MATH 18.01 Problem Set 5 - Spring 2009

Due Thursday, Mar. 12 at 1:00

Part I (10 points)

Lecture 13. (*Thurs., Mar. 5*) Summation formulas, Evaluation of definite integrals, First fundamental theorem of calculus.

Read: Simmons 6.3, 6.6 *Work:* 3B-7, 3C-2ab, 3a, 5

Lecture 14. (*Fri., Mar. 6*) Second fundamental theorem of calculus, antiderivatives.

Read: Simmons 6.6, 5.3, Notes FT *Work:* 3A-2acfj, 3D-1a, 3D-7ab

Lecture 15. (*Tues., Mar. 10*) Properties of integrals, change of variables.

Read: Simmons 6.7, Notes PI *Work:* 3A-2degl, 3E-3ab, 4

Part II (15 points)

Try each problem alone for 15 minutes before collaborating, and write up solutions independently. The problems are given in order according to the lecture schedule above.

Problem 1. (*4 pts: 1+2+1*) In this problem you will derive the summation formula for $\sum_{k=1}^{n} k^3$ by using the formulas for lower powers:

$$\sum_{k=1}^{n} k^0 = 1 + 1 + \cdots + 1 = n$$

$$\sum_{k=1}^{n} k^1 = 1 + 2 + \cdots + n = \frac{n^2}{2} + \frac{n}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

a) Based on the formulas above, it is natural to expect that the sum of cubes will involve n^4. Notice that

$$\begin{aligned}(n+1)^4 &= (n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1\\ &= ((n-1)+1)^4 + 4n^3 + 6n^2 + 4n + 1\\ &= (n-1)^4 + 4(n^3 + (n-1)^3) + 6(n^2 + (n-1)^2) + 4(n + (n-1)) + (1+1).\end{aligned}$$

Continue this process by writing $(n-1)^4 = ((n-2)+1)^4$ and expanding, and then $(n-2)^4 = ((n-3)+1)^4$ and so on. What is the result after this has been repeated as many times as possible? You should end up with $(n+1)^4$ equal to several sums of powers.

b) Use the formulas for the sums of k^0, k^1, and k^2 to evaluate and simplify part of the expression. You should now have the sum of the cubes and several polynomials in n.

c) Solve for the sum of the cubes and simplify the evaluation formula.

Problem 2. (*8 pts: 2+2+2+2*) In problem 3A-2 from Part I you found anti-derivatives for several functions. In this problem you must evaluate definite integrals in two different ways:

- Using the anti-derivatives directly with the first Fundamental Theorem of Calculus.
- Using a substitution and change of variables to simplify the integrands before evaluating.

a) $\displaystyle\int_0^{1/2} x^3(1-12x^4)^{1/8}\,dx$

c) $\displaystyle\int_1^2 7x^4 e^{x^5}\,dx$

b) $\displaystyle\int_{-1}^1 \frac{x}{\sqrt{8-x^2}}\,dx$

d) $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

Problem 3. (*3 pts: 1+2*) Define the function $F(t) := \displaystyle\int_0^t \frac{x}{1+x^4}\,dx$ (Note that $F(t) \neq \frac{t}{1+t^4}$ – it is instead the area under the graph of this quotient!). Although it is seemingly difficult to understand this function, the second Fundamental Theorem of Calculus allows us to calculate its derivatives and sketch the graph as usual.

a) Find and classify the critical points and inflection points of $F(t)$. What do you think the behavior is as $t \to \infty$?

b) Use the substitution $u = x^2$ to transform the integral into a more recognizable form. Evaluate it and find a formula for $F(t)$. Compare the result with the graph that you sketched in part a).