MATH 18.01 Problem Set 10 Solutions

Problem 1. (*8 pts: 3+2+3*) In this problem you will consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)\cdots(n+k)}$$

for any $k \geq 1$.

a) Calculate the partial fraction decompositions of $\frac{1}{n(n+1)}$ and $\frac{1}{n(n+1)(n+2)}$. Try to make an educated guess at the decomposition for a general k (recall Pascal's triangle!).

Solution. We saw the identity for $\frac{1}{n(n+1)}$ in class, but it should be derived properly using partial fractions. The partial fraction decomposition for linear terms in the denominator has the form

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1)+Bn}{n(n+1)} = \frac{(A+B)n+A}{n(n+1)}.$$

Now equate the polynomial terms in the numerators to find that $A = 1$ and $B = -1$. Thus

$$\boxed{\frac{\mathbf{1}}{\boldsymbol{n(n+1)}} = \frac{\mathbf{1}}{\boldsymbol{n}} - \frac{\mathbf{1}}{\boldsymbol{n+1}}}.$$

For the next case, there are three terms:

$$\begin{aligned}\frac{1}{n(n+1)(n+2)} &= \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2} = \frac{A(n+1)(n+2)+Bn(n+2)+Cn(n+1)}{n(n+1)(n+2)} \\ &= \frac{(A+B+C)n^2+(3A+2B+C)n+2A}{n(n+1)(n+2)}.\end{aligned}$$

The system of equations is then

$$\begin{aligned} A+B+C &= 0 \\ 3A+2B+C &= 0 \\ 2A &= 1, \end{aligned}$$

which has the solution $A = 1/2, B = -1, C = 1/2$. Thus the partial fraction decomposition is

$$\boxed{\frac{\mathbf{1}}{\boldsymbol{n(n+1)(n+2)}} = \frac{\mathbf{1/2}}{\boldsymbol{n}} - \frac{\mathbf{1}}{\boldsymbol{n+1}} + \frac{\mathbf{1/2}}{\boldsymbol{n+2}}}.$$

For the general situation, it is **not** a good idea to use partial fractions, and it is convenient to rewrite the above formula as

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right).$$

The general decompositions can then be found recursively. For example, the next case ($k = 3$) can be written as

$$\frac{1}{n(n+1)(n+2)(n+3)} = \frac{1}{3}\left(\frac{1}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}\right),$$

which can then be separated into individual terms by the $k = 2$ formula above, giving

$$\frac{1}{n(n+1)(n+2)(n+3)} = \frac{1}{6}\left(\frac{1}{n} - \frac{3}{n+1} + \frac{3}{n+2} - \frac{1}{n+3}\right).$$

Note that the numerators are exactly the numbers that appear in the 3rd row of Pascal's triangle, associated to $(x-1)^3 = \mathbf{1} \cdot x^3 - \mathbf{3} \cdot x^2 + \mathbf{3} \cdot x - \mathbf{1}$. These numbers are also known as the *binomial coefficients*, and the coefficient of x^j in $(x+1)^k$ is denoted by $\binom{k}{j}$. The general formula (which you were not required to get exactly right!) is

$$\boxed{\boldsymbol{\frac{1}{n(n+1)\cdots(n+k)} = \frac{1}{k!}\left(\frac{\binom{k}{0}}{n} - \frac{\binom{k}{1}}{n+1} + \cdots + (-1)^k \frac{\binom{k}{k}}{n+k}\right)}}.$$

b) Use a comparison to determine whether the sums converge or not for any given k.

Solution. The simplest comparison is to notice that $n(n+1)\cdots(n+k) > n^{k+1}$, and thus $\frac{1}{n(n+1)\cdots(n+k)} < \frac{1}{n^{k+1}}$. Thus

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)\cdots(n+k)} < \sum_{n=1}^{\infty} \frac{1}{n^{k+1}}.$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for any $p > 1$ (by the integral test and comparison with $f(x) = 1/x^p$); in this case we are only considering $p = k+1 \geq 2$, so $\boxed{\textbf{all sums are convergent}}$.

c) Use part a) and telescoping to evaluate the sums for $k = 1$ and $k = 2$. Guess at the formula for general k.

