18.01 Exam 2 Solutions

Problem 1. (*Short answer; 5 pts each*) Unless asked otherwise, you are not required to show detailed work for these questions, and need only give a brief explanation.

(a) Evaluate the integral $\int_{x=0}^{2} x^2\,dx$.

Solution.

$$\int_{x=0}^{2} x^2\,dx = \left.\frac{x^3}{3}\right|_{x=0}^{2} = \boxed{\mathbf{\frac{8}{3}}}.$$

(b) Write down or explain how to approximate the area under the curve $f(x) = 4x^3$ from $x = 1$ to 2 using rectangles.

Solution. Divide the interval into N equal segments of width $\Delta x = 1/N$. The i-th segment begins at the point $x_i = 1 + i \cdot \Delta x = 1 + i/N$. On this segment, take a rectangle of height $f(x_i)$. Thus the area is approximated by the Riemann sum

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = \boxed{\boldsymbol{\sum_{i=1}^{n} 4\left(1+\frac{i}{N}\right)^3 \cdot \frac{1}{N}}}.$$

(c) Find the derivative $\frac{d}{dt}\int_{x=1}^{t} x^2 + 7\,dx$.

Solution. By the second Fundamental Theorem of Calculus, the rate of change at the endpoint is given by the integrand. Thus

$$\frac{d}{dt}\int_{x=1}^{t} x^2 + 7\,dx = \boxed{\mathbf{t^2+7}}.$$

(d) Evaluate the integral $\int_{x=0}^{\pi/2} \left(\frac{d}{dx} e^{\sqrt{\sin(x)+\cos(x)}}\right) dx$.

Solution. By the first Fundamental Theorem of Calculus, the integral is evaluated by using an anti-derivative of the integrand. In this case, the anti-derivative is obvious:

$$\int_{x=0}^{\pi/2} \left(\frac{d}{dx} e^{\sqrt{\sin(x)+\cos(x)}}\right) dx = \left.\left(e^{\sqrt{\sin(x)+\cos(x)}}\right)\right|_{x=0}^{\pi/2} = e^{\sqrt{1+0}} - e^{\sqrt{0+1}} = \boxed{\mathbf{0}}.$$

(e) Find a function $f(x)$ such that $f'(x) = \sin(x)$ and $f(\pi) = 0$.

Solution. The general anti-derivative of $\sin(x)$ is $f(x) = -\cos(x) + C$ for an arbitrary constant C. However, the given point implies $0 = f(\pi) = -(-1) + C$, so $C = -1$. Thus

$$\boxed{\boldsymbol{f(x) = -\cos(x) - 1}}\,.$$

(f) Set up (but do **not** evaluate!) an integral to calculate the arc-length of $\ln(x)$ from $x = 1$ to 10.

Solution. The derivative of the function is $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$, so the arc-length is

$$\boxed{\boldsymbol{s = \int_{x=1}^{10} \sqrt{1 + \left(\frac{1}{x}\right)^2}\,dx}}\,.$$

Problem 2. (*15 pts: 5+5+5*)

(a) Evaluate the integral $\displaystyle\int_{x=0}^{1} \frac{e^x}{\sqrt{1+e^x}}\,dx$.

Solution. The substitution $u = e^x$ has differential $du = e^x\,dx = u\,dx$, and the limits $x = 0$ and 1 correspond to $u = 1$ and e, respectively. Therefore

$$\int_{x=0}^{1} \frac{e^x}{\sqrt{1+e^x}}\,dx = \int_{u=1}^{e} \frac{u}{\sqrt{1+u}}\,\frac{du}{u} = \int_{u=1}^{e} \frac{1}{\sqrt{1+u}}\,du$$
$$= 2\sqrt{1+u}\,\Big|_{u=1}^{e} = \boxed{\boldsymbol{2\sqrt{1+e} - 2\sqrt{2}}}\,.$$

(b) Find an anti-derivative of $f(x) = x\cos(x^2+1)$.

Solution. Since the derivative of $x^2 + 1$ is $2x$, it is natural to make the substitution $u = x^2 + 1$. This leads to the anti-derivative $\boxed{\boldsymbol{\dfrac{\sin(x^2+1)}{2}}}$.

(c) Find an anti-derivative of $f(x) = 2^x$.

Solution. The derivative of $f(x)$ is

$$\frac{d}{dx}2^x = \frac{d}{dx}e^{x\ln 2} = \ln 2 \cdot f(x).$$

An anti-derivative must therefore account for the constant $\ln 2$ introduced by differentiation, and is thus $\boxed{\boldsymbol{\dfrac{2^x}{\ln 2}}}$.

Problem 3. (*20 pts*) Find the solution to the differential equation

$$\frac{dy}{dx} = \frac{2y}{x}$$

that passes through the point $(1, 3)$.

