MATH 18.01 Problem Set 9 Solutions

Problem 1. (*6 pts: 2 each*) Use integration by parts to evaluate the integrals:

a) $\int x \arctan(x-1)\, dx$

Solution. The anti-derivative of x is much simpler than that of $\arctan x$, so it makes sense to set $du = x\,dx$ and $v = \arctan(x-1)$. Then $u = x^2/2$ and $dv = \dfrac{1}{(x-1)^2+1}\,dx$. Using integration by parts we find

$$\begin{aligned}\int x\arctan(x-1)\,dx &= \int du\,v = uv - \int u\,dv \\ &= \frac{x^2}{2}\arctan(x-1) - \int \frac{x^2}{2}\cdot\frac{1}{(x-1)^2+1}\,dx.\end{aligned}$$

The rational function in the final integral is not reduced (the degree of the numerator is not less than the degree of the numerator), so we should take the quotient to obtain

$$\frac{x^2}{x^2-2x+2} = 1 + \frac{2x-2}{x^2-2x+2}.$$

Finally, the substitution $w = x^2-2x+2$ has the differential $dw = (2x-2)\,dx$, so

$$\begin{aligned}\int x\arctan(x-1)\,dx &= \frac{x^2\arctan(x-1)}{2} - \int\frac{x^2}{2}\cdot\frac{1}{(x-1)^2+1}\,dx \\ &= \frac{x^2\arctan(x-1)}{2} - \frac{1}{2}\int 1 + \frac{2x-2}{x^2-2x+2}\,dx = \frac{x^2\arctan(x-1)}{2} - \frac{x}{2} + \int\frac{1}{w}\,dw \\ &= \frac{x^2\arctan(x-1)}{2} - \frac{x}{2} + \ln w = \boxed{\boldsymbol{\frac{x^2\arctan(x-1)}{2} - \frac{x}{2} + \ln(x^2-2x+2)}}\,.\end{aligned}$$

b) $\int \sin^4 x\,dx$

Solution. It is easiest to write down a general reduction formula for powers of sine and then to apply it as necessary. Consider the integral $\int \sin^a x\,dx$, and let

$$du = \sin x\,dx, \qquad v = \sin^{a-1} x,$$

so

$$u = -\cos x, \qquad dv = (a-1)\sin^{a-2} x\cos x\,dx.$$

Then integration by parts implies that

$$\int \sin^a x\,dx = \int du\,v = uv - \int u\,dv = -\sin^{a-1} x\cos x + \int (a-1)\sin^{a-2} x\cos^2 x\,dx.$$

The identity $\sin^2 x + \cos^2 x = 1$ then gives

$$\int \sin^a x\,dx = -\sin^{a-1} x\cos x + \int (a-1)\sin^{a-2} x(1-\sin^2 x)\,dx,$$

and moving the final term from the right side to the left we obtain

$$a\int \sin^a x\,dx = -\sin^{a-1} x\cos x + (a-1)\int \sin^{a-2} x\,dx,$$

which finally implies that

$$\int \sin^a x\,dx = -\frac{\sin^{a-1} x\cos x}{a} + \frac{a-1}{a}\int \sin^{a-2} x\,dx.$$

The specific problem now reduces as (applying the formula twice, first with $a = 4$ and then with $a = 2$)

$$\begin{aligned}\int \sin^4 x\,dx &= -\frac{\sin^3 x\cos x}{4} + \frac{3}{4}\int \sin^2 x\,dx \\ &= -\frac{\sin^3 x\cos x}{4} + \frac{3}{4}\left(-\frac{\sin x\cos x}{2} + \frac{1}{2}\int 1\,dx\right) \\ &= \boxed{\boldsymbol{-\frac{\sin^3 x\cos x}{4} - \frac{3\sin x\cos x}{8} + \frac{3x}{8}}}.\end{aligned}$$

c) $\displaystyle\int\limits_0^1 x\ln(x)\,dx.$

Solution. The anti-derivative of $\ln x$ is complicated, so it is best to let $du = x\,dx$ and $v = \ln(x)$; then $u = x^2/2$ and $dv = \dfrac{1}{x}\,dx$. Integration by parts gives

