyclic bar constructions

$(\mathcal{C}, \boxtimes, 1)$ symm monoidal cat

$N^{cyc}_{\boxtimes}(R;M)$ $R \in \mathcal{C}$ is a monoid

$R \boxtimes R \to R$

$M \in \mathcal{C}$ R-bimod

$M \boxtimes R \longrightarrow M$, $R \boxtimes M \longrightarrow M$

$\rightsquigarrow$

simplicial object:

$$[k] \longmapsto M \boxtimes \underbrace{R \boxtimes \cdots \boxtimes R}_{k}$$

e.g.

$$(m, x, y) \longmapsto (mx, y)$$
$$\longmapsto (m, xy)$$
$$\longmapsto (ym, x)$$

3 contexts:

R = top monoid

M = 2-sided R-space

$$[k] \longmapsto M \times R \times \cdots \times R$$

$$N_{\times}^{cyc}(R; *) \;=\; N_\bullet R$$

$$|N_\bullet R| = BR$$

Algebra:

$R =$ k-alg

$M =$ R-bimodule

$$[k] \longmapsto M \otimes_k R \otimes_k \cdots \otimes_k R$$

$\rightsquigarrow$ "Hochschild homology"

In spectra

$R = S$-alg. (A_∞-ring spectrum)

$N^{cyc}_\wedge(R; M)$

$$[k] \longmapsto R \wedge M \wedge \cdots \wedge M$$

Algebra

This is Hochschild homology!

$$HH_*(R;M) := \mathrm{Tor}^{R\otimes R^{op}}(R, M)$$

$$= R \otimes^L_{R\otimes R^{op}} M$$

Cyclic bar construction arises when using

$$B_\bullet(R, R\otimes R^{op}, R\otimes R^{op})$$

to compute.

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2 questions!

(1) Why are we looking at this

(2) Why this resolution

(1)

K hard to compute

THH, TC, etc $\underline{\underline{\text{computable}}}$.

K maps to these things

$$K_*(R) \longrightarrow HH_*(R)$$

(lift to THH, TC)

$$M_n(R) \xrightarrow[\mathrm{Tr}]{} R$$

<u>Dennis Trace</u>:

w/ coef in $GL_n(R)$

$$\begin{tikzcd} N_\cdot GL_n(R) \arrow[r] & N_\times^{cyc} GL_n(R) \arrow[d] \\ & N_\otimes^{cyc} M_n(R) \arrow[d] \\ & N_\otimes^{cyc}(R) \end{tikzcd}$$

K

HH

$$N_\cdot GL_n(R) \longrightarrow N_\times^{cyc} GL_n(R)$$

$$(m_1, \dots, m_\ell) \longmapsto ((m_1 \cdots m_\ell)^{-1}, m_1, \dots, m_\ell)$$

$$N_\times^{cyc} GL_n(R) \longrightarrow N_\otimes^{cyc} M_n(R)$$

$$(m_0, \dots, m_\ell) \longmapsto (m_0 \otimes \cdots \otimes m_\ell)$$

$$N_\otimes^{cyc} M_n(R) \longrightarrow N_\otimes^{cyc} R$$

multi-trace:

$$m_0 \otimes m_1 \otimes \cdots \otimes m_\ell \mapsto \sum_{i_0 \cdots i_\ell} m_0(i_0, i_1) \otimes m_1(i_1, i_2) \otimes \cdots \otimes m_\ell(i_\ell, i_0)$$

gives "Morita equivalence" of HH

Q2: Why this particular resolution?

A: equivariance: S^1-action

tr will be S^1-invariant

$\Rightarrow$ tr factors through S^1-fixed points

THH: R = assoc. alg in spectra.

M = R-bimodule

$$THH(R;M) = |N^{cy}_{\wedge}(R;M)|$$

$$THH(R) := THH(R;R)$$

What if A is a ring.

HA = Eilenberg - MacLane spectrum

M = A-bimodule

$$THH(A; M) := THH(HA, HM)$$

"Changing ground ring from $\mathbb{Z}$ to S"

THH sees far more than HH.

Might want to do this for

MODULE CATEGORIES

$\{R\text{-modules}\} \hookleftarrow \{\text{finite cell } R\text{-modules}\}$

"bounded cx's of projectives"

Algebraically

"perfect cx's"

$\underline{\mathrm{Mod}}_R$ is a $\underline{\text{Spectral}}$ $\underline{\text{Cat}}$.

