MATH 18.01 Problem Set 1 Solutions

Part II (15 points)

Problem 1. (*5 pts: 1+1+1+1+1*) A function is *even* if $f(-x) = f(x)$, and it is *odd* if $f(-x) = -f(x)$.

a) Determine whether the functions $f(x) = x^4+2$, $f(x) = x^3\sin(x)$, and $f(x) = \frac{1}{(x+1)^2} - \frac{1}{(x-1)^2}$ are even or odd.

Solution. Plug in $f(-x)$ for each function:

i. $(-x)^4 + 2 = x^4 + 2$, so $\boxed{\boldsymbol{x^4 + 2 \text{ is even}}}$.

ii. $(-x)^3 \sin(-x) = -x^3 \cdot -\sin(x) = x^3 \sin(x)$, so $\boxed{\boldsymbol{x^3 \sin(x) \text{ is even}}}$.

iii.

$$\frac{1}{(-x+1)^2} - \frac{1}{(-x-1)^2} = \frac{1}{(x-1)^2} - \frac{1}{(x+1)^2} = -\left(\frac{1}{(x+1)^2} - \frac{1}{(x-1)^2}\right),$$

so $\boxed{\boldsymbol{\frac{1}{(x+1)^2} - \frac{1}{(x-1)^2} \text{ is odd}}}$.

b) Is

$$f(x) = \frac{xe^x}{x+1} + \frac{(x+1)e^{-x}}{x}$$

even, odd, or neither?

Solution. Can simplify

$$f(x) = \frac{x^2e^x + (x+1)^2e^{-x}}{x(x+1)},$$

so

$$f(-x) = \frac{x^2e^{-x} + (x-1)^2e^x}{x(x-1)}.$$

This is clearly different than $\pm f(x)$, so $\boxed{\textbf{neither}}$.

c) Show that the product of two odd functions is even.

Solution. Suppose that $f(x)$ and $g(x)$ are odd. Then

$$f \cdot g(-x) = f(-x)g(-x) = -f(x) \cdot -g(x) = f(x)g(x) = f \cdot g(x).$$

d) It is easy to rewrite $f(x)$ algebraically as

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.$$

Show that the first term is always an even function, and the second term is always an odd function.

Solution. For the first part,

$$\frac{f(-x)+f(x)}{2} = \frac{f(x)+f(-x)}{2}.$$

For the second,

$$\frac{f(-x)-f(x)}{2} = -\left(\frac{f(x)-f(-x)}{2}\right).$$

e) Use part d) to decompose $f(x) = \frac{x}{x+1}$ as the sum of an even function and an odd function.
Solution. The even component is (note the even exponents after simplifying)

$$\frac{f(-x)+f(x)}{2} = \frac{\frac{x}{x+1} + \frac{-x}{-x+1}}{2} = \boxed{\boldsymbol{\frac{x^2}{x^2-1}}}.$$

The odd component is

$$\frac{f(-x)-f(x)}{2} = \frac{\frac{x}{x+1} - \frac{-x}{-x+1}}{2} = \boxed{\boldsymbol{\frac{-x}{x^2-1}}}.$$

Problem 2. (*3 pts*) (Simmons 2.2.9) Prove that there is no line passing through the point $(1, -2)$ that is tangent to the curve $y = f(x) = x^2 - 4$.

Solution. One approach is to find all of the tangent lines to the curve and show that $(1, -2)$ is not on any of them. Ther other approach would be to consider all lines through $(1, -2)$ and show that none of them are tangent to the curve.

Following the first approach, calculate $f'(x) = 2x$, so the tangent line at $x = a$ has slope $f'(a) = 2a$. This tangent line has the equation $y = 2ax + b$ for some b, and contains the point of tangency $(a, a^2 - 4)$. Plugging in,

$$a^2 - 4 = 2a^2 + b,$$

so $b = -a^2 - 4$, and the tangent line at $(a, a^2 - 4)$ is $y = 2ax - a^2 - 4$.

If $(1, -2)$ were to lie on such a line, then $-2 = 2a \cdot 1 - a^2 - 4$ must have a solution for some a. This simplies to $a^2 - 2a + 2 = 0$. But this equation has no real solutions since the left side is $(a-1)^2 + 1 > 0$. Therefore none of the tangent lines of $y = x^2 - 4$ go through the point $(1, -2)$.

Problem 3. (*4 pts: 2+2*) The graph of the function $y = f(x) = \sqrt{a^2 - x^2}$ is a semicircle of radius a centered at the origin.

a) For any fixed $0 \le x_0 \le a$, find the equation of the tangent line of the function at the point $(x_0, f(x_0))$.

