Toly

Note Title 2/12/2009

$X$ = variety / $\mathbb{C}$

$X^{an}$ = associated cx analytic variety
$\shortparallel$
$X(\mathbb{C})$

$H^*(X^{an}; \mathbb{Z})$ = singular cohom

f.g. ab gps

$\leadsto$ ranks, betti #'s, p-torsion for various primes p

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Key observation:

These quantities can admit purely alg descriptions, and have geometric consequences.

even not over $\mathbb{C}$

example: $C$ = smooth connected curve / $\mathbb{C}$

$$\mathrm{rk}\ H^1(C^{an}; \mathbb{Z}) = 2g$$

$$g = \text{genus} = \dim H^0(C, \Omega_C)$$
$$= \dim H^1(C, \mathcal{O}_C)$$
$$= 1 - \chi(\mathcal{O}_C)$$

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<u>Dream</u>: Create (purely alg) $H^*_?(X)$ recovering $H^*_{sing}(X^{an}; \mathbb{Z})$ for $X/\mathbb{C}$ but defined & is as well behaved as possible for $X$ $\begin{cases} \text{not in char } 0 \\ \text{not smooth} \end{cases}$

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<u>Prop</u>: $H^*(X^{an}; \mathbb{Z}) = H^*(X^{an}; \underbrace{\underline{\mathbb{Z}}}_{\text{constant sheaf}})$

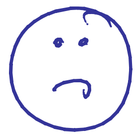
"pf"

$$\underline{\mathbb{Z}} \xrightarrow{\sim} \underbrace{C^\bullet_{sing}}_{\uparrow \text{ acyclic resolution}}(X, \mathbb{Z})$$

$\square$

what about

$H^*_{Zar}(X;\mathbb{Z})$ ? Vanishes for $* > \dim_{\mathbb{C}} X$



"The Zariski topology is too course to compute correct coh of local systems."

not enough opens

1st idea étale coh

finer top'ly

$\Rightarrow$ some locally constnt sheaves get correct coh

2nd idea Alg. De Rham coh

Replace loc. const. sheaves by things for which Zariski is fine enough

**Def:** A <u>Groth</u> (<u>Pre</u>)-<u>topology</u> (almost same as site)

is a ctgy $\mathcal{C}$, together

w/ a collection of collections of morphisms

$\{U_\alpha \to U\}$ which are called "coverings"

0) $f: X \to Y$ iso $\Rightarrow$ $\{X \to Y\}$ is a covering

1) $\{U_\alpha \to U\}$ covering

and

$\{U_{\beta\alpha} \to U_\alpha\}$ covering

$\Rightarrow \{U_{\beta\alpha} \to U_\alpha \to U\}$ covering

2) $\left.\begin{array}{l} f: V \to U \\ \{U_\alpha \to U\} \text{ covering} \end{array}\right\} \Rightarrow$ $U_\alpha \times_U V$ exists & $\{U_\alpha \times_U V \to V\}$

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<u>pre</u>-topology: can generalize to cats w/o pullbacks

A <u>presheaf</u> on $\mathcal{C}$ is a functor

$$\mathcal{C}^{op} \xrightarrow[F]{} \mathrm{Set}$$

A <u>sheaf</u> is a presheaf s.t. for all covers $\{U_\alpha \to U\}$

$$F(U) \to \prod_\alpha F(U_\alpha) \rightrightarrows \prod_{\alpha\beta} F(U_\alpha \times_U U_\beta)$$

is an equalizer.

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A <u>topos</u> is a category of the form

$$Sh(\mathcal{C})$$

site

Reason why you want this def:

There are maps of topoi which do <u>NOT</u> arise from maps of sites.

Ex: $X_{zar}$ objects: $U \overset{\text{zar open}}{\hookrightarrow} X$

morph: $U \underset{\text{open}}{\hookrightarrow} V$

$X_{an}$ objects: $U \hookrightarrow X(\mathbb{C})$

morph: 
$$\begin{tikzcd} U \arrow[rr, hook, "\text{open}"'] \arrow[dr] & & V \arrow[dl] \\ & X(\mathbb{C}) & \end{tikzcd}$$

Covers = set theoretically.

