MATH 18.01 Problem Set 8 Solutions

Problem 1. (*8 pts: 1+2+2+1+1+1*) Consider the function $f(x) = \dfrac{1}{4-x^2}$. In this problem you will calculate anti-derivatives and integrals of $f(x)$ in a number of ways.

a) Find the partial fraction decomposition of $f(x)$.

Solution. Factor the denominator to get $f(x)\dfrac{1}{(2+x)(2-x)}$. Now write

$$\frac{1}{(2+x)(2-x)} = \frac{A}{2+x} + \frac{B}{2-x}.$$

Multiply by the left-side denominator $(2+x)(2-x)$ and obtain

$$1 = A(2-x) + B(2+x).$$

Plugging in $x = 2$ implies that $B = 1/4$, and plugging in $x = -2$ implies that $A = 1/4$ as well. Thus

$$f(x) = \boxed{\frac{\mathbf{1/4}}{\mathbf{2}+\boldsymbol{x}} + \frac{\mathbf{1/4}}{\mathbf{2}-\boldsymbol{x}}}.$$

b) Use part a) to find an anti-derivative of $f(x)$.

Solution. The anti-derivative is

$$\int f(x)\,dx = \frac{1}{4}\left(\ln(2+x) - \ln(2-x)\right) = \boxed{\frac{\mathbf{1}}{\mathbf{4}}\,\mathbf{ln}\left(\frac{\mathbf{2}+\boldsymbol{x}}{\mathbf{2}-\boldsymbol{x}}\right)}.$$

c) Use a trigonometric substitution to find an anti-derivative of $f(x)$.

Solution. The natural substitution is $x = 2\sin(u)$, with differential $dx = 2\cos(u)\,du$. Then the anti-derivative is

$$\begin{aligned}\int \frac{1}{4-x^2}\,dx &= \int \frac{1}{4-4\sin^2(u)}\,2\cos(u)\,du = \int \frac{2\cos u}{4\cos^2(u)}\,du = \int \frac{1}{2\cos u}\,du \\ &= \frac{1}{2}\int \sec u\,du = \frac{1}{2}\ln(\sec u + \tan u).\end{aligned}$$

Now reverse the substitution to obtain an anti-derivative in terms of x. The inverse is $u = \arcsin(x/2)$, which corresponds to a triangle with opposite leg $x/2$, adjacent leg $\sqrt{1-(x/2)^2}$ and hypotenuse 1. Thus $\sec(\arcsin(x/2)) = 1/\sqrt{1-x^2/4}$ and $\sec(\arcsin(x/2)) = x/2\sqrt{1-x^2/4}$. The final solution is

$$\frac{1}{2}\ln(\sec u + \tan u) = \frac{1}{2}\ln\left(\frac{1+x/2}{\sqrt{1-x^2/4}}\right) = \boxed{\frac{\mathbf{1}}{\mathbf{2}}\,\mathbf{ln}\left(\frac{\sqrt{\mathbf{1}+\boldsymbol{x}/\mathbf{2}}}{\sqrt{\mathbf{1}-\boldsymbol{x}/\mathbf{2}}}\right)}.$$

d) Explain the relationship between your answers in parts a) and c).

Solution. In general, two different anti-derivatives of the same function might differ by a constant. However, in this case, parts a) and c) are identical (after a bit of algebra)!

e) Calculate the integral $\int\limits_{x=0}^{1} f(x)\,dx$.

Solution. Using the First Fundamental Theorem of Calculus,

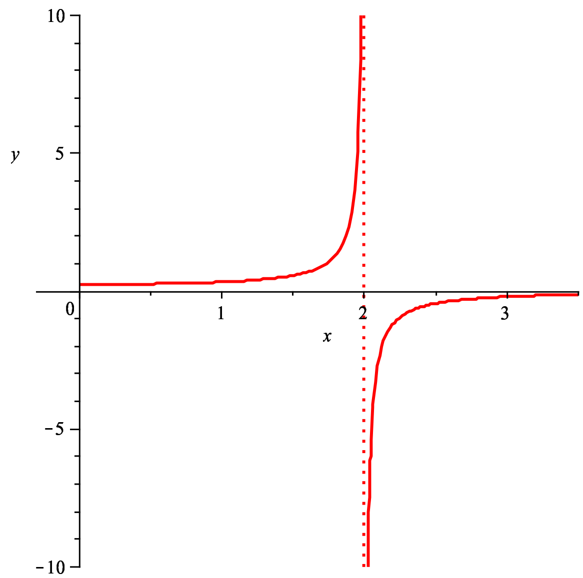
$$\int\limits_{x=0}^{1} f(x)\,dx = \frac{1}{4}\ln\left(\frac{2+x}{2-x}\right)\Bigg|_0^1 = \frac{1}{4}\left(\ln\left(frac31\right) - \ln(1)\right) = \boxed{\mathbf{\frac{\ln(3)}{4}}}.$$

f) Try to evaluate the integral $\int\limits_{x=1}^{3} f(x)\,dx$. Does your answer make any sense geometrically? Sketch the graph of $f(x)$ and explain your reasoning in terms of areas under curves.

