

Blumberg - THH + TC

Note Title

4/23/2009

Cycle bar constructions

$(\mathbb{C}, \otimes, 1)$ *symm* *residid* *cat*

$N_{\otimes}^{\text{cyc}}(R; M)$ $R \in \mathbb{C}$ is a monoid

$$R \otimes R \rightarrow R$$

$M \in \mathbb{C}$ R -bmod

$$M \otimes R \rightarrow M, \quad R \otimes M \rightarrow M$$

}
↓

symptical object!

$$[k] \longmapsto M \otimes R \otimes \cdots \otimes R$$

$\underbrace{\hspace{10em}}_k$

e.g.

$$\begin{array}{ccc}
 (m, x, y) & \xrightarrow{\quad} & (m, x, y) \\
 & \xrightarrow{\quad} & (m, xy) \\
 & \xrightarrow{\quad} & (y^m, x)
 \end{array}$$

3 contexts:

$R = \text{top monoid}$

$M = 2\text{-sided } R\text{-space}$

$$[k] \xrightarrow{\quad} M \times R \leftarrow \cdots \leftarrow R$$

$$N_x^{c, zc}(R; \#)$$

\parallel

$N \cdot R$

$$|N \cdot R| = BR$$

Algebra:

$$R = k\text{-alg}$$

$$M = R\text{-modul}$$

$$[k] \longmapsto M \otimes_k R \otimes_k \dots \otimes_k R$$

\rightsquigarrow "Hochschild Kohomologie"

In spectra

$$R = S\text{-alg. (A}_{\infty}\text{-alg spectrum)}$$

$$N_n^{\text{cyc}}(R; M)$$

$$[k] \longmapsto R \wedge M \wedge \dots \wedge M$$

Algebras

This $\stackrel{\cong}{\cong}$ Hochschild homology!

$$HH_*(R; M) := \text{Tor}^{R \otimes R^{op}}(R, M)$$

$$= R \otimes_{R \otimes R^{op}}^L M$$

Cycle bar construction arises when using

$$B_*(R, R \otimes R^{op}, R \otimes R^{op})$$

to compute.

2 questions!

- (1) Why are we looking at this
- (2) Why this resolution

①

K had to compute

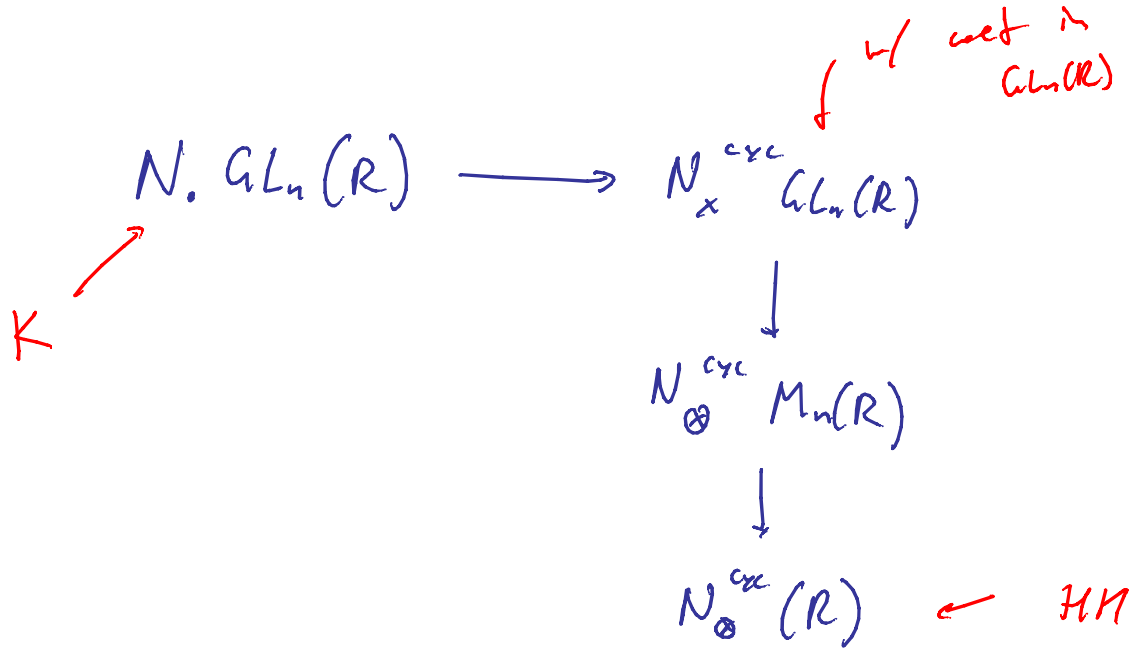
THM, TC, etc compatible.

