MATH 18.01 Problem Set 3 Solutions

Part II (15 points)

Problem 1. (*7 pts: 1+1+2+2+1*)
a) The volume of a spherical balloon of radius r is $V(r) = \frac{4\pi r^3}{3}$, and the surface area is $A(r) = 4\pi r^2$. Calculate the derivative of the volume as a function of radius.

Solution. The derivative is $\frac{d}{dr}V(r) = \boxed{\boldsymbol{4\pi r^2}}$.

b) Suppose that the balloon is being blown up such that the volume increases by 10 cubic inches per second. Assuming that the balloon was empty at time $t = 0$, what is the rate of change of surface area as a function of time?

Solution. Since both area and volume are easily expressed as functions of radius, it is easy to relate the relevant rates of change. First, differentiate the volume formula to find that

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt},$$

so

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{10}{4\pi r^2} = \frac{5}{2\pi r^2}.$$

Now use the area formula to find that

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} = 8\pi r \cdot \frac{5}{2\pi r^2} = \boxed{\frac{\mathbf{20}}{\boldsymbol{r}}}.$$

Alternatively, we could have related volume and surface area directly by noticing that $A^3 = 36\pi V^2$.

c) A cylindrical jello mold is 30 cm tall and has an adjustable band around its circumference that allows its radius to be altered. If the tank is set to radius r and is filled to height h, then the total volume of jello is $V = \pi r^2 h$. The portion of the side of the mold that is covered by jello has surface area $A = 2\pi r h$. Calculate the derivative of the volume as a function of radius. Based on your answer here and in part a), conjecture a general relationship between volume and surface area.

Solution. The derivative is $\frac{d}{dr}V = \boxed{\boldsymbol{2\pi r h}}$.

Part a) and this problem suggest that surface area is the derivative of volume, but this is surprisingly **not** true in general. It is essentially only true for objects that are very "circular" or "cylindrical". For example, a circular cone of height h and radius r has volume $V = \frac{\pi r^2 h}{3}$, but has surface area $A = \pi r\sqrt{r^2 + h^2}$.

d) Suppose that at time $t = 0$, the mold is set to an initial radius of $r = 10$ cm, and is filled to the full height. The band is then loosened, and the mold is allowed to expand under the pressure of the jello; as a function of time, the radius is $r = 10 + \sqrt{t}$. If the total volume of jello remains unchanged as the mold expands, find the surface area as a function of time.

Solution. The total volume remains constant at the initial value of $300\pi\text{cm}^3$. Thus

$$A = 2\pi r h = 2\frac{\pi r^2 h}{r} = \frac{2V}{r} = \boxed{\frac{\mathbf{600}\boldsymbol{\pi}}{\mathbf{10} + \sqrt{\boldsymbol{t}}}}.$$

e) Continuing part d), what is the rate of change of the outer surface area of the jello?

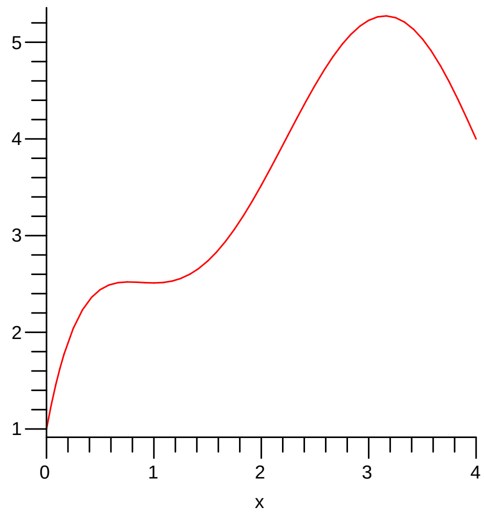
Solution. This is easiest to do directly, as the relations between all of the changing parameters (area, radius, height) become rather complicated.

$$\frac{d}{dt}A = 600\pi \cdot (-1)\frac{1}{(10+\sqrt{t})^2} \cdot \frac{1}{2\sqrt{t}} = \boxed{-\frac{\mathbf{300\pi}}{\sqrt{t}(\mathbf{10}+\sqrt{t})^{\mathbf{2}}}}.$$

Problem 2. (*3 pts*) Sketch a continuous, differentiable function that satisfies

- $f(0) = 1$,
- $f'(1) = 0, f''(1) = 1$,
- $f'(2) = 2$,
- $f''(3) = 0$,
- $f(4) = 4$.

