LEcTure 30: InducTed Maps Between clAssifying Spaces, H*(BU(n))

1. Induced Maps Between Classifying Spaces

Let $G$ be a topological group.

**Definition 1.1.** Let $X$ be a right $G$-space and $Y$ be a left $G$-space. The Borel construction is the quotient

$$X \times_G Y = X \times Y/(xg, y) \sim (x, gy).$$

Observe:

$$G \times G Y = Y$$

$$\ast \times_G Y = Y/G.$$

Suppose that $\alpha : G \to H$ is a homomorphism of topological groups. We can get an induced map

$$\alpha_* : BG \to BH$$

by providing a natural transformation between the functors that these spaces represent:

$$\alpha_* : \{G\text{-bundles over } X\} \to \{H\text{-bundles over } X\}.$$

Namely, send a $G$-bundle $E \to X$ to the $H$-bundle

$$H \times_G E \to X$$

where $G$ acts on $H$ through the homomorphism $\alpha$.

Thus $B$ may be viewed as a functor

Topological groups $\to$ homotopy category of CW-complexes.

There is one particular case where the induced map admits a more explicit description. Suppose that $H$ is a sub-Lie group of a Lie group $G$. On a homework problem you showed that $G \to G/H$ was an $H$-bundle. It follows that the quotient

$$EG \to EG/H$$

is an $H$-bundle, and is therefore classified by a map $\phi$:

$$\begin{array}{ccc}
EG & \longrightarrow & EH \\
\downarrow & & \downarrow \\
EG/H & \phi & BH
\end{array}$$

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Proposition 1.2. The map $\phi$ is an equivalence, and the map induced by the inclusion

$$i : H \hookrightarrow G$$

is the quotient map

$$i_* : BH \simeq EG/H \rightarrow EG/G = BG.$$ 

A proof of this proposition is deferred to the next lecture.

Corollary 1.3. There is a fiber sequence

$$G/H \rightarrow BH \xrightarrow{i_*} BG.$$

2. Classifying vector bundles

Given a paracompact space $X$, there are $1 - 1$ correspondences

$$\{U(n)\text{-bundles over }X\} \downarrow \{\text{hermitian }n\text{-dimensional complex vector bundles over }X\} \downarrow \{n\text{-dimensional complex vector bundles over }X\}.$$

The first correspondence associates to a principle $U(n)$-bundle $P$ over $X$ the vector bundle

$$P \times_{U(n)} \mathbb{C}^n \rightarrow X$$

with a fixed hermitian structure on $\mathbb{C}^n$.

Using partitions of unity, every vector bundle admits a hermitian structure, unique up to isomorphism.

3. Calculation of $H^*(BU(n))$

Theorem 3.1. There is an isomorphism

$$H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_n].$$

The generator $c_i$ lies in degree $2i$, and is called the $i$th Chern class. The induced map

$$H^*(BU(n)) \rightarrow H^*(BU(n - 1))$$

is the quotient by the ideal $(c_n)$.

Proof. We prove this theorem by induction on $n$, using the Serre spectral sequence of the fiber sequence

$$S^{2n-1} \rightarrow BU(n - 1) \rightarrow BU(n).$$

The $E_2$-term takes the form

$$E_2^{s,t} = H^s(BU(n)) \otimes \Lambda[e_{2n-1}]$$

where $H^s(BU(n))$ lies in $E_2^{s,0}$ and $|e_{2n-1}| = (0,2n-1)$. Because this is a 2-line spectral sequence, the only possible non-trivial differential is $d_2$, and the spectral sequence degenerates into a long exact sequence (the Gysin sequence). Because the inductive hypothesis implies that $H^*(BU(n - 1))$ is concentrated in
even dimensions, we may deduce that the differential $d_{2n}(e_{2n-1})$ is non-trivial, and we define $c_n \in H^p(BU(n))$ to be the image of this differential

\[ c_n := d_{2n}(e_{2n-1}) \]

The multiplicative structure implies that $d_{2n}$ is given by cupping with $c_n$. Thus the $E_\infty$-term is comprised of the elements of $H^*(BU(n))$ annihilated by $c_n$, and the quotient of $H^*(BU(n))$ by the ideal $(c_n)$. By inducting over $k$, using the fact that $H^*(BU(n-1))$ is concentrated in even dimensions, we can argue that $H^*(BU(n))$ must be generated in even dimensions, implying that in fact there can be no elements annihilated by $c_n$. Thus $H^*(BU(n))$ is generated by $c_1, \ldots, c_n$, with

\[ H^*(BU(n))/(c_n) = \mathbb{Z}[c_1, \ldots, c_{n-1}] \]

We deduce that there is an isomorphism $H^*(BU(n)) \cong \mathbb{Z}[c_1, \ldots, c_n]$. \qed

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