K maps to these things

$$K_*(R) \longrightarrow HH_*(R) \quad \left(\text{link to } \begin{matrix} \text{THM, TC} \end{matrix} \right)$$

$$M_n(R) \xrightarrow{\text{Tr}} R$$

Dennis Trace:



$$N \cdot GL_n(R) \longrightarrow N_x^{cyc} GL_n(R)$$

$$(m_1, \dots, m_\ell) \longmapsto ((m_1, \dots, m_\ell)^{-1}, m_1, \dots, m_\ell)$$

$$N_x^{cyc} GL_n(R) \longrightarrow N_{\otimes}^{cyc} M_n(R)$$

$$(m_0, \dots, m_\ell) \longmapsto (m_0 \otimes \dots \otimes m_\ell)$$

$$N_{\otimes}^{cyc} M_n(R) \longrightarrow N_{\otimes}^{cyc} R$$

m_0 (+ trace?)

$$m_0 \otimes m_1 \otimes \dots \otimes m_\ell \longmapsto \sum_{i_0, \dots, i_\ell} m_0(i_0, i_1) \otimes m_1(i_1, i_2) \otimes \dots \otimes m_\ell(i_\ell, i_0)$$

gives "Maurer equation"
of HH

Q2: Why this particular result?

A: equivariance: S^1 -action

tr will be S^1 -invariant

$\Rightarrow \text{tr}$ factors through S^1 -fixed points

THH: $R =$ assoc. alg in spectra.

$M = R$ -bimodule

$$\text{THH}(R; M) = |N_{\wedge}^{\text{or}}(R; M)|$$

$$\text{THH}(R) := \text{THH}(R; R)$$

What if A is \mathbb{Z} mod n .

$HA =$ Filenberg - Mac Lane spectrum

$M = A$ -bimodule

$TMM(A; M) := TMM(HA, MM)$

"Changing ground ring from
 \mathbb{Z} to \mathbb{S} "

TMM sees far more than HH .

Might want to do this for

MODULE CATEGORIES

$\{R\text{-modules}\} \longleftrightarrow \{\text{finite cell } R\text{-modules}\}$

Algebraically "banded cxs of projectives"

"perfect cxs"

Mod_R is a Spectral Cart.

Spectral category = "Rly spectrum w/
may objects"

$$TMM(\underline{C}) = \left| [k] \mapsto \bigvee_{c_0, \dots, c_k} \underline{C}(c_0, c_1) \wedge \dots \wedge \underline{C}(c_{k-1}, c_k) \right|$$

C = spectral cat

(admits algebraic description as well)

Properties:

(1) TMM is Morita invariant!

$$TMM(\text{Mod}_R) \simeq TMM(R)$$

(2) TMM preserves cofiber/fiber
sequences in
bimodule modules.

Philosophical Questions!

In what ways is THM like
K-thy?

properties that are same: { additivity
approximate

different: { localization
dense

"THM is invariant under
thick closure"

$$\text{THM}(R) \cong \text{THM}(\text{Mod}_R^f)$$

$\text{Mod}_R^f = \text{thick closure of } R$

How to calculate?

Tor spectral sequence

AKA Bökstedt spectral sequence

$E =$ comm ring spectrum

$$E_2 = HH_*(E_+(R), E_+(M)) \Rightarrow E_*(THH(R, M))$$

usually $E = H\mathbb{F}_p$ or $H\mathbb{Z}$ ----

Example $M =$ top'd module

$$\sum_1^{\infty} M_+ = \text{ring spectrum.}$$

\parallel

$S[M]$

"gp ring"

$THH(S[M])$??

$$\mathrm{THH}(\Sigma^\infty M_+) = N_n^{\mathrm{cyc}}(\Sigma^\infty M_+) \cong \Sigma^\infty (N_x^{\mathrm{cyc}} M)_+$$

$$\Sigma_+^\infty : (\mathrm{Spaces}, \times) \longrightarrow (\mathrm{Spectrum}, \wedge)$$