Solution. The important features are the **minima** at $x = 1$, the **positive slope** at $x = 2$, and the **inflection point** at $x = 3$. The graph shown is one of the simplest possible examples, but there are other possibilities!



Problem 3. (*5 pts: 1+1+1+2*) The goal of this problem is to study the strange function $f(x) = e^{-1/x^2}$ near $x = 0$.

a) The function $f(x)$ is not defined at $x = 0$, but this is a *removable* discontinuity. Find the limiting value by evaluating $\lim_{x\to 0} e^{-1/x^2}$.

Solution. As x goes to 0, it's clear that $\frac{1}{x^2}$ goes to ∞. Thus the limit is $e^{-\infty} = \boxed{\mathbf{0}}$.

b) Calculate the first and second derivatives of $f(x)$. Describe the general shape of all such derivatives.

Solution. First, by the chain rule and exponent rule

$$f'(x) = e^{-1/x^2} \cdot -\frac{-2}{x^3} = \boxed{\frac{\mathbf{2e^{-1/x^2}}}{\mathbf{x^3}}}.$$

Now by the quotient rule,

$$f''(x) = \frac{4e^{-1/x^2}x^{-3} \cdot x^3 - 2e^{-1/x^2} \cdot 3x^2}{x^6} = \boxed{\frac{\mathbf{4e^{-1/x^2}}}{\mathbf{x^6}} - \frac{\mathbf{6e^{-1/x^2}}}{\mathbf{x^4}}}.$$

In general, all derivatives will be the sum of terms of the form $\frac{e^{-1/x^2}}{x^n}$ for various powers n.

c) The derivatives are also not defined at $x = 0$, and the limits involve indeterminate forms $x^{-n}e^{-1/x^2}$ (for positive n). What happens if you apply L'Hospital's rule to

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^n}?$$

What if you apply L'Hospital's rule many times – does the calculation simplify?

Solution. The limit is the indeterminate form $\frac{0}{0}$. Applying L'Hospital's rule gives

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^n} = \lim_{x\to 0} \frac{2e^{-1/x^2}x^{-3}}{nx^{n-1}} = \boxed{\lim_{x\to 0} \frac{\mathbf{2e^{-1/x^2}}}{\mathbf{nx^{n+2}}}}.$$

Each time that L'Hospital's rule is applied, the exponent on the denominator will rise by 2, which always results in the indeterminate form $\frac{0}{0}$, so it $\boxed{\textbf{does not simplify}}$.

d) An alternative approach for the limit in c) is to first change the limit variable: let $y = 1/x^2$, so that $y \to \infty$ as $x \to 0$. Rewrite the limit using y, and then use L'Hospital to show that the limit in c) is 0 for every n.

Solution. The substitution means that $x = \frac{1}{\sqrt{y}}$, so $x^n = y^{-n/2}$. Thus the limit is

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^n} = \lim_{y\to\infty} \frac{e^{-y}}{y^{-n/2}} = \lim_{y\to\infty} \frac{y^{n/2}}{e^y}.$$

This is now the indeterminate form $\frac{\infty}{\infty}$, and L'Hospital gives

$$\lim_{y\to\infty} \frac{y^{n/2}}{e^y} = \lim_{y\to\infty} \frac{\frac{n}{2} \cdot y^{n/2-1}}{e^y} = \lim_{y\to\infty} \frac{\frac{n}{2} \cdot (\frac{n}{2} - 1) \cdot y^{n/2-1}}{e^y} = \cdots$$

Now the power of y is decreasing, so after L'Hospital has been applied enough times to reduce the power to 0 or less, the limit has the shape $\frac{C}{e^y}$ or $\frac{Cy^{-1/2}}{e^y}$ (depending on whether n is odd or even) for some constant C. Thus

$$\lim_{y\to\infty} \frac{y^{n/2}}{e^y} = \lim_{y\to\infty} \frac{C}{e^y} = \boxed{\mathbf{0}}.$$