18.01 Practice Final Exam Solutions

1a) $\frac{d}{dx}\frac{\ln x}{x^2} = \frac{d}{dx}(x^{-2}\ln x) = \left(\frac{d}{dx}x^{-2}\right)\ln x + x^{-2}\left(\frac{d}{dx}\ln x\right)$

$= -2x^{-3}\ln x + x^{-2}x^{-1} = \boxed{\frac{1-2\ln x}{x^3}} \left(= \frac{\ln\left(\frac{e}{x^2}\right)}{x^3}\right)$

b) $\frac{d}{du}\sqrt{3\sin^2 u + 2} = \frac{d}{du}(3\sin^2 u + 2)^{\frac{1}{2}} = \frac{1}{2}(3\sin^2 u + 2)^{-\frac{1}{2}}$

$\cdot \frac{d}{du}(3\sin^2 u + 2) = \frac{1}{2}\frac{3\cdot 2\sin u\cos u}{\sqrt{3\sin^2 u + 2}} = \boxed{\frac{3\sin u\cos u}{\sqrt{3\sin^2 u + 2}}}$

c) $\frac{d}{dx}e^{kx} = ke^{kx}$ $\qquad \frac{d^2}{dx^2}e^{kx} = k^2e^{kx}$

$\frac{d^n}{dx^n}e^{kx} = (k)^n e^{kx}$ $\qquad \boxed{\left.\frac{d^n}{dx^n}e^{kx}\right|_{x=0} = (k^n)}$

2. $x^2y^2 + y^3 = 2$ at $p = (1, 1)$

Implicit differentiation $2xy^2\,dx + 2yx^2\,dy + 3y^2\,dy = 0$.

slope $\frac{dy}{dx} = -\frac{2xy^2}{2yx^2 + 3y^2}$. At $(1,1)$ slope $= -\frac{2}{5}$

Tangent line
$y = -\frac{2}{5}x + b$ To pass through $(1,1)$ $1 = -\frac{2}{5}\cdot 1 + b$
$b = \frac{7}{5}$

$\boxed{y = -\frac{2}{5}x + \frac{7}{5}}$

3. $y = \cos^{-1} x$ $\cos y = x$

Differentiate: $-\sin y \, dy = dx$

$$dy = \frac{dx}{-\sin y} = \frac{dx}{-\sqrt{1-\cos^2 y}} = \boxed{\frac{dx}{-\sqrt{1-x^2}}}$$

4. To be continuous at 0:

$f(0) = \lim_{x \to 0^+} f(x)$ $a = 2$

To be differentiable at 0:

$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x)$

For $x < 0$

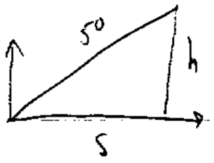
$f'(x) = (x^2 + x + a)' = 2x + 1$

For $x > 0$ $f'(x) = b$

$b = 2 \cdot 0 + 1 = 1$

$\boxed{a = 2 \quad b = 1}$

5.



If "falling" means (prefered interpretation) loosing height, then

$$\frac{dh}{dt} = -2 \text{ f/s} \qquad \text{and} \qquad \text{as} \quad h^2 + s^2 = 50^2$$

$$h = 30f \qquad s = 40f$$

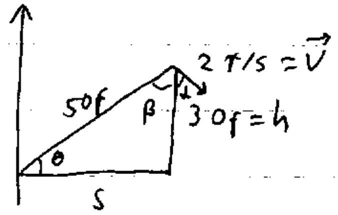
$$2h\frac{dh}{dt} + 2s\frac{ds}{dt} = 0$$

$$\frac{ds}{dt} = -\frac{h}{s}\frac{dh}{dt} = \frac{30f}{40f} \, 2 \text{ f/s} = \boxed{\frac{3}{2}} \text{ f/s}$$

