MATH 18.01 Problem Set 4 Solutions

Problem 1. (*5 pts: 2+1+2*) Suppose that $f(x)$ is a differentiable function such that

$$f(0) = 1 \quad \text{and} \quad f'(x) \geq 1 \text{ for all } x.$$

a) Show that $f(x) \geq x + 1$ for $x \geq 0$.

Solution. The Mean Value Theorem implies that for any $x \geq 0$, there is some c such that $f'(c) = \dfrac{f(x) - f(0)}{x - 0} = \dfrac{f(x) - 1}{x}$. The bound on the derivative then implies that

$$\frac{f(x) - 1}{x} = f'(c) \geq 1,$$

so $f(x) \geq x + 1$.

b) If $f(2) = 3$, what can you conclude about $f(x)$ in the range $0 \leq x \leq 2$?

Solution. The points $(0, 1)$ and $(2, 3)$ are both part of the graph of $f(x)$, and the line that joins them has slope 1. This line is $y = x + 1$, as in part a). Furthermore, if $f(x) > x + 1$ for some $x \in (0, 2)$, then a similar argument shows that $\dfrac{f(2) - f(x)}{2 - x} \geq 1$, which implies that

$$f(2) \geq f(x) + (2 - x) > x + 1 + 2 - x = 3,$$

which contradicts the value of $f(2)$. Thus $\boxed{\boldsymbol{f(x) = x + 1 \text{ for } x \in (0, 2)}}$.

c) Is it possible that $f(-2) = 0$?

Solution. If the function had this value, then the Mean Value Theorem would imply that there is some $c \in (-2, 0)$ such that $f'(c) = \dfrac{f(0) - f(-2)}{2} = \dfrac{1}{2}$. But $f'(x) \geq 1$ for all values of x, so this is $\boxed{\textbf{not possible}}$.

Problem 2. (*5 pts: 2+2+1*) A retail store has paid expensive consultants to observe shopper patterns. The store is open from 10:00 A.M. to 8:00 P.M., and the report showed that the rate of sales behaves as the function $S(t) = 100 - 4(t - 5)^2$, where t is the number of hours after opening.

a) Approximate the total daily sales by hourly *initial* segments: the sales for the entire hour $t \in [n, n + 1]$ are assumed to be at the same rate as $S(n)$.

Solution. This means that for $t = 0, 1, \ldots, 9$, the t-th hourly sales are approximated by $S(t)$. So the total daily sales are approximately

$$\begin{aligned} &S(0) + S(1) + S(2) + S(3) + S(4) + S(5) + S(6) + S(7) + S(8) + S(9) \\ &\quad = 0 + 36 + 64 + 84 + 96 + 100 + 96 + 84 + 64 + 36 = \boxed{\mathbf{660}}. \end{aligned}$$

The exact sales figure is given by the integral

$$\int_0^{10} 100 - 4(t - 5)^2 dt = 100t - \frac{4(t - 5)^3}{3}\bigg|_0^{10} = 1000 - 2 \cdot \frac{4 \cdot 125}{3} = 1000 \cdot \frac{2}{3} \sim 667.$$

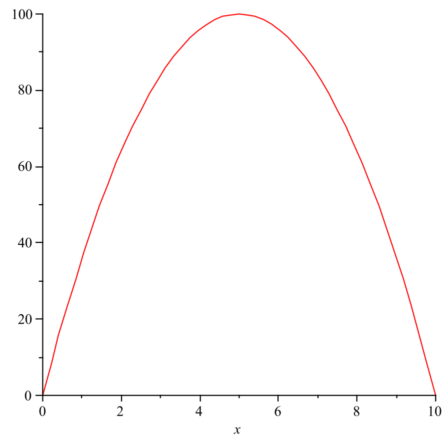
b) Approximate the total daily sales by hourly *final* segments: the hour $t \in [n, n+1]$ is assumed to be the same as $S(n+1)$.

Solution. Now the approximation is

$$\begin{aligned} &S(1)+S(2)+S(3)+S(4)+S(5)+S(6)+S(7)+S(8)+S(9)+S(10) \\ &= 36+64+84+96+100+96+84+64+36+0 = \boxed{\mathbf{660}}. \end{aligned}$$

c) Compare your answers from parts a) and b), and explain the result geometrically by looking at Riemann rectangles on the graph of $S(t)$.

Solution. The answers from $\boxed{\textbf{a) and b) are identical}}$. This is because $S(t)$ is symmetric around the midpoint $t = 5$:



Problem 3. (*5 pts: 2+2+1*) In this problem you will calculate the area under the line $y = mx$ over the range $a \le x \le b$.

a) Write down a Riemann sum with n rectangles that approximates the area.

Solution. The rectangles each have width $\Delta x = \frac{b-a}{n}$, and the base points are $x_k = a + k \cdot \Delta x$. Therefore the area is approximated by

$$\sum_{k=1}^{n} m x_k \cdot \Delta x = \boxed{\boldsymbol{m\Delta x \sum_{k=1}^{n} a + k\Delta x}}.$$

b) Evaluate the sum from a) using the fact that $1 + 2 + \cdots + n = \dfrac{(n+1)n}{2}$.

Solution. The sum can be evaluated by splitting the summand and using the given formula:

$$\begin{aligned} m\Delta x \sum_{k=1}^{n} a + k\Delta x &= am\Delta x \sum_{k=1}^{n} 1 + m(\Delta x)^2 \sum_{k=1}^{n} k = am\Delta x \cdot n + m(\Delta x)^2 \cdot \frac{(n+1)n}{2} \\ &= am(b-a) + m(b-a)^2 \frac{(n+1)n}{2n^2} \\ &= \boxed{\boldsymbol{am(b-a) + m(b-a)^2 \cdot \frac{n+1}{2n}}}. \end{aligned}$$

c) Find the area of the trapezoid using basic geometry and check that your answer in b) is consistent as $n \to \infty$.

Solution. The trapezoid has base $b - a$ and parallel sides of length ma and mb. Thus the area is

$$(b-a) \cdot \frac{ma + mb}{2} = \frac{m(b^2 - a^2)}{2}.$$

In the limit as $n \to \infty$, the area from b) also approaches

$$am(b-a) + m(b-a)^2 \cdot \frac{1}{2} = m(ba - a^2) + m\left(\frac{b^2 - 2ba + a^2}{2}\right) = \frac{m(b^2 - a^2)}{2}.$$