HOMEWORK 7

DUE DATE: WEDNESDAY, APRIL 5

1. Argue that there exists a map $\alpha : S^2 \to S^2 \vee S^1$ so that the inclusion

$$S^1 \hookrightarrow (S^2 \vee S^1) \cup_\alpha D^3$$

induces an isomorphism on π_1 and $\widetilde{H}_*$, but is not a homotopy equivalence.

2. (Hatcher) Consider the equivalence relation $\sim_w$ generated by weak homotopy equivalence: $X \sim_w Y$ if there are spaces $X = X_1, X_2, \dots, X_n = Y$ with weak homotopy equivalences $X_i \to X_{i+1}$ or $X_i \leftarrow X_{i+1}$ for each i. Show that $X \sim_w Y$ iff X and Y have a common CW-approximation.

3. Show that $p : E \to B$ is a principle G-bundle, and $f : X \to B$ is a map, then the pullback $f^*E = E \times_B X \to X$ is a principle G-bundle. Show that if $g : Y \to X$ is another map, then there is an isomorphism of G-bundles

$$g^* f^* E \cong (f \circ g)^* E.$$

4. Suppose that X is a pointed, connected CW complex. A trivialization of a principle G-bundle

$$p : E \to X$$

is a section $s : X \to E$, satisfying $ps = Id_X$. An isomorphism of trivialized bundles is a bundle isomorphism which preserves the trivialization.

(a) Argue that a trivialization is the same thing as an isomorphism with the trivial G-bundle.
(b) Let $EG \to BG$ be the universal G-bundle. Argue that there is an isomorphism

$$[X, EG]_* \cong \{\text{isomorphism classes of trivialized } G\text{-bundles}\}.$$

(c) Show that any two trivialized bundles are isomorphic. Conclude that EG is contractible, and that ΩBG is weakly equivalent to G.

WARNING: This problem turns out to be subtler than I initially imagined. You may for simplicity assume that G is discrete!