If falling means "has speed of", then

see next page

$5'$ Soln 1:



$$\frac{dh}{dt} = (V \text{ projected to vertical})$$

$$= -V \cos\alpha$$

$\alpha = 90 - \beta \qquad \beta = 90 - \theta$, so

$\alpha = \theta$, $\quad \frac{dh}{dt} = -v\cos\theta$

$h^2 + s^2 = 50^2 \qquad$ Differentiate: $\quad 2h\,dh + 2s\,ds = 0$

$$\frac{ds}{dt} = -\frac{h\frac{dh}{dt}}{s} = \frac{+h\,v\cos\theta}{s}$$

$h = 30f$

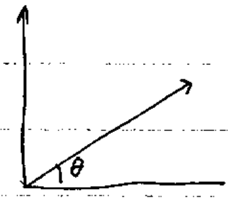
$s = 40f$

$v = 2$

$\cos\theta = \frac{40}{50}$

$$\frac{ds}{dt} = \frac{30\cdot 2\cdot \frac{40}{50}}{40} = \boxed{\frac{6}{5}}$$

Sol 2:



$s = r\cos\theta \qquad ds = (dr)\cos\theta + r\sin\theta\, d\theta =$

$= r\sin\theta\, d\theta$

Also $\qquad$ (as $r = \text{const} = 50$)

$\vec{V} = r'\hat{r} + r\theta'\hat{\theta} =$

$= r\theta'\hat{\theta}$

$|\vec{V}| = r\frac{d\theta}{dt}$

So $\quad \frac{ds}{dt} = r\sin\theta\frac{d\theta}{dt} = \sin\theta\,|\vec{V}| = \frac{3}{5}\cdot 2 = \boxed{\frac{6}{5}}$

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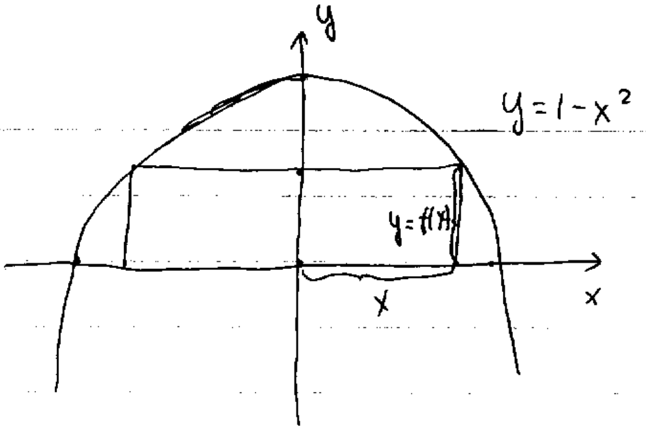


$$A = (\overset{\text{base}}{2x})\, \overset{\text{height}}{f(x)} =$$

$$= 2x(1-x^2)$$

Extrema: $\left(2x(1-x^2)\right)' = 0$

$$(x-x^3)' = 0$$

$$1-3x^2 = 0 \qquad x = \pm \frac{1}{\sqrt{3}}$$

This is a maximum (can check A'' or note that at bdry $x=0$ or $x=1$ $A=0$).

$$A = \frac{2}{\sqrt{3}} \; \frac{2}{3} = \boxed{\frac{4}{3\sqrt{3}}}$$

7. a) $p = (1,0)$. Line from (x_0, y_0) to $(1,0)$ has slope $k = \frac{y_0}{x_0-1}$. Line $\perp$ to it has slope

$-\frac{1}{k} = \frac{1-x_0}{y_0}$. This is tangent to $y = y(x)$, at (x_0, y_0) so

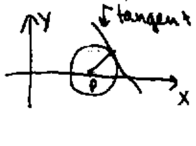
$$y'(x) = \frac{1-x}{y}$$

b) $\frac{dy}{dx} = \frac{1-x}{y}$ $\qquad y\,dy = (1-x)\,dx$. Integrate

$$\frac{y^2}{2} = \left(x - \frac{x^2}{2}\right) + C$$

c) $y^2 + x^2 - 2x = 2C$ $\qquad y^2 + (x-1)^2 = C_1$ -

circles centered at $p=(1,0)$ of varying radii



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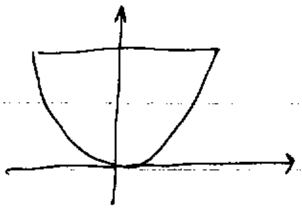