$\underline{\text{Spectral}}$ $\underline{\text{category}}$ = "Ring spectrum w/ many objects"

$$\mathrm{THH}(\underline{\mathcal{C}}) = \left| [k] \longmapsto \bigvee_{c_0, \dots, c_k} \underline{\mathcal{C}}(c_0, c_1) \wedge \dots \wedge \underline{\mathcal{C}}(c_k, c_0) \right|$$

$\underline{\mathcal{C}}$ - spectral cat

(admits algebraic description as well)

$\underline{\text{Properties}}$:

(1) THH is Morita invariant!

$$\mathrm{THH}(\mathrm{Mod}_R) \simeq \mathrm{THH}(R)$$

(2) THH preserves cofiber/fiber sequences in bimodule variable.

Philosophical Question:

In what ways is THH like K-thy?

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properties that are same: $\begin{cases} \text{additivity} \\ \text{approximation} \end{cases}$

different: $\begin{cases} \text{localization} \\ \text{devissage} \end{cases}$

"THH is invariant under thick closure"

$$THH(R) \simeq THH(Mod_R^f)$$

Mod_R^f = thick closure of R

How to calculate?

Tor spectral sequence

AKA Bökstedt spectral sequence

E = commutative ring spectrum

$$E_2 = HH_*(E_*(R), E_*(M)) \Rightarrow E_*(THH(R,M))$$

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usually $E = H\mathbb{F}_p$ or $H\mathbb{Z}$ ----

Example M = topological monoid

$\Sigma^\infty M_+$ = ring spectrum.

$\shortparallel$

$S[M]$ "group ring"

$THH(S[M])$??

$$THH(\Sigma^\infty M_+) = N^{cyc}_{\wedge}(\Sigma^\infty M_+) \simeq \Sigma^\infty (N^{cyc}_{\times} M)_+$$

$$\Sigma^\infty_+ : (\text{Spaces}, \times) \longrightarrow (\text{Spectra}, \wedge)$$

$N^{cyc}_{\times} M$ is a <u>cyclic</u> <u>object</u>

(not just simplicial)

extra operator t_n permutes n-simplices

$C_{n+1} \circlearrowright$ n-simplices

get a functor

$$\Lambda^{op} \longrightarrow \text{sets, Spaces, etc...}$$

$$\Delta^{op} \hookrightarrow \Lambda^{op}$$

<u>Claim</u> X = cyclic set / space

$|X|$ has an S^1-action

Q: do cyclic sets model S^1-spaces?

A:

w.e. = underlying equivalence

w.e. = w.e. on all $H \leq S^1$

w.e. = w.e. on all finite fixed points

$\cong$

Cyclic sets

How does $|X|$ get an S^1-action

$$|\Lambda_n| \cong \Delta_n \times S^1$$

$\Longrightarrow$ get an S^1 action.

Rmk:

$$S^1\text{-spaces} \longrightarrow \mathrm{Set}^{\Lambda^{op}}$$

w.e. on all finite subgp fixed points

S^{cyc} ↑ cyc singular chains funct

$$S^1 \times |N^{cyc}_\infty M| \longrightarrow |N^{cyc}_\infty M| \longrightarrow BM$$

adjoint:

$$|N^{cyc}_\infty M| \xrightarrow[\simeq]{} LBM$$

↑ on all finite fixed points

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i.e.

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\simeq"] & {|N^{cyc}_{*} M|} \arrow[r] \arrow[d, dashed, "\simeq"] & BM \arrow[d, "\simeq"] \\ \Omega BM \arrow[r] & LBM \arrow[r] & BM \end{tikzcd}$$

$$THH(\Sigma^\infty_{\cdot} M_+) = \Sigma^\infty_{\cdot}(LBM)_+$$

S^1-action

"special kind of S^1-space"

$$\simeq (\Sigma^\infty_{\cdot} M_+) \wedge BM_+$$

Rmk: $f : X \longrightarrow BGL_1 S$

$$\Rightarrow \quad THH(Mf) = Mf \wedge BX_+$$

"lots of things appear as Thom spectra"

$$H\mathbb{Z}/2 = Mf \qquad f : \Omega^2 S^3 \longrightarrow BO$$

$$THH(H\mathbb{Z}/2) \simeq H\mathbb{Z}/2 \wedge \Omega S^3_+$$

Properties of $\Sigma^\infty_+(LBM)_+$ as an S^1-spectrum

Axiomatization of special structure on fixed points.