Solution. We saw in class how the first sum telescopes:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = \boxed{1},$$

since all terms after the first cancel out.

For the $k = 2$ case the telescoping is a bit more complicated;

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+1}\right)$$

$$= \frac{1}{2}\left[\left(\frac{1}{1} - 2\cdot\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{2} - 2\cdot\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{3} - 2\cdot\frac{1}{4} + \frac{1}{5}\right) + \left(\frac{1}{4} - 2\cdot\frac{1}{5} + \frac{1}{6}\right) + \dots\right].$$

Now group all terms with the same denominators (note that in a general infinite series the terms can **not** be rearranged, and it is very important that we already know that the series is a convergent telescoping sum; it would be more proper to truncate the sum at N and then take the limit as $N \to \infty$):

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left[\left(\frac{1}{1}\right) + \left(-2\cdot\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} - 2\cdot\frac{1}{3} + \frac{1}{3}\right) + \left(\frac{1}{4} - 2\cdot\frac{1}{4} + \frac{1}{4}\right) + \dots\right]$$
$$= \frac{1}{2}\left(1 - \frac{1}{2}\right) = \boxed{\mathbf{\frac{1}{4}}}.$$

The terms involving $\frac{1}{3}$ and all higher denominators canceled out, leaving only the initial terms (with denominators 1 or 2).

It is well beyond the scope of this course to show the (amazing) general formula

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)\cdots(n+k)} = \frac{1}{k\cdot k!}.$$

Problem 2. (*7 pts: 1+2+2+2*) *The Fly Problem.* Two trains are approaching each other on the same track, each traveling at 50mph. When they are exactly 100 miles apart, a fly begins traveling back and forth between them at 100mph, changing direction each time he hits one of the trains.

a) Calculate the time before the fly first turns around. How far did the fly travel during this time? How much distance remains between the two trains at this time?

Solution. The fly turns around when it meets the other train. The fly is traveling at 100mph, and the train at 50mph, so the distance between them is effectively diminishing at a speed of 150mph. The time until meeting is therefore $\frac{100}{150} = \boxed{\textbf{2/3 hrs}}$.

In this time period, the fly has traveled $\boxed{\textbf{200/3 mi}}$. Each train travels 100/3 mi, so the remaining distance is reduced to $100 - 2\cdot 100/3 = \boxed{\textbf{100/3 mi}}$.

b) Calculate the time between the fly's first direction change from part a) and the next direction change. How far does the fly travel during this time period?

Solution. After the fly changes direction, it now approaches the other train at 100mph, and the train travels at 50mph. Therefore the distance of 100/3 mi is covered in a time of $\frac{100/3}{150} = \boxed{\textbf{2/9 hrs}}$.

During this time the fly travels $\boxed{\textbf{200/9 mi}}$.

c) Identify a geometric series and evaluate to find the total distance traveled by the fly before the trains meet.

Solution. In the first segment, the fly traveled 200/3 mi; in the second segment, 200/9 mi, and similarly, the next segment is 200/27 mi (since after part b) the trains are 100/9 mi apart, and hence the fly travels for 2/27 hrs before the next meeting). Therefore the total distance traveled by the fly is

$$\frac{200}{3} + \frac{200}{9} + \frac{200}{27} + \dots = 200\left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots\right).$$

The general geometric series evaluation (for $a < 1$) is $a + a^2 + \cdots = \dfrac{a}{1-a}$, so the distance is

$$200 \cdot \frac{1/3}{1-1/3} = 200 \cdot \frac{1/3}{2/3} = \boxed{\textbf{100 mi}}\,.$$

d) Check your answer by doing the problem the "easy way": calculate the total time before the trains collide, and use the fly's constant velocity of 100mph to find the total distance traveled.

Solution. The two trains travel at 50mph, so the distance between them decreases at a total rate of 100mph. Thus they meet after exactly one hour. Regardless of direction changes, the fly travels at a constant speed of 100mph throughout the hour before collision, confirming that it travels a total of $\boxed{\textbf{100 mi}}$.