$N_x^{\mathrm{cyc}} M$ is a cyclic object
(not just simplicial)

extra operator t_n permutes n -simplices

$$C_{n+1} \curvearrowright n\text{-simplices}$$

get a functor

$$\Lambda^{\mathrm{op}} \longrightarrow \text{sets, Spaces, etc...}$$

$$\Lambda^{\mathrm{op}} \hookrightarrow \Lambda^{\mathrm{op}}$$

Claim $X = \text{cyclic set/space}$

$|X|$ has an S^1 -action

Q: do cycle sets model S^1 -spaces?

A:

w.e. = uniquely equivalent

w.e. = w.e. on all $H \leq S^1$

w.e. = w.e. on all finite
sized parts

12

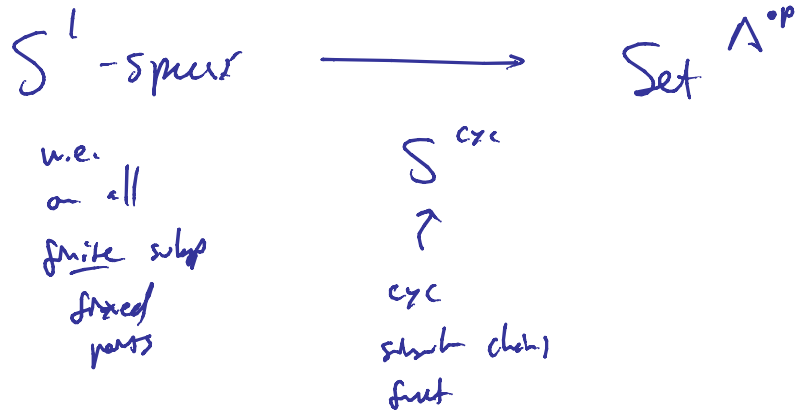
Cycle sets

How does $|X|$ get an S^1 -action

$$|\Lambda_n| \cong \Delta_n \times S^1$$

\implies get an S^1 action.

Rank:



$$S^1 \times |N_n^{\text{cyc}} M| \longrightarrow |N_n^{\text{cyc}} M| \longrightarrow \text{BM}$$

adpint:

$$|N_n^{\text{cyc}} M| \longrightarrow \text{LBM}$$

\cong

\uparrow
on all finite fixed parts

7d

$$\begin{array}{ccccc}
 M & \longrightarrow & (M^{\mathbb{Z}/2} \times M) & \longrightarrow & BM \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 \Omega BM & \longrightarrow & LBM & \longrightarrow & BM
 \end{array}$$

$$\begin{aligned}
 THH(\Sigma_+^{\infty} M_+) &= \Sigma_+^{\infty} (LBM)_+ \quad \begin{array}{l} \curvearrowright S^1\text{-actn} \\ \text{"special kind} \\ \text{of } S^1\text{-space"} \end{array} \\
 &\simeq (\Sigma_+^{\infty} M_+) \wedge BM_+
 \end{aligned}$$

Rank: $f: X \longrightarrow BGL_1 S$

$$\Rightarrow THH(Mf) = Mf \wedge BX_+$$

"lots of thys appear as Thom Spectra"

$$H\mathbb{Z}/2 = Mf$$

$$f: \Omega^2 S^3 \longrightarrow B\mathbb{O}$$

$$THH(H\mathbb{Z}/2) \simeq H\mathbb{Z}/2 \wedge \Omega S^3_+$$

Properties of $\Sigma^0(\text{LBM})_+$ as an
 S' -spectrum

Axiomatization of spectral structure on
fixed points.

$$P_H: S' \xrightarrow{\cong} S'/H \quad H \subseteq S' \text{ finite}$$

"finitely roots"

$$X = S' \text{-space}$$

$$X^H = S'/H \text{-space}$$

$$P_H^* X^H = S' \text{-space}$$

On LBM

$$P_c^* (\text{LBM})^H \cong \text{LBM}$$

S' -equivariant homeomorphism

S'-Spectra: (a few words)

$U =$ universal

infinite dimensional vector space
w/ S' -actions

Spectra: indexed on f.d. S' -reps

$V = S'$ -rep

$S^V =$ 1-point compactification

$$\sum_{S'}^{\infty} (LBM)_+$$

"

X

$$X_0 = \text{colim}_{V \subset U} \Omega^V \Sigma^V LBM_V$$

Think about fixed points.