$\rho_H : S^1 \xrightarrow[\cong]{} S^1/H$ $\quad H \leq S^1$ finite

"taking roots"

$X = S^1$-space

$X^H = S^1/H$ - space

$\rho_H^* X^H = S^1$-space

On LBM

$$\rho_c^* (LBM)^H \cong LBM$$

S^1-equivariant homeomorphism

$\underline{\underline{S}}^1$-$\underline{\text{Spectra}}$: (a few words)

$\mathcal{U}$ = universe

infinite dimensional vector space

w/ S^1-action

Spectra: indexed on f.d. S^1-reps

$V = S^1$-rep

S^V = 1-point compactification

$$\Sigma^\infty_{S^1} (LBM)_+$$

$$\|$$

$$X \qquad\qquad X_0 = \operatorname{colim}_{V \subset \mathcal{U}} \Omega^V \Sigma^V LBM_+$$

Think about fixed points.

Fix prime p

$\{C_{p^n}\}$

$$\left(\Sigma_{S^1}^{\infty}(LBM)_+\right)^{C_{p^n}} \rightrightarrows \left(\Sigma_{S^1}^{\infty}(LBM)_+\right)^{C_{p^{n-1}}}$$

two maps

(1) Inclusion of fixed points (easy)

(2) <u>Restriction</u>

$F(X,Y)$ X, Y are G-spaces

$$F(X,Y)^G \longrightarrow F(X^G, Y^G)$$

"relates ordinary and geometric fixed points"

$$\begin{tikzcd}
\left(\Sigma^\infty_{S^1}(LBM)_+\right)^{C_{p^n}} \arrow[r] & \left(\left[\Sigma^\infty_{S^1}(LBM)_+\right]^{C_p}\right)^{C_{p^n}/C_p} \arrow[d] \\
& \left(\Sigma^\infty_{S^1}\left(LBM_+^{C_p}\right)\right)^{C_{p^n}/C_p} \arrow[d, "\simeq"] \\
& \left(\Sigma^\infty_{S^1} LBM_+\right)^{C_{p^{n-1}}}
\end{tikzcd}$$

$$F(S^V, S^V \wedge LBM_+)^{C_p} \longrightarrow F\left(S^{V^{C_p}}, S^{V^{C_p}} \wedge (LBM)_+^{C_p}\right)$$

brings $(\ )^{C_p}$ inside $\Sigma^\infty_{S^1}$

have maps

F_n, R_n on $\left(\Sigma^\infty_{S^1}(LBM_+)\right)^{C_{p^n}}$

(THH of anything has these maps)

$$TC(R) = \underset{\overleftarrow{F_n, R_n}}{\mathrm{holim}} \; THH(R)^{C_{p^n}}$$

$$TR(R) = \underset{\overleftarrow{R_n}}{\mathrm{holim}} \; THH(R)^{C_{p^n}}$$

HC tells about <u>rational</u> info

TC tells us about <u>p-adic</u> info

$\tilde{J}$ = cat objects = $\mathbb{N}$

$n \geq m$ maps $R, F : n \to m$

$$R_1 = F_1 = id \qquad R_r R_s = R_{rs}$$

$$R_r F_s = F_s R_r$$

$\boxed{\tilde{J}_p = \text{subcat } \{1, p, p^2, \ldots\}}$

$THH(R)^{C_{p^n}}$ define a functor

$\tilde{J},\ \tilde{J}_p \longrightarrow$

idea

$$\begin{tikzcd} X \arrow[r] \arrow[dr, equal] & X^{\times k} \\ & (X^{\times k})^{C_k} \arrow[u, hook] \end{tikzcd}$$

Does not hold in spectra

Technical Rmk on cyclotomic structure

Instead, to get cyclotomic structure

$$F = FSP$$

$$F : \mathrm{Spaces}_* \longrightarrow \mathrm{Spaces}_*$$

$$F(X) \wedge F(Y) \longrightarrow F(X \wedge Y)$$

$$F(S^n) \wedge F(S^m) \longrightarrow F(S^{n+m})$$

$K \wedge K$:

$$\operatorname*{hocolim}_{\substack{I^2 \\ \ni \\ (n_1, n_2)}} F(S^{n_1} \wedge S^{n_2}, \; K(S^{n_1}) \wedge K(S^{n_2}))$$