Upper rim $r=1$, so $x=1$, $y=1$

Need to compute volume

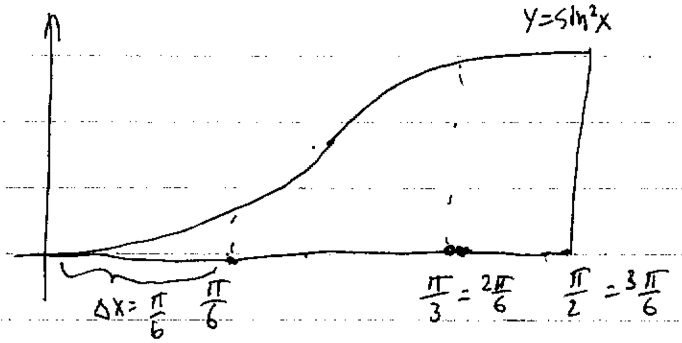
Method 1: Discs $\displaystyle\int_{y_{min}}^{y_{max}} \pi x^2 \, dy$ $\qquad x=\sqrt{y}$, $y_{min}=0$, $y_{max}=1$

$$\int_0^1 \pi y \, dy = \pi \left.\frac{y^2}{2}\right|_0^1 = \boxed{\frac{\pi}{2}}$$

Method 2: Shells $\displaystyle\int_{x_{min}}^{x_{max}} 2\pi x \; y \, dx$ $\qquad y=1-x^2$, $x_{min}=0$, $x_{max}=1$

$$\int_0^1 2\pi \, (x-x^3) \, dx = 2\pi \left(\frac{x^2}{2}-\frac{x^4}{4}\right)\Big|_0^1 = 2\pi \, \frac{1}{4} = \boxed{\frac{\pi}{2}}$$

9.



x	y
0	0
$\frac{\pi}{6}$	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$
$\frac{2\pi}{6} = \frac{\pi}{3}$	$\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$
$\frac{3\pi}{6} = \frac{\pi}{2}$	$1^2 = 1$

Trapezoidal $\displaystyle\sum_{n=1}^{3} \frac{f(\text{left}) + f(\text{right})}{2} \cdot \frac{\pi}{6} = \frac{\pi}{6}\left(\frac{f(0)}{2} + f\left(\frac{\pi}{6}\right) + f\left(\frac{2\pi}{6}\right) + \frac{f\left(\frac{3\pi}{6}\right)}{2}\right)$

$$= \frac{\pi}{6}\left(0 + \frac{1}{4} + \frac{3}{4} + \frac{1}{2}\right) = \frac{\pi}{6}\,\frac{3}{2} = \boxed{\frac{\pi}{4}}$$

Compare: $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$

Answer is exactly correct
because of symmetry

10 $F(x) = \int_0^x e^{-t^2}\,dt$

a) By Fundamental Theorem $\boxed{F'(1) = e^{-1}}$ $F'(x) = e^{-x^2}$ $F''(x) = \frac{d}{dx}(e^{-x^2})$

b) $A = \int_1^2 e^{-u^2/4}\,du$ $= -2xe^{-x^2}$, $\boxed{F''(1) = -2e^{-1}}$

Let $u = v/2$ $du = \frac{dv}{2}$. Substitute

$$A = \int_{v_{start}}^{v_{finish}} e^{-v^2}\,\frac{dv}{2} = \frac{1}{2}\int_{\frac{1}{2}}^{1} e^{-v^2}\,dv =$$

$$= \boxed{\frac{1}{2}\left(F(1) - F(\tfrac{1}{2})\right)}$$

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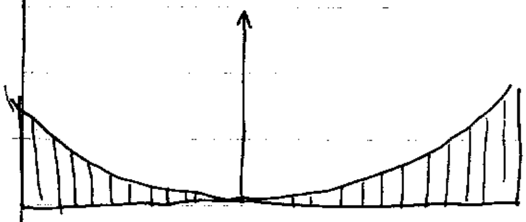


a) length of main cable = arc length

$$2\int_0^1 ds = 2\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx =$$

$$= \boxed{2\int_0^1 \sqrt{1+\left(\frac{1}{5}x\right)^2}\,dx}$$

b) Average of $\frac{1}{10}x^2 = \dfrac{\int_0^1 \frac{1}{10}x^2\,dx}{1} = \dfrac{\frac{x^3}{30}\Big|_0^1}{1} = \boxed{\frac{1}{30}}$

