MATH 18.01 Problem Set 6 - Spring 2009

Due Thursday, Mar. 19 at 1:00

Part I (10 points)

Lecture 16. (*Thurs., Mar. 12*) Differential equations, separation of variables, exponential growth and decay.
Read: Simmons 5.4, 8.5 *Work:* 3F-1ac, 2abe, 4, 7
Lecture 17. (*Fri., Mar. 13*) Area between two curves, volumes, disk method.
Read: Simmons 7.1, 7.2, 7.3 *Work:* 4A-1bcd, 4, 4B-1bce, 2bce
Lecture 18. (*Tues., Mar. 17*) Cylindrical shells.
Read: Simmons 7.4 *Work:* 4C-2, 4bce, 5bce

Part II (15 points)

Problem 1. (*3 pts: 2+1*) a) Use separation of variables in order to find the function $y = f(x)$ that satisfies the differential equation

$$\frac{dy}{dx} = (x-2)(y-1),$$

with initial condition $f(0) = 0$. Verify your answer by computing $f'(x)$.

b) What is the solution to the differential equation in a) for the arbitrary initial condition $f(0) = c$?

Problem 2. (*7 pts: 1+1+2+2+1*) In this problem you will use calculus to find volume formulas for a variety of related shapes. Given any plane figure (such as a circle, triangle, square, etc.) that has area A, a *pyramid* of height h is constructed by drawing a single point above the center of the base, and then connecting the base region to the point by straight lines (in the examples listed earlier, this gives a cone, tetrahedron, and square-pyramid, respectively).

a) First, consider a *prism* of height h, which is the region that is swept out as the base moves a distance of h (a loaf of bread is, roughly speaking, a "prism" whose base is a very thin slice of bread!). Explain why the volume of such a prism is approximated by the Riemann sum

$$\sum_{k=1}^{n} A\,\Delta x,$$

and thus the volume is given by the integral evaluation $\int_0^h A\,dx = Ah$.

b) Suppose that a cross-section of a square-pyramid is sliced at a distance x from the peak. Show using similar triangles that the area of the cross-section is $A \cdot \left(\frac{x}{h}\right)^2$.

c) Explain why the total volume of the square-pyramid can then be approximated by

$$\sum_{k=1}^{n} A \left(\frac{k \cdot h/n}{h}\right)^2 \frac{h}{n}.$$

As $n \to \infty$, this sum corresponds to an integral that gives the volume exactly. Evaluate this integral.

d) Go through a similar process for a tetrahedron and cone (you may skip steps if you understand clearly how they will work).

e) Compare your answers to the known formulas for the volume of a square-pyramid, tetrahedron, and cone. In general, what do you think that the volume of a pyramid of height h over an arbitrary base of area A will be? How does this compare with part a)?

Problem 3. (*5 pts: 1+1+2+1*) Consider the region in the first quadrant bounded by $x = 0, y = 0$ and $y = 1 - \sqrt{x}$. In this problem you will calculate the volume of revolution about the y-axis.

a) Set up an integral to calculate the volume of revolution about the y-axis using the disc method.

b) Set up an integral to calculate the volume of revolution about the y-axis using the cylindrical shells method.

c) Calculate the volume by evaluating whichever of the two integrals seems easier. As a reality check, you should compare your answer with 2d).

d) Now consider the region bounded by $x = 1, y = 1$ and the same curve $y = 1 - \sqrt{x}$. Calculate the volume of revolution about the y-axis using whatever method you like (you do not necessarily need to even evaluate an integral!).