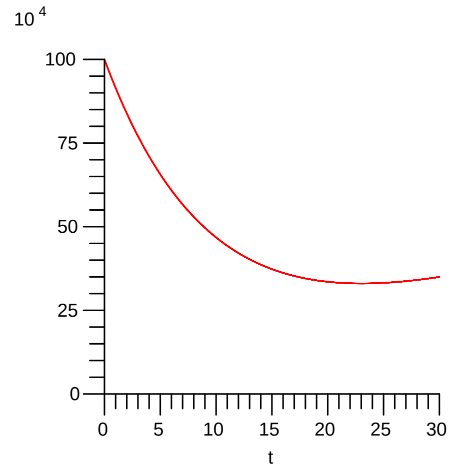
# MATH 18.01 Problem Set 2 Solutions

## Part II (15 points)

**Problem 1.** (*4 pts: 1+1+1+1*) At the beginning of retirement, a worker withdraws his life-savings of $1,000,000 to hold in cash. With a steady inflation rate of 10%, this money has *real* value $1,000,000e^{-.1t}$ after $t$ years. As part of his pension, he also receives an inflation-indexed payment of $10,000 each year.

a) Sketch the graph of his total real wealth as a function of time,

$$f(t) = 1,000,000e^{-.1t} + 10,000t.$$




b) What is the average rate of growth of his wealth during his first 10 years of retirement?

*Solution.* He initially has $f(0) = 1,000,000$ dollars, and after 10 years he has $f(10) = 1,000,000e^{-1} + 100,000 \sim 467,000$. Thus his average rate of change was

$$\frac{f(10) - f(0)}{10} \sim \frac{-533,000}{10} \sim \boxed{\mathbf{-\$53,300/year}}.$$

c) In which year is his real wealth declining most quickly?

*Solution.* The rate of decline in his real wealth is $f'(t) = -100,000e^{-.1t} + 10,000$. This is minimized at $\boxed{\boldsymbol{t = 0}}$.

d) In which year will his wealth be at a minimum?

*Solution.* His wealth is minimized when $f'(t) = 0$. This occurs when $100,000e^{-.1t} = 10,000$, which means that $t = -10\ln(1/10) = 10\ln(10) \sim \boxed{\textbf{23 years}}$.

**Problem 2.** (*4 pts: 1+1+1+1*) Calculate the 101-st derivative of the following functions:

a) $f(x) = x^{101}$

*Solution.* Each additional derivative reduces the exponent by 1 and pulls it down as a multiplicative factor, giving

$$\frac{d^{101}}{dx^{101}}x^{101} = 101 \cdot 100 \cdots 2 \cdot 1 = \boxed{\mathbf{101!}}.$$

b) $f(x) = x^{102}$

*Solution.* Similarly,

$$\frac{d^{101}}{dx^{101}} x^{102} = \boxed{\mathbf{102! \cdot x}} .$$

c) $f(x) = e^{-x}$

*Solution.* Here $f'(x) = -e^{-x} = -f(x)$ and $f''(x) = e^{-x} = f(x)$. So the derivatives alternate between $-f(x)$ and $f(x)$. Since 101 is odd,

$$\frac{d^{101}}{dx^{101}} e^{-x} = \boxed{\boldsymbol{-e^{-x}}} .$$

d) $f(x) = \cos 2x$.

*Solution.* The derivatives of the cosine function also repeat cyclically as $\cos \to -\sin \to -\cos \to \sin \to \cos \ldots$. Furthermore, by the chain rule, each derivative brings out a factor of 2. Thus

$$\frac{d^{101}}{dx^{101}} \cos 2x = \boxed{\boldsymbol{-2^{101} \sin 2x}} .$$

**Problem 3.** (*2 pts*) Compute the derivative of both sides of the double angle formula for cosine,

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta.$$

Can you identify the resulting identity?

*Solution.* The derivatives are

$$-2 \sin 2\theta = -2 \cos \theta \sin \theta - 2 \sin \theta \cos \theta = -4 \sin \theta \cos \theta.$$

This is the $\boxed{\textbf{sine double-angle formula}}$ .

**Problem 4.** (*3 pts*) The equation $y^2 = x^3 - x$ is an example of an *elliptic curve* (which are very important in number theory and cryptography). Find all points on the curve where the tangent line has slope 0. The graph to the right should give you a hint for the number of solutions!

*Solution.* Using implicit differentiation, calculate the differential:

$$2y dy = 3x^2 dx - dx.$$

Simplifying and solving the equation gives

$$\frac{dy}{dx} = \frac{3x^2 - 1}{2y}.$$

This tangent slope is 0 when $3x^2 - 1 = 0$, which occurs when $x = \pm\frac{1}{\sqrt{3}}$.

However, plugging in $x = \frac{1}{\sqrt{3}}$ to the curve equation gives $y = -\frac{2}{3\sqrt{3}} < 0$, so there are no points on the curve for the positive square root. For the negative square root, there are two points

$$\boxed{\left(-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3\sqrt{3}}}\right) \quad \textbf{and} \quad \left(-\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{3\sqrt{3}}}\right)} .$$

**Problem 5.** (*2 pts*) Use the rule for derivatives of inverse functions to calculate the derivative of $\arccos(x)$.

*Solution.* In general, $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(y)}$, where $y = f^{-1}(x)$. Here that means that

$$\frac{d}{dx}\arccos(x) = \frac{1}{-\sin(y)} = -\frac{1}{\sin(\arccos(x))}.$$

Simple trigonometry shows that the angle $y$ corresponds to a right triangle with adjacent side $x$ and hypotenuse 1, so the opposite leg is $\sqrt{1-x^2}$. Thus

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\frac{\sqrt{1-x^2}}{1}} = \boxed{-\frac{\mathbf{1}}{\sqrt{\mathbf{1}-\boldsymbol{x}^{\mathbf{2}}}}}.$$