12 a) $x^2+3x+2 = (x+1)(x+2)$

Now by cover-up

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \qquad A = \frac{1}{(-1)+2} = 1$$

$$B = \frac{1}{(-2)+1} = -1$$

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

$$\int_0^1 \frac{1}{(x+1)(x+2)}\,dx = \int_0^1 \left(\frac{1}{(x+1)} - \frac{1}{(x+2)}\right)dx = \ln(x+1)\Big|_0^1 - \ln(x+2)\Big|_0^1$$

$$= -\ln 3 + \ln 2 + \ln 2 - \ln 1 = \boxed{\ln\frac{4}{3}}$$

b) $\int x^2 \ln x\,dx =$ | By parts $v = \ln x$ $u' = x^2$

$v' = \frac{1}{x}$ $u = \frac{x^3}{3}$

$$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{1}{x}\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C = \boxed{\frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + C}$$

13 $x = \tan u$ $dx = \sec^2 u\,du$ $u_{start} = \tan^{-1} 0 = 0$

$u_{finish} = \tan^{-1} 1 = \frac{\pi}{4}$

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_{u_{start}}^{u_{finish}} \frac{\sec^2 u\,du}{(1+\tan^2 u)^2} = \int_0^{\frac{\pi}{4}} \frac{\sec^2 u}{\sec^4 u}\,du$$

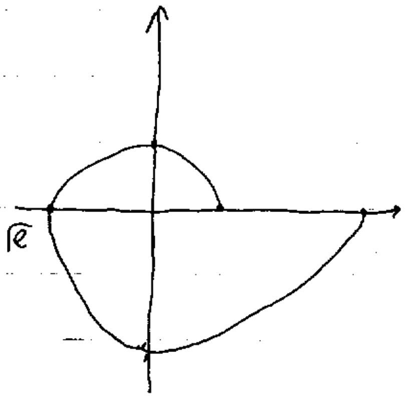
$$= \int_0^{\frac{\pi}{4}} \cos^2 u\,du = \int_0^{\frac{\pi}{4}} \frac{\cos(2u)+1}{2}\,du = \left(\frac{u}{2} + \frac{\sin(2u)}{4}\right)\Big|_0^{\frac{\pi}{4}} = \boxed{\frac{\pi}{8} + \frac{1}{4}}$$

14 $\quad r = e^{\theta/2\pi}$ $\qquad r=1 \Rightarrow e^{\theta/2\pi} = 1$

$\theta/2\pi = 0, \ \theta = 0$

As θ goes from 0 to 2π

r increases from 1 to $e^{\frac{2\pi}{2\pi}} = e$



$1.6 \approx \sqrt{e}$

$$A = \int_0^{2\pi} \frac{1}{2} (r(\theta))^2 \, d\theta =$$

$$\int_0^{2\pi} \frac{1}{2} e^{\theta/\pi} \, d\theta = \frac{\pi}{2} e^{\theta/\pi}\Big|_0^{2\pi} = \boxed{\frac{\pi}{2}(e^2-1)}$$

15 a) $$\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x} = \lim_{x\to 0} \frac{1-\cos^2 x}{1-\cos x} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{1-\cos x} = \lim_{x\to 0} (1+\cos x) = \boxed{2}$$

OR

$$\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x} \overset{\text{l'Hopital}}{=} \lim_{x\to 0} \frac{2\sin x \cos x}{\sin x} = \lim_{x\to 0} 2\cos x = \boxed{2}$$