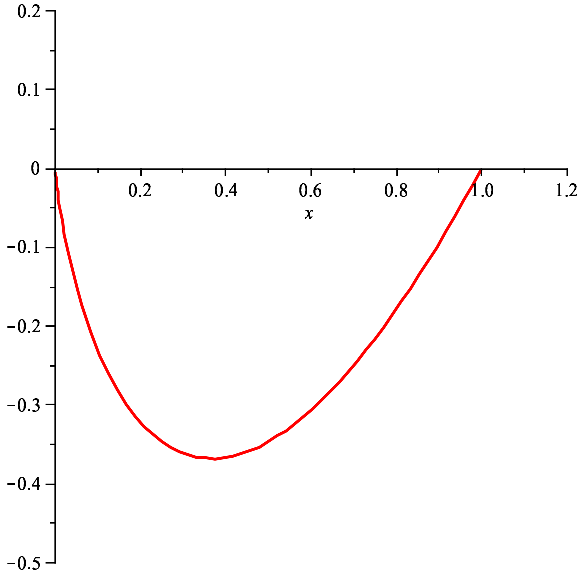
$$\begin{aligned}\int_0^1 x\ln(x)\,dx &= \int_0^1 du\,v = uv\,\Big|_0^1 - \int_0^1 u\,dv \\ &= \frac{x^2}{2}\ln x\,\Big|_0^1 - \int_0^1 \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \frac{x^2}{2}\ln x\,\Big|_0^1 - \int_0^1 \frac{x}{2}\,dx \\ &= \frac{x^2}{2}\ln x\,\Big|_0^1 - \frac{x^2}{4}\,\Big|_0^1.\end{aligned}$$

Finally, note that the first evaluation cannot be evaluated directly at the end point $x = 0$, as $x^2\ln x$ has the indeterminate form $0\cdot(-\infty)$. However, L'Hospital's rule implies that

$$\lim_{x\to 0^+}\frac{\ln x}{1/x^2} = \lim_{x\to 0^+}\frac{1/x}{-2/x^3} = \lim_{x\to 0^+} -x^2/2 = 0,$$

so this is not an improper integral (and we do not need to take the limit of an endpoint). This means that the integral evaluates to (grouping terms for clarity)

$$\int_0^1 x\ln(x)\,dx = \left(\frac{x^2}{2}\ln x\,\Big|_0^1\right) - \left(\frac{x^2}{4}\,\Big|_0^1\right) = \left(\frac{1}{2}\cdot 0 - 0\right) - \left(\frac{1}{4} - \frac{0}{4}\right) = \boxed{\boldsymbol{-\frac{1}{4}}}.$$



Problem 2. (*6 pts: 3+3*)

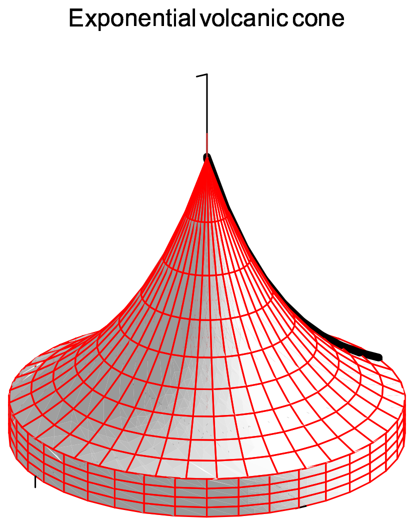
a) Find a reduction formula for $\int x^n e^{Cx}\,dx$.

Solution. Differentiation reduces the degree of a polynomial, but leaves an exponential unchanged. Therefore, in order to simplify the integral we must set $u = x^n, dv = e^{Cx}\,dx$, so $du = nx^{n-1}\,dx$ and $v = e^{Cx}/C$. Thus the reduction formula is

$$\begin{aligned}\int x^n e^{Cx}\,dx &= \int u\,dv = uv - \int v\,du \\ &= \frac{x^n e^{Cx}}{C} - \int \frac{e^{Cx}}{C} \cdot nx^{n-1}\,dx = \boxed{\boldsymbol{\frac{x^n e^{Cx}}{C} - \frac{n}{C}\int x^{n-1}e^{Cx}\,dx}}\,.\end{aligned}$$

b) Simmons Problem 10.7.26.

