MATH 18.01 Problem Set 6 Solutions

Problem 1. (*3 pts: 2+1*) a) Use separation of variables in order to find the function $y = f(x)$ that satisfies the differential equation

$$\frac{dy}{dx} = (x-2)(y-1),$$

with initial condition $f(0) = 0$. Verify your answer by computing $f'(x)$.

Solution. Separating the variables yields the equation

$$\frac{dy}{y-1} = (x-2)\,dx.$$

Now find an anti-derivative for each side, being sure to add in a generic constant C:

$$\ln(y-1) = \frac{x^2}{2} - 2x + C.$$

Exponentiate to get $y - 1 = e^{x^2/2-2x+C} = C'\,e^{x^2/2-2x}$, where $C' = e^C$ is also an unknown constant. Finally, the initial condition at $x = 0$ implies that $0 - 1 = C' \cdot e^0 = C'$, so the solution is

$$\boxed{\boldsymbol{y = f(x) = -e^{x^2/2-2x} + 1}}\,.$$

Check this by calculating $f'(x) = -e^{x^2/2-2x} \cdot (x-2) = (y-1)(x-2)$ as required.

b) What is the solution to the differential equation in a) for the arbitrary initial condition $f(0) = c$?

Solution. The general form of the solution is the same, satisfying $y - 1 = C'\,e^{x^2/2-2x}$ for some C'. The initial condition then implies $c - 1 = C'$, so the solution is

$$\boxed{\boldsymbol{y = f(x) = (c-1)\,e^{x^2/2-2x} + 1}}\,.$$

Problem 2. (*7 pts: 1+1+2+2+1*) In this problem you will use calculus to find volume formulas for a variety of related shapes. Given any plane figure (such as a circle, triangle, square, etc.) that has area A, a *pyramid* of height h is constructed by drawing a single point above the center of the base, and then connecting the base region to the point by straight lines (in the examples listed earlier, this gives a cone, tetrahedron/triangular-pyramid, and square-pyramid, respectively).

a) First, consider a *prism* of height h, which is the region that is swept out as the base moves a distance of h (a loaf of bread is, roughly speaking, a "prism" whose base is a very thin slice of bread!). Explain why the volume of such a prism is approximated by the Riemann sum

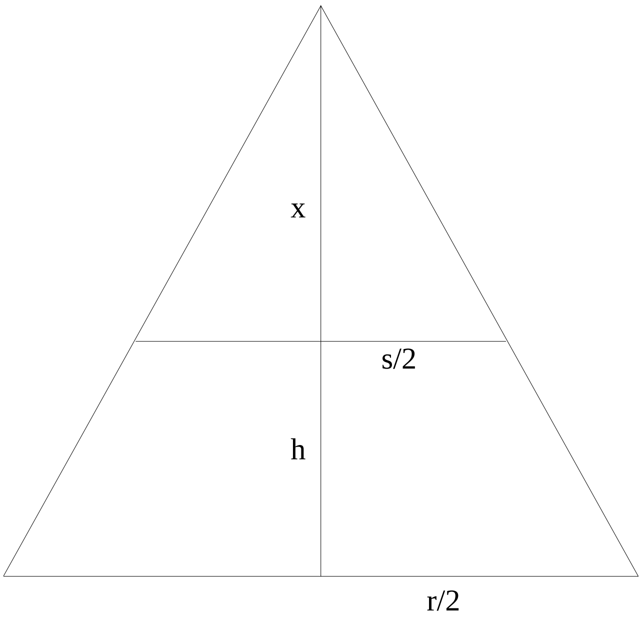
$$\sum_{k=1}^{n} A\,\Delta x,$$

and thus the volume is given by the integral evaluation $\int_0^h A\,dx = Ah$.

Solution. The sum represents cutting the region into very thin volume elements, so that the cross-section slices (all of which are the same shape, with area A) are given width Δx. Note that the equally spaced widths are $\Delta x = h/n$. This is exactly the Riemann sums that show up in the limiting process of the stated integral.

b) Suppose that a cross-section of a square-pyramid is sliced at a distance x from the peak. Show using similar triangles that the area of the cross-section is $A \cdot \left(\frac{x}{h}\right)^2$.

Solution. Suppose that the base square has side-length r, so $A = r^2$. Then the central point, which lies directly under the peak, is a distance $r/2$ from the edge. Now the cross-section at distance x from the peak is also a square, with some side-length s. The similar triangles in the following figure imply that $s = \frac{x}{h} \cdot r$.



Thus the area of the square is $s^2 = \left(\frac{x}{h}\right)^2 r^2 = \boxed{\left(\boldsymbol{\frac{x}{h}}\right)^{\mathbf{2}} \cdot \boldsymbol{A}}$.

c) Explain why the total volume of the square-pyramid can then be approximated by

$$\sum_{k=1}^{n} A \left(\frac{k \cdot h/n}{h}\right)^2 \frac{h}{n}.$$

As $n \to \infty$, this sum corresponds to an integral that gives the volume exactly. Evaluate this integral.

Solution. If a cross-section at height x is stretched into a small volume of height Δx, then the volume is $\left(\frac{x}{h}\right)^2 \cdot A\,\Delta x$. If the height h is split into n equal segments of width $\Delta x = h/n$, then the given sum adds up such rectangles at points $x_k = k \cdot \Delta x$. The integral (and hence volume) is

$$\lim_{n\to\infty} \sum_{k=1}^{n} A \cdot \left(\frac{x_k}{h}\right)^2 \Delta x = \int_0^h A\left(\frac{x}{h}\right)^2 dx = \frac{A}{h^2}\frac{x^3}{3}\Big|_0^h = \frac{A}{h^2}\cdot\frac{h^3}{3} = \boxed{\boldsymbol{\frac{Ah}{3}}}.$$

d) Go through a similar process for a triangular-pyramid and cone (you may skip steps if you understand clearly how they will work).

