MATH 18.01 Problem Set 5 Solutions

Problem 1. (*4 pts: 1+2+1*) In this problem you will derive the summation formula for $\sum_{k=1}^{n} k^3$ by using the formulas for lower powers:

$$\sum_{k=1}^{n} k^0 = 1 + 1 + \cdots + 1 = n$$
$$\sum_{k=1}^{n} k^1 = 1 + 2 + \cdots + n = \frac{n^2}{2} + \frac{n}{2}$$
$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

a) Based on the formulas above, it is natural to expect that the sum of cubes will involve n^4. Notice that

$$\begin{aligned}(n+1)^4 &= (n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1 \\ &= ((n-1)+1)^4 + 4n^3 + 6n^2 + 4n + 1 \\ &= (n-1)^4 + 4(n^3 + (n-1)^3) + 6(n^2 + (n-1)^2) + 4(n + (n-1)) + (1+1).\end{aligned}$$

Continue this process by writing $(n-1)^4 = ((n-2)+1)^4$ and expanding, and then $(n-2)^4 = ((n-3)+1)^4$ and so on. What is the result after this has been repeated as many times as possible? You should end up with $(n+1)^4$ equal to several sums of powers.

Solution. In the end all of the fourth powers have been successively written as lower order powers, leaving

$(n+1)^4 =$

$$\boxed{\mathbf{4(n^3 + (n-1)^3 + \cdots + 2^3 + 1^3) + 6(n^2 + \cdots + 1^2) + 4(n + \cdots + 1) + (1 + \cdots + 1)}}.$$

Careful bookkeeping shows that the final sum has $n+1$ 1s!

b) Use the formulas for the sums of k^0, k^1, and k^2 to evaluate and simplify part of the expression. You should now have the sum of the cubes and several polynomials in n.

Solution. The known formulas for the lower powers imply that

$$(n+1)^4 = 4(n^3 + \cdots + 1^3) + 6\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) + 4\left(\frac{n^2}{2} + \frac{n}{2}\right) + n + 1.$$

Expanding the left side as a polynomial in n and simplifying the right side gives

$$\boxed{\mathbf{n^4 + 4n^3 + 6n^2 + 4n + 1 = 4(n^3 + \cdots + 1^3) + 2n^3 + 5n^2 + 4n + 1}}.$$

c) Solve for the sum of the cubes and simplify the evaluation formula.

Solution. Shifting all of the polynomial terms to the left yields

$$n^4 + 2n^3 + n^2 = 4(n^3 + \cdots + 1^3).$$

Thus the formula is $\boxed{\mathbf{n^3 + \cdots + 1^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}}}$.

Problem 2. (*8 pts: 2+2+2+2*) In problem 3A-2 from Part I you found anti-derivatives for several functions. In this problem you must evaluate definite integrals in two different ways:

- Using the anti-derivatives directly with the first Fundamental Theorem of Calculus.
- Using a substitution and change of variables to simplify the integrands before evaluating.

a) $\int_0^{1/2} x^3(1-12x^4)^{1/8}\,dx$

Solution. The anti-derivative is $-\frac{1}{54}(1-12x^4)^{9/8}$, so FTC(1) implies that

$$\int_0^{1/2} x^3(1-12x^4)^{1/8}\,dx = -\frac{1}{54}(1-12x^4)^{9/8}\Big|_0^{1/2} = \boxed{-\frac{\mathbf{1}}{\mathbf{54}}\left(\left(\frac{\mathbf{1}}{\mathbf{4}}\right)^{\mathbf{9/8}}+\mathbf{1}\right)}.$$

Alternatively, the substitution $u = 1-12x^4$ and corresponding differential $du = -48x^3\,dx$ let us calculate the integral as

$$\int_{x=0}^{1/2} x^3(1-12x^4)^{1/8}\,dx = \int_{u=1}^{1/4} -\frac{1}{48}u^{1/8}\,du = \int_{u=1/4}^{1} \frac{1}{48}u^{1/8}\,du$$
$$= \frac{1}{48}\cdot\frac{u^{9/8}}{9/8}\Big|_{u=1/4}^{1} = \boxed{\frac{\mathbf{1}}{\mathbf{54}}-\frac{\mathbf{1}}{\mathbf{54}}\left(\frac{\mathbf{1}}{\mathbf{4}}\right)^{\mathbf{9/8}}}.$$

Remark: There are often many possible choices for substitution (for example, $u = x^4$ also would have simplified the above integral), and any one is a valid approach as long as it simplifies the integrand.

b) $\int_{-1}^{1} \frac{x}{\sqrt{8-x^2}}\,dx$

Solution. The anti-derivative is $-\sqrt{8-x^2}$, so FTC(1) implies

$$\int_{-1}^{1} \frac{x}{\sqrt{8-x^2}}\,dx = -\sqrt{8-x^2}\Big|_{-1}^{1} = -\sqrt{7}+\sqrt{7} = \boxed{\mathbf{0}}.$$

Note that the function is *odd*, so this is the expected answer!