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$X_{ét}$ objects: $U \xrightarrow[\text{étale}]{} X$ "iso on tangent spaces"

morphisms:
$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

$\{U_\alpha \longrightarrow U\}$ is covering

if $\bigcup f(U_\alpha) = U$.

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Assume $X$ is a smooth var$/\mathbb{C}$

$X_{an, ét}$ objects: $U \xrightarrow[\text{étale}]{\text{analytic}} X$

mor: comm's as before

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$X$ smooth $/\mathbb{C}$

$$\begin{tikzcd}
 & X_{an,et} \arrow[dl, "\text{pretty much same coh}"'] \arrow[dr, "\text{same coh for torsion sheaves}"] & \\
X_{an} & & X_{ét}
\end{tikzcd}$$

want to study

# Cohomology

$\mathcal{C}$ = site, $U \in \mathcal{C}$

$Sh_{AbGp}(\mathcal{C})$ = ab cat w/ enough injectives

$\cup\!\!\!\!\!\!\;\;\mathcal{F}$

$$H^*(U, \mathcal{F}) := R^*\Gamma(U, \mathcal{F})$$

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Thm $X$ smooth$/\mathbb{C}$

$$H^*(X_{\acute{e}t}; \mathbb{Z}/\ell^n) \cong H^*(X_{an}, \mathbb{Z}/\ell^n)$$

So get all ranks, all torsion.

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Defects: • Not very well behaved for non-smooth

but MORE IMPORTANTLY

- $p$-torsion is VERY BAD in char $p$

Sub-dream: Fix this.

(Crystalline)

Starting Rmk:

$X/\mathbb{C}$ projective

GAGA $\longrightarrow$ Zariski coh gives correct answers for coherent sheaves of $\mathcal{O}_X$-mods.

e.g. vector bundles.

Starting Rmk 2: (after de Rham thm)

i) $M$ is a smooth mfld

$TM$ = tangent sheaf

$\Omega^1_{M^{smooth}}$ = dual

$\Omega^n_{M^{smooth}} = \Lambda^n \Omega_{M^{smooth}}$

$$\underbrace{\mathcal{O}_{M^{sm}} \xrightarrow{d} \Omega^1_{M^{sm}} \longrightarrow \Omega^2_{M^{sm}} \to \cdots \to \Omega^{\dim M}_{M^{sm}}}_{=:\ \Omega^\bullet_{M^{sm}}}$$

- Poincaré lemma:

  $\Omega^\bullet_{M^{sm}}$ is exact, except at beginning

  $$\ker d\big|_{\mathcal{O}_{M^{sm}}} = \underline{\mathbb{R}}$$

- (Partitions of unity) Each

  $\Omega^n_{M^{sm}}$ is "fine" (so acyclic)
  no higher coh

$$\Longrightarrow H^*(M,\mathbb{R}) = H^*(M,\underline{\mathbb{R}}) = H^*(\Gamma(M,\Omega^\bullet_{M^{sm}}))$$

De Rham Thm

(ii) $M$ = $\mathbb{R}$ or $\mathbb{C}$-analytic mfld

$$\Omega^\bullet_{M^{an}}$$

- Poincaré lemma: still true!

$$\underline{\mathbb{K}} \to \Omega^\bullet_{M^{sm}} \quad \text{quasi-iso}$$

$$\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}$$

- Acyclic: NO!

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Thm:

$$H^*(M, \mathbb{K}) = H^*(M, \underline{\mathbb{K}}) = \mathbb{H}^*(M, \Omega^\bullet_{M^{an}})$$

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(iii) $X$ = smooth proj var. $/\mathbb{C}$

$$(\Omega^\bullet_X)^{an} = \Omega^\bullet_{X^{an}}$$

GAGA $\Rightarrow$ $\mathbb{H}^*(X, \Omega^\bullet_X)$ } – purely algebraic!

$$= \underbrace{\mathbb{H}^*(X^{an}, \Omega^\bullet_{X^{an}})}_{*}$$

(iv) $X$ smooth$/\mathbb{C}$ but not nec proper.

<u>Thm (Groth)</u> uses res'n of singularities

(*) above still <u>true</u>.

