LECTURE 29: LINE BUNDLES

[The first part of this lecture was about the multiplicative structure of the Serre spectral sequence, which followed Hatcher. I also presented Hatcher’s example of the cohomological Serre spectral sequence for $S^1 \to ES^1 \to CP^\infty$.]

We have equivalences

$$\mathbb{C}P^\infty \simeq BU(1) \simeq K(\mathbb{Z}, 2).$$

Recall:

$$H^*(\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z}[c_1]$$

where $|c_1| = 2$. This generator is called the “first Chern class”.

Given a space $X$, there are 1–1 correspondences

$$\{U(1)\text{-bundles over } X\}$$

$$\downarrow$$

$$\{\text{hermitian complex line bundles over } X\}$$

$$\downarrow$$

$$\{\text{complex line bundles over } X\}.$$ 

The first correspondence associates to a principle $U(1)$-bundle $P$ over $X$ the line bundle

$$P \times_{U(1)} \mathbb{C} \to X$$

with a fixed hermitian structure on $\mathbb{C}$. Since every hermitian structure on $\mathbb{C}$ takes the form

$$(z, w) = az \bar{w}$$

for $a$ a positive real, hermitian structures on a line bundle $L \to X$ are the same thing as positive values real functions on $X$, and any two hermitian structures on $L$ are equivalent.

Given a line bundle $L/X$, there is a classifying map

$$\xymatrix{ L \ar[r] & L_{\text{univ}} \ar[d] \ar[r] & \ar[d] \ar[r] & BU(1) \ar[d] \ar[r] & }$$

which recovers $L$ as the pullback of $L_{\text{univ}}$.

**Definition 0.1.** The first Chern class $c_1(L) \in H^2(X; \mathbb{Z})$ is defined to be the class $f^*_L(c_1)$.

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*Date: 4/24/06.*
Proposition 0.2. The association
\[ L \mapsto c_1(L) \]
gives an isomorphism
\[ \{ \text{complex line bundles over } X \} \cong H^2(X; \mathbb{Z}). \]

Proof. Under the equivalence \( BU(1) \simeq K(\mathbb{Z}, 2) \), \( c_1 \) gives the fundamental class of \( H^2(K(\mathbb{Z}, 2), \mathbb{Z}). \) \( \square \)

Remark 0.3. A homework problem you were assigned indicates that if the left hand side of the isomorphism in Proposition 0.2 is given the structure of an abelian group by \( \otimes \), then this is an isomorphism of groups.