OR $\quad$ Taylor series $\quad$ top $= \sin^2 x = \left(x - \frac{x^3}{3} + \dots\right)^2 = x^2 - \frac{2x^4}{3} + \dots$

bot $= 1-\cos x = 1 - \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} \dots\right) = \frac{x^2}{2} + \frac{x^4}{4!} + \dots$

$$\frac{\text{top}}{\text{bot}} = \frac{x^2 - \frac{2x^4}{3} \dots}{\frac{x^2}{2} - \frac{x^4}{4!} \dots} = \frac{1 - \frac{2x^2}{3} + \dots}{\frac{1}{2} - \frac{x^2}{4!} + \dots} \qquad \lim_{x\to 0} \frac{\text{top}}{\text{bot}} = \frac{1}{\frac{1}{2}} = \boxed{2}$$

b) $$\cancel{\lim_{x\to 1} \frac{(\ln x)^2}{x-1} = \lim_{h\to 0} \frac{(\ln(h+1))^2}{h}}$$

$h = x-1$

$x = h+1$

by definition
$$\cancel{= \left((\ln(h+1))^2\right)'\Big|_{h=0} = 2\ln(h+1)/h+1\Big|_{h=0} = 2\ln 1} = 0$$

~~l'Hopital~~
$=$ $\quad$ 2

b) $\lim\limits_{x \to 1} \dfrac{(\ln(x))^2}{x-1} \overset{\text{by definition}}{=} \left((\ln(x))^2\right)'\Big|_{x=1} =$

$= \dfrac{2\ln x}{x}\Big|_{x=1} = \boxed{0}$

OR $\lim\limits_{x \to 1} \dfrac{(\ln(x))^2}{x-1} = \lim\limits_{\substack{h \to 0 \\ h = x-1 \\ x = h+1}} \dfrac{(\ln(h+1))^2}{h} \overset{\text{L'Hopital}}{=} \lim\limits_{h \to 0} 2\ln(h+1)\dfrac{1}{h+1} =$

$\dfrac{2\ln 1}{1} = \boxed{0}$

c) $\lim\limits_{x \to \infty} x^2 e^{-x} = \lim\limits_{x \to \infty} \dfrac{x^2}{e^x} \overset{\text{L'Hopital}}{=} \lim\limits_{x \to \infty} \dfrac{2x}{e^x} \overset{\text{L'Hopital}}{=} \lim\limits_{x \to \infty} \dfrac{2}{e^x} = \boxed{0}$

16 $\displaystyle\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^{3/2}} = \lim_{b \to \infty} \left(\frac{x^{-1/2}}{(-1/2)}\right)\Bigg|_1^b =$

$= \lim\limits_{b \to \infty} \left(+2 - 2 b^{-1/2}\right) = \boxed{2}$

17. Rough estimate $\sqrt{4+n^p} \sim \sqrt{n^p} = n^{p/2}$

$\dfrac{n}{\sqrt{4+n^p}} \sim n^{1-p/2}$. So, let's do limit comparison

$b_n = n^{1-p/2}$ $\qquad \lim\limits_{n \to \infty} \dfrac{b_n}{a_n} = \dfrac{n}{\sqrt{n^p}} \Big/ \dfrac{n}{\sqrt{n^p+4}} = \dfrac{\sqrt{1+\frac{4}{n^p}}}{\sqrt{1}} =$

$= \sqrt{1+4n^{-p}} = 1$ for $p > 0$

Hence the series converges if and only if $1 - \frac{p}{2} < -1$, $\boxed{p > 4}$

Note: for $p < 0$

$\dfrac{n}{\sqrt{4+n^p}} > \dfrac{n}{\sqrt{5}}$, does not go to 0, so the series can not converge

18 a) $f(x) = (1+x)^{\frac{1}{2}}$ $f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ $f'''(x) = \frac{3}{8}(1+x)^{-5/2}$

b) $(1+x)^{\frac{1}{2}} = f(x) \approx 1 + \frac{1}{2}x + \frac{1}{2} \cdot -\frac{1}{4}x^2 + \frac{1}{6}\frac{3}{8}x^3$

$(1.2)^{\frac{1}{2}} = 1 + \frac{1}{2}(0.2) - \frac{1}{2}\frac{1}{4}(0.2)^2 + \frac{1}{6}\frac{3}{8}(0.2)^3 =$

$= 1 + 0.1 - 0.005 + 0.0005 = \boxed{1.0955}$

19 $(\tan^{-1}(x))' = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$

$\tan^{-1}(x) = \int (\tan^{-1}(x))' \, dx = \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

(no constant, as $\tan^{-1}(0) = 0$)