MATH 18.01 Problem Set 3 - Spring 2009

Due Thursday, Feb. 26 at 1:00

Part I (10 points)

Lecture 7. (*Thurs., Feb. 19*) Related rates.
Read: Simmons 4.5 *Work:* 2E-1, 2, 4, 7

Lecture 8. (*Fri., Feb. 20*) Curve sketching. Maxima and minima.
Read: Simmons 4.1 - 4.4 *Work:* 2C-1, 2, 4, 13

Lecture 9. (*Tues., Feb. 24*) Indeterminate limits and L'Hospital's rule, linear and quadratic approximation.
Read: Simmons 2.6, 12.1 - 12.3, Notes A *Work:* 6A-1acdfg, 2abdh, 2A-3, 6

Part II (15 points)

Try each problem alone for 15 minutes before collaborating, and write up solutions independently. The problems are given in order according to the lecture schedule above.

Problem 1. (*7 pts: 1+1+2+2+1*)
a) The volume of a spherical balloon of radius r is $V(r) = \frac{4\pi r^3}{3}$, and the surface area is $A(r) = 4\pi r^2$. Calculate the derivative of the volume as a function of radius.

b) Suppose that the balloon is being blown up such that the volume increases by 10 cubic inches per second. Assuming that the balloon was empty at time $t = 0$, what is the rate of change of surface area as a function of time?

c) A cylindrical jello mold is 30 cm tall and has an adjustable band around its circumference that allows its radius to be altered. If the tank is set to radius r and is filled to height h, then the total volume of jello is $V = \pi r^2 h$. The portion of the side of the mold that is covered by jello has surface area $A = 2\pi rh$. Calculate the derivative of the volume as a function of radius. Based on your answer here and in part a), conjecture a general relationship between volume and surface area.

d) Suppose that at time $t = 0$, the mold is set to an initial radius of $r = 10$ cm, and is filled to the full height. The band is then loosened, and the mold is allowed to expand under the pressure of the jello; as a function of time, the radius is $r = 10 + \sqrt{t}$. If the total volume of jello remains unchanged as the mold expands, find the surface area as a function of time.

e) Continuing part d), what is the rate of change of the outer surface area of the jello?

Problem 2. (*3 pts*) Sketch a continuous, differentiable function that satisfies

- $f(0) = 1$,
- $f'(1) = 0, f''(1) = 1$,
- $f'(2) = 2$,
- $f''(3) = 0$,
- $f(4) = 4$.

Problem 3. (*5 pts: 1+1+1+2*) The goal of this problem is to study the strange function $f(x) = e^{-1/x^2}$ near $x = 0$.

a) The function $f(x)$ is not defined at $x = 0$, but this is a *removable* discontinuity. Find the limiting value by evaluating $\lim_{x\to 0} e^{-1/x^2}$.

b) Calculate the first and second derivatives of $f(x)$. Describe the general shape of all such derivatives.

c) The derivatives are also not defined at $x = 0$, and the limits involve indeterminate forms $x^{-n}e^{-1/x^2}$ (for positive n). What happens if you apply L'Hospital's rule to

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^n}?$$

What if you apply L'Hospital's rule many times – does the calculation simplify?

d) An alternative approach for the limit in c) is to first change the limit variable: let $y = 1/x^2$, so that $y \to \infty$ as $x \to 0$. Rewrite the limit using y, and then use L'Hospital to show that the limit in c) is 0 for every n.

Remark: This means that $f(0) = 0, f'(0) = 0, f''(0) = 0,$ *and all other derivatives are also* 0*! This will be a problematic example when we study Taylor series much later in the semester.*