Definition 0.1. A map $f : X \to Y$ is a fibration if it satisfies the covering homotopy property (CHP): for all maps $f$ and homotopies $H$ making the square commute

$$
\begin{array}{ccc}
Z \times \{0\} & \xrightarrow{f} & X \\
\downarrow & \nearrow \tilde{H} & \\
Z \times I & \xrightarrow{H} & Y
\end{array}
$$

there exists a lift $\tilde{H}$ making the diagram commute.

We saw that cofibrations $f : X \to Y$ had the property that the canonical map

$$C(f) \to Y/X$$

is a homotopy equivalence. On the homework, you will verify:

Lemma 0.2. Suppose that $f : X \to Y$ is a fibration, and suppose that $Y$ is pointed. The canonical map

$$f^{-1}(\ast) \to F(f)$$

is a homotopy equivalence.

The infinite fiber sequence therefore yields the following corollary:

Corollary 0.3. For a fibration $f : X \to Y$ with fiber $F = f^{-1}(\ast)$, there is a long exact sequence of homotopy groups

$$\cdots \to \pi_n(F) \to \pi_n(X) \xrightarrow{f_*} \pi_n(Y) \to \pi_{n-1}(F) \to \cdots$$

Examples of fibrations:

1. Covering spaces: the CHP is easily obtained from the homotopy lifting properties of covering spaces.
2. Products: a projection $X \times F \to X$ is easily seen to be a fibration.
3. Locally trivial bundles: a map $f : X \to Y$ is a locally trivial bundle with fiber $F$ if there is an open cover $\{U_i\}$ of $Y$ such that there are homeomorphisms $f^{-1}(U_i) \approx U_i \times F$. If $Y$ is paracompact, then $f$ is a fibration.

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