Solution. There are again similar triangles that imply the following important property: if the base area is A, then the area of the cross-section at distance x from the top has area $\left(\frac{x}{h}\right)^2 \cdot A$. Thus the integration process is identical to part c), and the volume formula is the same: $\boxed{\boldsymbol{\frac{Ah}{3}}}$.

e) Compare your answers to the known formulas for the volume of a square-pyramid, tetrahedron, and cone. In general, what do you think that the volume of a pyramid of height h over an arbitrary base of area A will be? How does this compare with part a)?

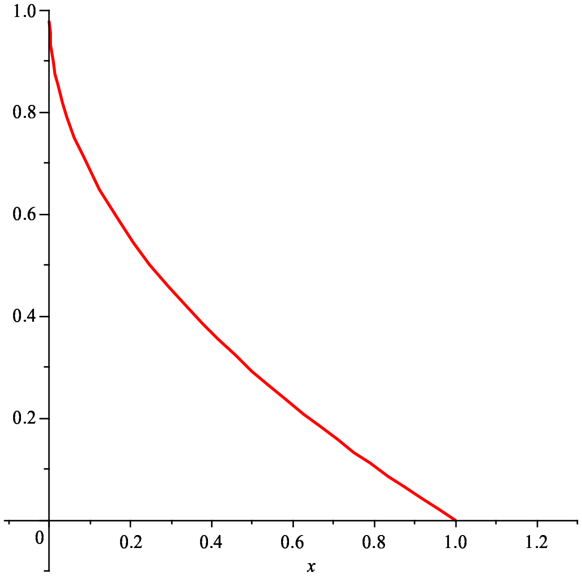
Solution. A square-pyramid with base side-length r and height h has volume $\frac{r^2h}{3}$. A cone of base radius r and height h has volume $\frac{\pi r^2 h}{3}$. Both of these clearly correspond to $\frac{Ah}{3}$. The formula for triangular-pyramids is less well-known, and has several variants. For example, in a true tetrahedron all sides have the same length r, and the volume formula is $\frac{\sqrt{2}r^3}{12}$.. In this case the base area is $\frac{\sqrt{3}r^2}{4}$ and the height is $\sqrt{\frac{2}{3}}$; the volume formula again corresponds with $\frac{Ah}{3}$.

In fact, the formula $\boxed{\boldsymbol{V = \frac{Ah}{3}}}$ holds in general (although the similarity argument from c) and d) is more subtle for more complicated shapes). Compared to part a), this means that a pyramid has one-third the area of the prism with the same base and height!

Problem 3. (*5 pts: 1+1+2+1*) Consider the region in the first quadrant bounded by $x = 0, y = 0$ and $y = 1 - \sqrt{x}$. In this problem you will calculate the volume of revolution about the y-axis.

a) Set up an integral to calculate the volume of revolution about the y-axis using the disc method.

Solution. First, sketch the function so that the region can be accurately described:



Now consider discs centered on the y-axis. At a height of y, the disc has radius x, so the volume of revolution is $V_{\text{rev}} = \int_{y=0}^{1} \pi x^2 \, dy$. However, the integrand must be written solely in terms of y, and solving for x gives $x = (y-1)^2$. Thus the volume is

$$V_{\text{rev}} = \int\limits_{y=0}^{1} \pi x^2 \, dy = \boxed{\boldsymbol{\int\limits_{y=0}^{1} \pi (y-1)^4 \, dy}} .$$

b) Set up an integral to calculate the volume of revolution about the y-axis using the cylindrical shells method.

Solution. A thin cylinder at distance x has height $y = 1 - \sqrt{x}$. Thus the volume is

$$V_{\text{rev}} = \int\limits_{x=0}^{1} 2\pi x y \, dx = \boxed{\boldsymbol{\int\limits_{x=0}^{1} 2\pi x (1 - \sqrt{x}) \, dx}} .$$

c) Calculate the volume by evaluating whichever of the two integrals seems easier. As a reality check, you should compare your answer with 2d).

Solution. From part a),

$$\int\limits_{y=0}^{1} \pi (y-1)^4 \, dy = \pi \cdot \frac{(y-1)^5}{5} \Bigg|_0^1 = \boxed{\boldsymbol{\frac{\pi}{5}}} .$$

From part b),

$$\int\limits_{x=0}^{1} 2\pi x (1 - \sqrt{x}) \, dx = 2\pi \left(\frac{x^2}{2} - \frac{x^{5/2}}{5/2} \right) \Bigg|_0^1 = 2\pi \left(\frac{1}{2} - \frac{2}{5} \right) = \boxed{\boldsymbol{\frac{\pi}{5}}} .$$

In comparison with 2d), the volume of a cone of radius 1 and height 1 is $\frac{\pi}{3}$. Since the current volume of revolution lies inside the cone, it makes sense that the volume is less, and it's true that $\frac{\pi}{5} < \frac{\pi}{3}$.

d) Now consider the region bounded by $x = 1, y = 1$ and the same curve $y = 1 - \sqrt{x}$. Calculate the volume of revolution about the y-axis using whatever method you like (you do not necessarily need to even evaluate an integral!).

Solution. Note that this region combined with that from the earlier parts of the problem form a cylinder of height 1 and radius 1, with total volume π. Thus the complementary region here has volume $\pi - \pi/5 = \boxed{\boldsymbol{\frac{4\pi}{5}}}$.