MATH 18.01 Problem Set 11 Solutions

Problem 1. (*8 pts: 2+2+2+2*) In this problem you will use Taylor series to evaluate certain infinite sums.

a) Differentiate the power series expansion

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

to obtain a new power series formula.

Solution. Differentiation gives

$$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}\left(1 + x + x^2 + \dots\right)$$

$$\boxed{\frac{\mathbf{1}}{(\mathbf{1}-\boldsymbol{x})^{\mathbf{2}}} = \sum_{\boldsymbol{n=0}}^{\boldsymbol{\infty}} \boldsymbol{n x^{n-1}} = \mathbf{1} + \mathbf{2}\boldsymbol{x} + \mathbf{3}\boldsymbol{x}^{\mathbf{2}} + \dots}\ .$$

Note that the $n = 0$ term vanishes, so the sum really begins at $n = 1$.

b) Multiply the equality from a) through by x. You should now have a formula for evaluating the sum $\sum_{n=1}^{\infty} nx^n$. Use it to calculate the sum

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \dots.$$

Solution. Multiply the formula from part a) by x to obtain

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n.$$

The series in question can be evaluated by setting $x = 1/2$, which gives

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \dots = \frac{1/2}{(1-1/2)^2} = \boxed{\mathbf{2}}\ .$$

c) Now integrate the power series expansion

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \dots;$$

it will help to use partial fractions or a trigonometric substitution on the left.

Solution. The easiest way to find an anti-derivative for the left side is to use partial fractions. We can write

$$\frac{1}{1-x^2} = \frac{1}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x} = \frac{(B-A)x + A + B}{(1+x)(1-x)}.$$

Equating the corresponding powers of x in the numerators gives the two equations $0 = B - A$ and $1 = A + B$, which have the solution $A = B = 1/2$.

Thus integration gives

$$\int \frac{1}{1-x^2}\,dx = \int \frac{1/2}{1+x} + \frac{1/2}{1-x}\,dx = \int \sum_{n=0}^{\infty} x^{2n}\,dx$$
$$\frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots.$$

This simplifies to

$$\boxed{\ln\left(\sqrt{\frac{1+x}{1-x}}\right) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}}\,.$$

Alternatively, the trigonometric substitution $x = \sin u$ will also lead to the anti-derivative.

d) Use your answer from c) (or a modified version) to calculate the sum

$$\sum_{n=1}^{\infty} \frac{(1/3)^{2n}}{2n} = \frac{1}{2\cdot 3^2} + \frac{1}{4\cdot 3^4} + \frac{1}{6\cdot 3^6} + \dots.$$

Solution. Actually, the simplest approach is to modify the argument in part c) and integrate the series identity

$$\int \frac{x}{1-x^2}\,dx = \int \sum_{n=0}^{\infty} x^{2n+1}\,dx$$
$$-\frac{1}{2}\ln(l - x^2) = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{2n+2} = \frac{x^2}{2} + \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} \frac{x^{2n}}{2n}.$$

The series in question is the specialization to $x = 1/3$, and hence

$$\sum_{n=1}^{\infty} \frac{(1/3)^{2n}}{2n} = -\frac{1}{2}\ln\left(1 - \frac{1}{9}\right) = \boxed{\frac{1}{2}\ln\left(\frac{9}{8}\right)} \sim 0.05889\dots.$$

Problem 2. (*7 pts: 2+3+2*) On the third midterm, you saw an integral of the form

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx.$$

This integral can be solved by making the trigonometric substitution $x = a\sec u$; the anti-derivative is $\ln\left(x + \sqrt{x^2 - a^2}\right)$. In this problem you will see that the integral also arises naturally in the context of hyperbolic functions.

a) Recall that the hyperbolic sine and cosine are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \qquad \text{and} \qquad \sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots.$$

Verify that they satisfy the hyperbolic equation

$$(\cosh x)^2 - (\sinh x)^2 = 1.$$

Solution. This follows from the exponential definitions:

$$(\cosh x)^2 - (\sinh x)^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1.$$

b) The inverse hyperbolic cosine is defined as the function of y that satisfies $\cosh x = y$, and is denoted by $\cosh^{-1}(y) = x$. Find the derivative of this function by using the inverse function formula from earlier in the semester. Recall that the formula states that if $y = f(x)$, then

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{\frac{d}{dx} f(x)}$$

In this case, $f(x) = \cosh x$, you will obtain an expression for $\frac{d}{dy}\cosh^{-1}(y)$. Use the hyperbolic equation from part a) to write this expression as a function of y; you should find that

$$\frac{d}{dy}\cosh^{-1}(y) = \frac{1}{\sqrt{y^2 - 1}}.$$

Solution. The inverse function derivative formula implies that

$$\frac{d}{dy}\cosh^{-1}(y) = \frac{1}{\frac{d}{dx}\cosh x} = \frac{1}{\sinh x} = \frac{1}{\sqrt{\cosh^2 x - 1}} = \frac{1}{\sqrt{y^2 - 1}},$$

which follows from the definition of $y = \cosh x$.

c) Consider the defining function formula

$$x = \cosh y = \frac{e^y + e^{-y}}{2}.$$

Let $u = e^y$ and use the quadratic formula to solve for u in terms of x. Use this expression to write $y = \cosh^{-1}(x)$ as an explicit function of x. This means that you've found an anti-derivative for $\frac{1}{\sqrt{x^2 - 1}}$!

Solution. Using the suggested auxiliary variable u, we have $x = (u + u^{-1})/2$. Multiply this equation by u and move all terms to the same side, which results in

$$u^2 - 2xu + 1 = 0.$$

The quadratic formula now implies that

$$u = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}.$$

Thus

$$y = \cosh^{-1}(x) = \ln(x \pm \sqrt{x^2 - 1}).$$

In fact, it is straightforward to check that only the positive option gives a correct anti-derivative, as

$$\frac{d}{dx}\ln(x + \sqrt{x^2 - 1}) = \frac{1}{x + \sqrt{x^2 - 1}} \cdot \left(1 + \frac{x}{\sqrt{x^2 - 1}}\right) = \frac{1}{\sqrt{x^2 - 1}}.$$

Thus

$$\boxed{\boldsymbol{y = \cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})}}$$

is an anti-derivative for $\frac{1}{\sqrt{x^2 - 1}}$.

The general case for arbitrary a proceeds analogously.