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<u>Def:</u> Suppose $f: X \to S$ smooth

$$H^*_{dR}(X/S) = Rf_*(\Omega^\bullet_{X/S})$$

$$H^*_{DR}(X) = \mathbb{H}^*(X, \Omega^\bullet_{X/S})$$

v) $X$ var$/\mathbb{C}$ not necc smooth.

$$X \underset{\text{closed}}{\hookrightarrow} Y \quad \text{smooth var}/\mathbb{C}$$

$$\hat{\underset{\sim}{I}} = \tilde{I}_{X \hookrightarrow Y}$$

$\hat{Y}_{/X}$ = locally ringed space

$$\left(X, \varprojlim \mathcal{O}_Y / \underset{\sim}{I}^n\right)$$

topologically like $X$

geometrically like $Y$

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$$\mathbb{H}^*(\hat{Y}_{/X}, \hat{\Omega}^\bullet_Y) = \text{"Good def"}$$

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Problems

- Not intrinsic for non-smooth case
- It would be nice to be able to vary coef's (instead of only $\mathbb{C}$)

e.g.

**Leray - Serre SS:**

$f : X \longrightarrow Y$ smooth

$$H^p(Y, \underbrace{Rf_*^q \underline{\mathbb{C}}}_{\text{different coef's}}) \Rightarrow H^{p+q}(X, \mathbb{C})$$

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Infinitesimal site solves both!

**Def:** (old) $X/S$ smooth

An $S$-connection on $M$ ($\mathcal{O}_X$-mod) is an $\pi^{-1}\mathcal{O}_S$-linear map

$$\nabla : M \longrightarrow \Omega^1_{X/S} \otimes_{\mathcal{O}_X} M$$

Satisfying: $\nabla(fm) = df \otimes m + f\nabla m$

Can extend to

$$\nabla^i : \Omega^i_{X/S} \otimes M \to \Omega^{i+1}_{X/S} \otimes M$$

$$\nabla^i(\omega \otimes m) = d\omega \otimes s + (-1)^i \omega \wedge \nabla m$$

$\nabla$ is <u>integrable</u> if

$$\nabla^1 \circ \nabla^0 = 0$$

Then $\nabla^{i+1} \circ \nabla^i = 0 \quad \forall i$

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$$\Omega^\bullet_{X/S} \otimes_{\mathcal{O}_X} M$$

wrong: need connection to make sense of this bad notation

<u>E.g.</u> $(\mathcal{O}, d) \rightsquigarrow$ Usual de Rham cx.

In analytic setting:

$$\left\{\begin{array}{c}\text{locally const} \\ \text{sheaves on } X^{an} \\ \text{(fin'd } \mathbb{C} \text{ v.s. fibers)}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Vector bundles} \\ \text{w/ integrable} \\ \text{connection}\end{array}\right\}$$

$$\begin{array}{c}\text{flat sections} \\ = \ker \nabla\end{array} \longleftarrow\!\shortmid \ (M, \nabla)$$

Riemann-Hilbert correspondence explains this wonderfully

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equivalence of "bounded derived cats"

$$\underbrace{R\mathrm{Hom}(\underline{\mathbb{C}}^{an}, \underline{\mathbb{C}}^{an})}_{\text{cohomology}} = \underbrace{R\mathrm{Hom}((\mathcal{O}, d), (\mathcal{O}, d))}_{\substack{\text{ctgy} \\ \text{of D-modules}}}$$

## But infinitesimal site suffices!

**Def:** (New) $(X/S)_{inf}$ the infinitesimal site

**objects:** $(U, T)$

$$\begin{tikzcd} U \arrow[r, hook, "\text{open}"'] \arrow[d, hook] & X \\ T & \end{tikzcd}$$

closed immersion w/ nilpotent ideal sheaf

$/S$

**morphisms:** maps of pairs

**Coverings:** $\{(U_\alpha, T_\alpha) \xrightarrow[f_\alpha]{} (U, T)\}$

$$\bigcup f_\alpha(T_\alpha) = T$$

$(X/S)_{Strat}$ objects: $X \supset U \hookrightarrow T$

locally admits a section

# Whats geom connection?

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\text{nilp.}"] & X \\ Y' \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] & \end{tikzcd}$$

integrable connection $\Longleftrightarrow$ $g_1^* M \cong g_2^* M$

+ transitivity