Solution. Naively, we would follow the same procedure as in part e), and use the First Fundamental Theorem of Calculus to calculate

$$\int\limits_{x=1}^{3} f(x)\,dx = \frac{1}{4}\ln\left(\frac{2+x}{2-x}\right)\Bigg|_1^3 = \frac{1}{4}\left(\ln\left(frac5{-1}\right) - \ln(3)\right) = ????$$

Thinking geometrically reveals the trouble:



The function is undefined at $x = 2$ and the area under the curve includes infinite regions, so the integral is also $\boxed{\textbf{undefined}}$.

Note. The integral of a function that goes to infinity is not necessarily undefined – later in the semester we will see examples of functions that go to infinity and yet bound a finite area!

Problem 2. (*3 pts*) Evaluate the integral

$$\int_{x=0}^{2} \frac{1}{x^4 + 8x^2 + 16}\,dx.$$

Hint: Factor the denominator and use a trigonometric substitution.

Solution. The integrand is a perfect square $\frac{1}{(x^2+4)^2}$. The form of the function suggests the substitution $x = 2\tan u$, so $dx = 2\sec^2 u\,du$. Furthermore, the point $x = 0$ corresponds to $u = 0$, and $x = 2$ corresponds to $u = \pi/4$. Therefore the integral is

$$\int_{x=0}^{2} \frac{1}{x^4+8x^2+16}\,dx = \int_{u=0}^{\pi/4} \frac{1}{(4\sec^2 u)^2}\,2\sec^2 u\,du = \int_{u=0}^{\pi/4} \frac{1}{(8\sec^2 u}\,du = \frac{1}{8}\int_{u=0}^{\pi/4} \cos^2 u\,du.$$

This final expression is an even power of cosine, and using the double-angle formula gives

$$\frac{1}{8}\int_{u=0}^{\pi/4} \frac{1+\cos(2u)}{2}\,du = \frac{1}{16}\left(u + \frac{\sin(2u)}{2}\right)\Bigg|_0^{\pi/4} = \frac{1}{16}\left(\frac{\pi}{4}+\frac{1}{2}-0-0\right) = \boxed{\boldsymbol{\frac{\pi+2}{64}}}.$$

Problem 3. (*4 pts: 1+3*) *Constrained growth* models occur very frequently in the study of populations that live in environments with a fixed, limited amount of natural resources. The population initially grows slowly when it is small, then begins to flourish and grow more rapidly, but eventually reaches a point of diminishing returns where it is difficult to achieve significant additional growth. The corresponding differential equation has the form

$$\frac{dP}{dt} = P\cdot(N-P),$$

where $P(t)$ is the population at time t, and N represents the maximum population that can be supported by the environment.

a) At what population is the growth rate maximized?

Solution. The growth rate depends on the population, and is given by the expression $G(P) = P(N-P) = NP - P^2$. This is a negative parabola, and achieves its maximum at the critical point when $G'(P) = N - 2P = 0$, which means $\boxed{\boldsymbol{P = \frac{N}{2}}}$.

Note. The fact that the population actually reaches $N/2$ is a bit more subtle, and requires the solution from part b) (and a positive initial condition n).

b) Solve the differential equation subject to the initial condition $P(0) = n$.

Solution. The differential equation can be separated, yielding

$$\frac{dP}{P(N-P)} = dt.$$

The partial fraction decomposition of the left-hand side is $\frac{1}{P(N-P)} = \frac{1/N}{P} + \frac{1/N}{N-P}$, so we need to find the anti-derivatives

$$\frac{1}{N}\int\left(\frac{1}{P}+\frac{1}{N-P}\right)dP = \int dt$$

These are

$$\frac{1}{N}\,(\ln P - \ln(N-P)) = t + C$$

for some constant C. This is equivalent to

$$\frac{P}{N-P} = C' e^{Nt}$$

for some other constant C', which must be $C' = \frac{n}{N-n}$ by the initial condition. Finally, solving for P gives the solution

$$P(t) = \frac{C' N e^{Nt}}{1 + C' e^{Nt}} = \boxed{\frac{\boldsymbol{N \cdot e^{Nt}}}{\boldsymbol{\left(\frac{N-n}{n}\right) + e^{Nt}}}}\,.$$

As $t \to \infty$, the limit of the fraction is 1, so $P(t)$ approaches N as claimed. An example is graphed is below.