18.01 Exam 3 Solutions

Problem 1. (*Short answer; 5 pts each*) Unless asked otherwise, you are not required to show detailed work for these questions, and need only give a brief explanation.

(a) If you were given the integral $\int \frac{1}{(4x^2-1)^3}\,dx$ to solve, what trigonometric substitution would be most appropriate?

Solution. The denominator is of the form $x^2 - a^2$, so a secant substitution is a good choice. In particular, if we set $\boxed{\boldsymbol{2x = \sec u}}$, then $4x^2 - 1 = \sec^2 u - 1 = \tan^2 u$, which is a helpful simplification.

(b) Set up (but do **not** completely solve) the partial fraction decomposition of $\frac{3x+1}{x^2(2x^2+1)}$.

Solution. The denominator has a linear term raised to a power and a single quadratic term. The partial fraction will have the form

$$\frac{3x+1}{x^2(2x^2+1)} = \boxed{\boldsymbol{\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{2x^2+1}}}.$$

The next steps would be to combine the right-hand terms over a common denominator, and then to solve for the constants by equating like polynomial coefficients.

(c) The integral $\int_{x=0}^{1} \frac{1}{(x-1)^2}\,dx$ is improper. Write down the limiting truncated integrals that you would need to calculate in order to determine whether it converges.

Solution. The integrand is undefined at $x = 1$, and thus must be calculated as the limit

$$\int_{x=0}^{1} \frac{1}{(x-1)^2}\,dx = \boxed{\boldsymbol{\lim_{a\to 1^-} \int_{x=0}^{a} \frac{1}{(x-1)^2}\,dx}}.$$

(d) Is $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ a convergent series?

Solution. Ignoring the sign, the terms $1/\sqrt{n+1}$ are decreasing toward 0 as $n \to \infty$. The series therefore satisfies the alternating test, so $\boxed{\textbf{it converges}}$.

Problem 2. (*20 pts: 10+10*)

(a) Calculate the anti-derivative $\int \cos^3 x\,dx$ using any method.

Solution. Since cosine occurs to an odd power, the easiest approach is to use the identity $\cos^2 x = 1 - \sin^2 x$. This gives

$$\int \cos^3 x\,dx = \int (1 - \sin^2 x)\cos x\,dx = \boxed{\boldsymbol{\sin x - \frac{\sin^3 x}{3}}}.$$

Integration by parts and/or a reduction formula are also valid techniques.

(b) Calculate the integral $\int_{x=0}^{1} x\sqrt{1-x^2}\,dx$ by making a trigonometric substitution.

Solution. The appropriate substitution to make is $x = \sin u$, with differential $dx = \cos u\,du$. The lower integration limit $x = 0$ corresponds to $u = 0$, and the upper limit $x = 1$ corresponds to $u = \pi/2$. Thus the integral is

$$\int_{x=0}^{1} x\sqrt{1-x^2}\,dx = \int_{u=0}^{\pi/2} \sin u\sqrt{1-\sin^2 u}\cos u\,du = \int_{u=0}^{\pi/2} \sin u\cos^2 u\,dx$$
$$= -\frac{\cos^3 u}{3}\Big|_0^{\pi/2} = 0 - -\frac{1}{3} = \boxed{\mathbf{\frac{1}{3}}}.$$

Problem 3. (*20 pts: 10 + 10*)

(a) Calculate the anti-derivative $\int \frac{1}{\sqrt{x^2+2x}}\,dx$ by completing the square in the denominator.

Solution. The integral becomes

$$\int \frac{1}{\sqrt{(x^2+2x+1)-1}}\,dx = \int \frac{1}{\sqrt{(x+1)^2-1}}\,dx.$$

Now make the trigonometric substitution $x+1 = \sec u$, with differential $dx = \sec u \tan u\,du$, obtaining

$$\int \frac{1}{\tan u}\sec u\tan u\,du = \int \sec u\,du = \ln(\sec u + \tan u).$$

Finally, use $u = \operatorname{arcsec}(x+1)$ to conclude that $\sec u = x+1$ and $\tan u = \sqrt{(x+1)^2-1}$. Thus

$$\int \frac{1}{\sqrt{x^2+2x}}\,dx = \boxed{\mathbf{\ln\left(x+1+\sqrt{x^2+2x}\right)}}.$$

(b) Calculate the anti-derivative $\int \frac{x+1}{x^2+2x}\,dx$ by finding the partial fraction decomposition of the integrand.

Solution. The denominator factors into linear terms $x(x+2)$, so the partial fraction decomposition has the form

$$\frac{x+1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} = \frac{A(x+2)+Bx}{x(x+2)} = \frac{(A+B)x+2A}{x(x+2)}.$$

Equating the numerators implies that $A = B = 1/2$.

The integral is thus

$$\int \frac{x+1}{x^2+2x}\,dx = \frac{1}{2}\int \frac{1}{x} + \frac{1}{x+2}\,dx = \frac{1}{2}(\ln x + \ln(x+2)) = \boxed{\mathbf{\ln\sqrt{x(x+2)}}}.$$

Problem 4. (*20 pts: 10+10*)

(a) Find the anti-derivative $\int x \sin x \, dx$.

Solution. Use integration by parts, with $u = x, dv = \sin x \, dx$, so $du = dx, v = -\cos x$. Now

$$\int x \sin x \, dx = \int u \, dv = uv - \int v \, du = -x\cos x + \int \cos x \, dx = \boxed{\boldsymbol{-x\cos x + \sin x}}.$$

(b) Evaluate the integral $\int\limits_{x=-\pi}^{\pi} x \cos x \, dx$.

Solution. Very similar to the above calculation, let $u = x, dv = \cos x \, dx$, so $du = dx, v = \sin x$. Then

$$\int\limits_{x=-\pi}^{\pi} x \cos x \, dx = \int\limits_{-\pi}^{\pi} u \, dv = uv \Big|_{-\pi}^{\pi} - \int\limits_{-\pi}^{\pi} v \, du$$

$$= x \sin x \Big|_{-\pi}^{\pi} - \int\limits_{-\pi}^{\pi} \sin x \, dx = 0 + \cos x \Big|_{-\pi}^{\pi} = \boxed{\mathbf{0}}.$$

Alternatively, observe that the integrand $x \cos x$ is an odd function, and the integration range is symmetric about 0, so the integral is automatically 0!

Problem 5. (*20 pts: 10 + 10*)

(a) Does the integral $\int\limits_{x=2}^{\infty} \frac{x}{x^3+1} \, dx$ converge?

Solution. Using the simple comparison $x^3 + 1 > x^3$, we have

$$\frac{x}{x^3+1} < \frac{x}{x^3} = \frac{1}{x^2}.$$

Thus the integral is bounded above by

$$\int\limits_{x=2}^{\infty} \frac{x}{x^3+1} \, dx < \int\limits_{x=2}^{\infty} \frac{1}{x^2} \, dx,$$

which $\boxed{\textbf{converges}}$ because the exponent on the denominator is larger than 1.

(b) Use part (a) to determine whether the series $\sum\limits_{n=2}^{\infty} \frac{n}{n^3+1}$ converges.

Solution. We know that the integral from part (a) converges, so it is natural to expect that the corresponding sum will too. The function $\frac{x}{x^3+1}$ is decreasing for $x \geq 1$. Therefore the integral comparison test applies, and states that

$$\sum_{n=2}^{\infty} \frac{n}{n^3+1} \leq \int_{x=1}^{\infty} \frac{x}{x^3+1}\, dx,$$

which also $\boxed{\textbf{converges}}$ because the integral from $x = 1$ to 2 makes a finite contribution.