Solution. The described volcanic cone is simply the volume of revolution about the y-axis of the curve $y = ae^{-bx}$ from $x = 0$ to c.



Exponential volcanic cone

This volume is easiest to calculate using cylindrical shells, as we already have a formula for y as a function of x. Specifically, the shell at an arbitrary distance x has height ae^{-bx}, so the infinitesimal shell volume is

$$dV = 2\pi xy\,dx = 2\pi x(ae^{-bx})\,dx.$$

Thus the total volume is

$$V = \int dV = \int\limits_{x=0}^{c} 2\pi axe^{-bx}\,dx.$$

The reduction formula from a) then implies

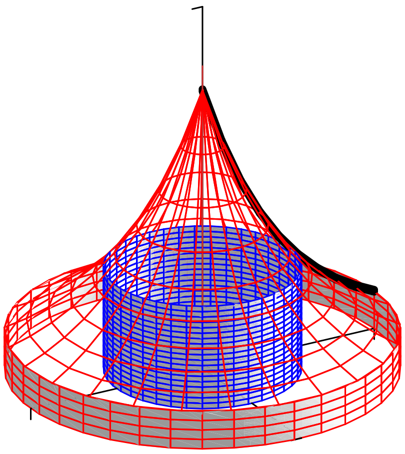
$$\begin{aligned} V &= -\frac{2\pi axe^{-bx}}{b}\bigg|_0^c + \frac{2\pi a}{b}\int\limits_0^c e^{-bx}\,dx = -\frac{2\pi ace^{-bc}}{b} + \frac{2\pi a}{b}\left(-\frac{e^{-bx}}{b}\right)\bigg|_0^c \\ &= -\frac{2\pi ace^{-bc}}{b} + \frac{2\pi a}{b}\left(-\frac{e^{-bc}}{b} + \frac{1}{b}\right) = \boxed{\boldsymbol{\frac{2\pi a}{b}\left(-ce^{-bc} - \frac{e^{-bc}}{c} + \frac{1}{b}\right)}}. \end{aligned}$$

Finally, we've seen before that for any positive constant b,

$$\lim_{y\to\infty} x^m e^{-bx} = 0.$$

This means that the limiting behavior as $c \to \infty$ gives $\boxed{\boldsymbol{V \to \frac{1}{b}}}$.

Remark. The calculation of the limit of the integral corresponds to the improper integral from $x = 0$ *to* ∞*. Physically, this represents the total volume of ash that erupted from the volcano.*



Exponential volcanic cone with cylindrical shell

Problem 3. (*3 pts*)

Describe the values of a such that $\int\limits_{x=0}^{\infty} e^{ax}\,dx$ converges. Calculate the limit in the convergent cases.

Solution. Following the definition of the improper integral and using the exponential integral rule,

$$\int\limits_{x=0}^{\infty} e^{ax}\,dx = \lim_{N\to\infty} \int\limits_{0}^{N} e^{ax}\,dx = \lim_{N\to\infty} \left.\frac{e^{ax}}{a}\right|_{0}^{N} = \lim_{N\to\infty} \frac{e^{aN}-1}{a}.$$

Note that in general, the powers of a fixed positive constant behave as

$$\lim_{N\to\infty} c^N = \begin{cases} \infty \text{ (divergent)} & \text{if } c < 0 \\ 1 & \text{if} c = 1 \\ 0 & \text{if} 0 \leq c < 1. \end{cases}$$

Thus the powers of e^a can be similarly classified, giving the following limits.

- If $a > 0$, then $e^a > 1$, so the integral $\boxed{\textbf{diverges}}$.
- If $a = 0$, then the integral $\boxed{\textbf{converges}}$ to $\dfrac{1-1}{a} = \boxed{\mathbf{0}}$.
- If $a < 0$, then the integral $\boxed{\textbf{converges}}$ to $\dfrac{0-1}{a} = \boxed{-\dfrac{\mathbf{1}}{\boldsymbol{a}}}$.