

## MATH 18.01 Problem Set 11 - Spring 2009

Due Thursday, Apr. 30 at 1:00

### Part I (10 points)

**Lecture 35.** (*Tues., May 5*) Introduction to power series, Intervals of convergence, Taylor series.

*Read:* Simmons 14.1 - 14.4

*Work:* 7C-1, 7D-1ade

**Lecture 36.** (*Thurs., May 7*) Remainder terms, hyperbolic functions.

*Read:* Simmons 14.5, 9.7

*Work:* 7C-2, 4

**The following problems are NOT to be turned in, and are only suggested as practice for the final exam.**

**Lecture 37.** (*Fri., May 8*) Probability, distributions.

*Read:* Notes P1 - P3

*Work:* 8B-2, 3, 5

**Lecture 38.** (*Tues., May 12*) Normal distribution.

*Read:* Notes P4, Simmons 12.5

*Work:* 8D-1, 2, 8E-1, 2

### Part II (15 points)

**Problem 1.** (*8 pts: 2+2+2+2*) In this problem you will use Taylor series to evaluate certain infinite sums.

a) Differentiate the power series expansion

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

to obtain a new power series formula.

b) Multiply the equality from a) through by  $x$ . You should now have a formula for evaluating the sum  $\sum_{n=1}^{\infty} nx^n$ . Use it to calculate the sum

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \dots$$

c) Now integrate the power series expansion

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \dots;$$

it will help to use partial fractions or a trigonometric substitution on the left.

d) Use your answer from c) (or a modified version) to calculate the sum

$$\sum_{n=1}^{\infty} \frac{(1/3)^{2n}}{2n} = \frac{1}{2 \cdot 3^2} + \frac{1}{4 \cdot 3^4} + \frac{1}{6 \cdot 3^6} + \dots$$

**Problem 2.** (7 pts: 2+3+2) On the third midterm, you saw an integral of the form

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx.$$

This integral can be solved by making the trigonometric substitution  $x = a \sec u$ ; the anti-derivative is  $\ln(x + \sqrt{x^2 - a^2})$ . In this problem you will see that the integral also arises naturally in the context of hyperbolic functions.

a) Recall that the hyperbolic sine and cosine are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

Verify that they satisfy the hyperbolic equation

$$(\cosh x)^2 - (\sinh x)^2 = 1.$$

b) The inverse hyperbolic cosine is defined to satisfy  $\cosh x = y$ , and is denoted by  $x = \cosh^{-1}(y)$ . Find the derivative of this function by using the inverse function formula from earlier in the semester. Recall that the formula states that if  $y = f(x)$ , then

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{\frac{d}{dx} f(x)}$$

In this case,  $f(x) = \cosh x$ , you will obtain an expression for  $\frac{d}{dy} \cosh^{-1}(y)$ . Use the hyperbolic equation from part a) to write this expression as a function of  $y$ ; you should have found

$$\frac{d}{dy} \cosh^{-1}(y) = \frac{1}{\sqrt{y^2 - 1}}.$$

c) Consider the defining function formula

$$x = \cosh y = \frac{e^y + e^{-y}}{2}.$$

Let  $u = e^y$  and use the quadratic formula to solve for  $u$  in terms of  $x$ . Use this expression to write  $y = \cosh^{-1}(x)$  as an explicit function of  $x$ . This means that you've found an anti-derivative for  $\frac{1}{\sqrt{x^2 - 1}}$ !