The substitution $u = 8-x^2$ has differential $du = -2x dx$, so

$$\int_{x=-1}^{1} \frac{x}{\sqrt{8-x^2}}\,dx = \int_{u=7}^{7} -\frac{1}{2\sqrt{u}}\,du = \boxed{\mathbf{0}}$$

since the integration range is a single point.

c) $\int_1^2 7x^4 e^{x^5}\,dx$

Solution. The anti-derivative is $\frac{7}{5}e^{x^5}$, so FTC(1) implies

$$\int_1^2 7x^4 e^{x^5}\,dx = \frac{7}{5}e^{x^5}\Big|_1^2 = \boxed{\mathbf{\frac{7}{5}\left(e^{32}-e^{1}\right)}}.$$

The substitution $u = x^5$ has differential $du = 5x^4\,dx$, so

$$\int_{x=1}^2 7x^4 e^{x^5} = \int_{u=1}^{32} \frac{7}{5}e^u\,du = \frac{7}{5}e^u\Big|_1^{32} = \boxed{\mathbf{\frac{7}{5}\left(e^{32}-e^{1}\right)}}.$$

d) $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

Solution. The anti-derivative is $\dfrac{\ln(x)^2}{2}$, so FTC(1) implies

$$\int_1^e \frac{\ln x}{x}\,dx = \frac{\ln(x)^2}{2}\Big|_1^e = \boxed{\mathbf{\frac{1}{2}}}.$$

The substitution $u = \ln(x)$ has differential dx/x, so

$$\int_{x=1}^e \frac{\ln x}{x}\,dx = \int_{u=0}^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \boxed{\mathbf{\frac{1}{2}}}.$$

Problem 3. (*3 pts: 1+2*) Define the function $F(t) := \displaystyle\int_0^t \frac{x}{1+x^4}\,dx$ (Note that $F(t) \neq \frac{t}{1+t^4}$ – it is instead the area under the graph of this quotient!). Although it may seem difficult to understand a function that is defined as the integral of another function, the second Fundamental Theorem of Calculus allows us to calculate its derivatives and sketch the graph of $F(t)$ as usual.

a) Find and classify the critical points and inflection points of $F(t)$. What do you think the behavior is as $t \to \infty$?

Solution. FTC(2) implies that the derivative of $F(t)$ can be read off of the integrand, so $F'(t) = \dfrac{t}{1+t^4}$. Thus $t = 0$ is the only critical point. The second derivative of $F(t)$ is now

$$\frac{d}{dt}\frac{t}{1+t^4} = \frac{1\cdot(1+t^4) - t\cdot(4t^3)}{(1+t^4)^2} = \frac{1-3t^4}{(1+t^4)^2}.$$

Thus there are inflection points at $\boxed{\mathbf{t = \pm\sqrt[4]{3}}}$ (they are inflection points because $F''(0) = 1$ but $F''(-1) = F''(1) = -1/2$, so the concavity actually changes). Furthermore, the parenthetical calculation means that $\boxed{\mathbf{t = 0 \text{ is a minima}}}$.

The behavior as $t \to \infty$ is more complicated. One approach is to observe that the integrand $\dfrac{t}{1+t^4} \to 0$, and for large t it is very similar to $1/t^3$. Therefore it is reasonable to conclude that $F(t)$ behaves something like $\displaystyle\int_1^t \frac{1}{x^3}\,dx = \frac{1}{2} + \frac{1}{2t^2}$, which $\boxed{\textbf{approaches a constant}}$ as $t \to \infty$. The exact solution in part b) shows the precise behavior.

As a final comment, the similar example $\int_0^t \frac{x}{1+x^2}\,dx$ would actually not converge to a constant, but instead approaches infinite area as t grows.

b) Use the substitution $u = x^2$ to transform the integral into a more recognizable form. Evaluate it and find a formula for $F(t)$. Compare the result with the graph that you sketched in part a).

Solution. The suggested substitution has differential $du = 2x\,dx$, so

$$F(t) = \int_{x=0}^{t} \frac{x}{1+x^4}\,dx = \int_{u=0}^{t^2} \frac{1}{2}\cdot\frac{1}{1+u^2}\,du = \frac{1}{2}\arctan(u)\Big|_{u=0}^{t^2} = \boxed{\frac{\arctan(t^2)}{2}}.$$

The function is graphed below, along with the line $y = \pi/2$, which is the asymptotic value as $t \to \infty$.