Fix point p

$\{C_{p^n}\}$

$$\left(\sum_{S^1}^{\infty} (LBM)_+ \right)^{C_{p^2}} \implies \left(\sum_{S^1}^{\infty} (LBM)_+ \right)^{C_{p^{2-1}}}$$

two maps

(1) Inclusion of fixed points (easy)

(2) Restriction

$F(X, Y)$ X, Y are G -spaces

$$F(X, Y)^G \longrightarrow F(X^G, Y^G)$$

"relates ordinary and geometric fixed points"

$$\left(\sum_{s^i}^{\infty} (\text{LBM})_{+} \right)^{C_p^n} \rightarrow \left(\left[\sum_{s^i}^{\infty} (\text{LBM})_{+} \right]^{C_p} \right)^{C_p^n / C_p}$$

↓

$$\left(\sum_{s^i}^{\infty} (\text{LBM})_{+}^{C_p} \right)^{C_p^n / C_p}$$

↪

$$\left(\sum_{s^i}^{\infty} \text{LBM}_{+} \right)^{C_p^{n-1}}$$

$$F(s^v, s^v \sim \text{LBM}_{+})^{C_p} \rightarrow F(s^{vCP}, s^{vCP} \sim (\text{LBM})_{+}^{C_p})$$

↑
bmys $()^{C_p}$ inside $\sum_{s^i}^{\infty}$

how maps

$$F_n, R_n \text{ on } \left(\sum_{s^i}^{\infty} (\text{LBM})_{+} \right)^{C_p^n}$$

(THM of anything has
these maps)

$$TC(R) = \underset{F_n, R_n}{\hookrightarrow} JHM(R)^{C_p^n}$$

$$TR(R) = \underset{R_n}{\longleftarrow} JHM(R)^{C_p^n}$$

HC tells about algebra info

TC tells us about probe info

$\tilde{J} = \text{cat}$ objects = \mathbb{N}

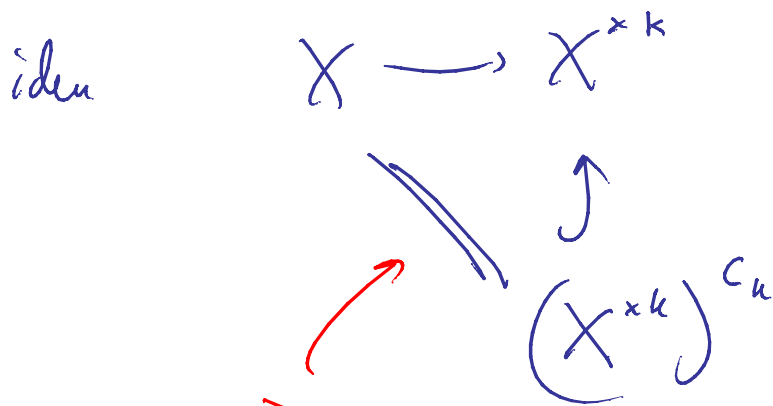
$u = v_m$ maps $R, F : u \rightarrow v$

$$R_i = F_i = \text{id} \quad R_r R_s = R_{rs}$$

$$R_r F_s = F_s R_r \quad \boxed{\tilde{J}_p = \text{subset } \{1, p, p^2, \dots\}}$$

$THH(R)^{C_p^n}$ defn = funct!

\sim , $\sim_p \longrightarrow$



Does
not hold
in spectra

Technical Rank on cyclotomic structure

Instead, to get cyclotomic structure

$$F = FSP$$

$$F: \text{Spres}_* \rightarrow \text{Spres}_*$$

$$F(X) \wedge F(Y) \rightarrow F(X \wedge Y)$$

$$F(S^n) \wedge F(S^m) \rightarrow F(S^{n+m})$$

$$K \wedge K :$$

$$\text{ho coh } F(S^{n_1} \wedge S^{n_2}, K(S^{n_1}) \wedge K(S^{n_2}))$$

$$\begin{array}{c} \mathbb{F}^2 \\ \cup \\ (n_1, n_2) \end{array}$$