18.01 Final Review Problems

Note that this sheet covers only material from the final two weeks of lecture (after Exam 3).

1. (a) Differentiate the Taylor series $\frac{1}{1-x} = 1+x+x^2+\ldots$ and then set $x = 1/3$ in order to evaluate the infinite series

$$1 + 2\cdot\frac{1}{3} + 3\cdot\frac{1}{3^2} + 4\cdot\frac{1}{3^3} + \ldots.$$

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   (b) Write down the Taylor series for $f(x) = xe^x$. Differentiate this function and plug in $x = 1$ to find an infinite series that sums to e. Compare this with the simple formula $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$.

   (c) Recall that $\frac{d}{dx}\arctan x = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$. Integrate this equation and plug in $x = -1$ to find a formula for π (recall that tangent is an odd function).

2. Calculate the first several terms in the Taylor series for the following functions (at $a = 0$ unless stated otherwise). Write down the general formula if you can identify a pattern.

   (a) e^{-x}

   (b) $\ln(x)$ at $a = 1$.

   (c) $(1+x)^4$

   (d) $sin(x)$ at $a = \pi/2$. Compare with the Taylor series for $\cos(x)$.

   (e) $\frac{1}{\sqrt{1-x}}$.

3. Evaluate the following limits by using Taylor series.

   (a) $\lim_{x\to 0}\frac{\ln(1+x)}{\sin x}$.

   (b) $\lim_{x\to 0}\frac{e^{2x}-1}{x}$.

   (c) $\lim_{x\to 0}\frac{\tan^2(x)}{x}$.

4. (a) What is the sum of the series $1 - 2\cdot\frac{1}{3} + 3\cdot\frac{1}{3^2} - 4\cdot\frac{1}{3^2} + \ldots$ (recall 1(a))? How many terms in the series are necessary to come within 0.0001 of the true sum?

   (b) From 1(c), the finite sum

$$4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots + \frac{4}{41}$$

   is close to a very well-known number. What is the error in this approximation? Try checking this in your calculator!