Solution. First use the chain rule to calculate

$$f'(x) = \frac{1}{2\sqrt{a^2 - x^2}} \cdot (-2x) = \frac{-x}{\sqrt{a^2 - x^2}}.$$

Thus the tangent line has slope $f'(x_0) = \frac{-x_0}{\sqrt{a^2 - x_0^2}}$ and passes through the point $(x_0, \sqrt{a^2 - x_0^2})$. This means that the line has the equation

$$y = \frac{-x_0}{\sqrt{a^2 - x_0^2}} \cdot x + b.$$

Plugging in the known point gives

$$\sqrt{a^2 - x_0^2} = \frac{-x_0^2}{\sqrt{a^2 - x_0^2}} + b$$
$$\Rightarrow \quad b = \frac{(a^2 - x_0^2) + x_0^2}{\sqrt{a^2 - x_0^2}} = \frac{a^2}{\sqrt{a^2 - x_0^2}}.$$

Overall the tangent line is

$$\boxed{\boldsymbol{y = \frac{-x_0}{\sqrt{a^2 - x_0^2}} \cdot x + \frac{a^2}{\sqrt{a^2 - x_0^2}}}}.$$

b) Consider the line that passes through the points $(0,3)$ and $(2,0)$. Find the largest circle centered at the origin that does not intersect this line (the circle may be tangent to the line).

Solution. Some thought shows that the maximal circle is the one that is tangent to the line (any smaller circle does not intersect the circle, and any larger circle is cut by the line). The restricting line is $y = -\frac{3}{2}x + 3$, and we must find the value of a such that this is the tangent line of the circle at some point x_0.

Using the equation from part a), we need

$$y = -\frac{3}{2}x + 3 = \frac{-x_0}{\sqrt{a^2 - x_0^2}} \cdot x + \frac{a^2}{\sqrt{a^2 - x_0^2}}.$$

So

$$-\frac{3}{2} = -\frac{x_0}{\sqrt{a^2 - x_0^2}} \qquad \text{and} \qquad 3 = \frac{a^2}{\sqrt{a^2 - x_0^2}}.$$

Squaring the first equation and solving for a^2 gives that $a^2 = \frac{13}{9}x_0^2$. Plug this in to the second equation to get

$$3 = \frac{\frac{13}{9}x_0^2}{\sqrt{\frac{13}{9}x_0^2 - x_0^2}} = \frac{\frac{13}{9}x_0^2}{\sqrt{\frac{4}{9}x_0^2}} = \frac{13}{6}x_0,$$

so $x_0 = \frac{18}{13}$.

Finally, this means that the largest possible radius is

$$a = \sqrt{\frac{13}{9}}\,x_0 = \sqrt{\frac{13}{9}} \cdot \frac{18}{13} = \boxed{\boldsymbol{\frac{6}{\sqrt{13}}}}.$$

Note: This problem is much easier to solve using vector calculus and a perpendicular line segment from the origin!

Problem 4. (*4 pts: 2+2*) Suppose that $f(x)$ and $g(x)$ are differentiable functions.

a) Calculate $\frac{d}{dx}(f(x)g(x))$, $\frac{d^2}{dx^2}(f(x)g(x))$, and $\frac{d^3}{dx^3}(f(x)g(x))$.

Solution. Successively compute using the product rule:

$$\begin{aligned}
\frac{d}{dx}(f(x)g(x)) &= \boxed{\boldsymbol{f'(x)g(x) + f(x)g'(x)}} \\
\frac{d^2}{dx^2}(f(x)g(x)) &= \frac{d}{dx}\left(\frac{d}{dx}(f(x)g(x))\right) = \frac{d}{dx}(f'(x)g(x) + f(x)g'(x)) \\
&= f''(x)g(x) + f'(x)g'(x) + f'(x)g'(x) + f(x)g''(x) \\
&= \boxed{\boldsymbol{f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)}}\ . \\
\frac{d^3}{dx^3}(f(x)g(x)) &= \boxed{\boldsymbol{f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + f(x)g'''(x)}}\ .
\end{aligned}$$

b) Expand and simplify $(a+b)$, $(a+b)^2$, and $(a+b)^3$ and compare to part a). Conjecture a general formula for the higher derivatives of $f(x)g(x)$.

Solution.

$$\begin{aligned}
(a+b) &= \boxed{\boldsymbol{a+b}} \\
(a+b)^2 &= \boxed{\boldsymbol{a^2 + 2ab + b^2}} \\
(a+b)^3 &= \boxed{\boldsymbol{a^3 + 3a^2b + 3ab^2 + b^3}}
\end{aligned}$$

The general principle is that for an arbitrary k, $\frac{d^k}{dx^k}(f(x)g(x))$ has terms corresponding to $(a+b)^k$, where the monomial $a^m b^n$ corresponds to $f^{(m)}(x)\,g^{(n)}(x)$, and the coefficients are also the same.