Solution. Separate the variables in the differential equation to get

$$\frac{dy}{2y} = \frac{dx}{x}.$$

Taking the anti-derivative of both sides gives

$$\frac{\ln(2y)}{2} = \ln(x) + C$$

for some constant C. Note that the left side is equal to $\ln\sqrt{2y}$. Exponentiation then gives the general solution

$$\sqrt{2y} = e^C \cdot (x),$$

or equivalently (after squaring and dividing by 2),

$$y = e^{2C} \cdot x^2 = C' \cdot x^2,$$

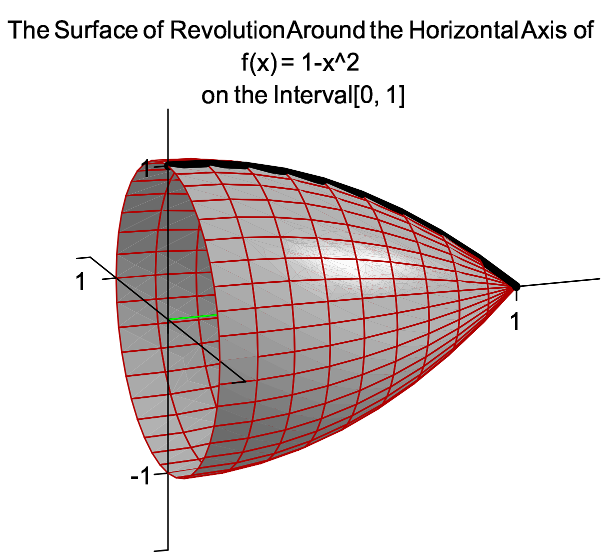
where C' is a (renamed) positive constant. For the given point, $3 = C' \cdot (1)^2 = C'$, so the solution is

$$\boxed{\boldsymbol{y = 3x^2}}.$$

Problem 4. (*15 pts: 10+5*) Consider the solid of revolution formed by rotating the curve $y = 1 - x^2$ around the x-axis between $x = 0$ and 1.

(a) Use the method of disks to write an integral that calculates the volume of the solid. Evaluate the integral.

Solution. The solid of revolution is a curved "conical" shape:



The disk method requires us to make slices along the axis of revolution. Specifically, the slice at an arbitrary x has radius $y = 1 - x^2$ and thickness dx, so the volume is

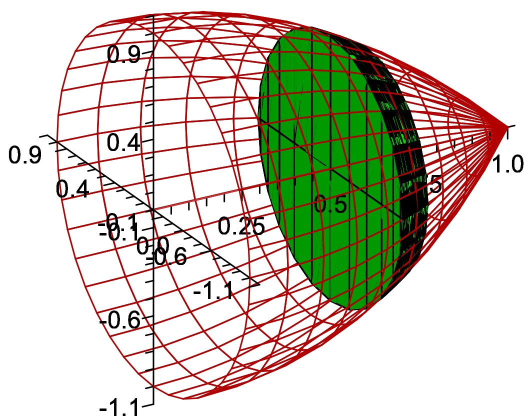
$dV = \pi y^2 \, dx = \pi(1 - x^2)^2 \, dx$. Thus the volume is

$$V = \int dV = \boxed{\boldsymbol{\int_{x=0}^{1} \pi(1 - x^2)^2 \, dx}}.$$

This integral evaluates as

$$V = \int_{x=0}^{1} \pi 1 - 2x^2 + x^4 \, dx = \pi \left(x - \frac{2x^3}{3} + \frac{x^5}{5} \right) \Big|_0^1 = \pi \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \boxed{\boldsymbol{\frac{8\pi}{15}}}.$$

Volume of Revolutionfor f(x) = 1-x^2



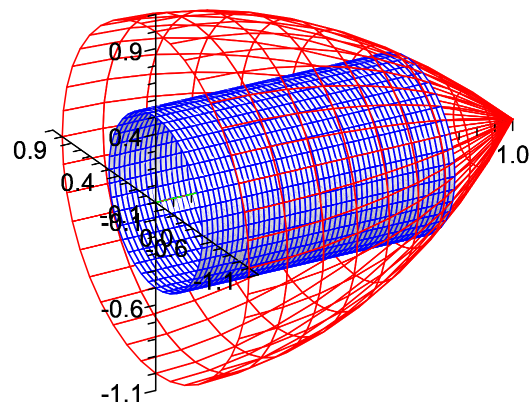
with a Disk

(b) Use the method of cylindrical shells to write down an integral that calculates the volume of the solid. Do **not** evaluate it.

Solution. To calculate the volume with the method of shells, pick an arbitrary height y, and create a cylindrical shell about the x-axis with infinitesimal thickness dy. The volume of such a shell is then $dV = 2\pi y \cdot x \cdot dy$, and we can solve for $x = \sqrt{1-y}$. Thus the total volume is the integral

$$V = \int dV = \boxed{\boldsymbol{\int_{y=0}^{1} 2\pi y \sqrt{1-y} \, dy}}.$$

Volume of Revolutionfor f(x) = 1-x^2



with a Cylindrical Shell

Problem 5. (*20 pts*) A 1000lb ice sculpture must be lifted to the top of a 200ft building on a hot summer day. If the elevator system rises 20ft/min and the ice sculpture loses 10lb/min due to melting, how much work total work is done in moving the sculpture?

Solution. Let h denote the height of the elevator, and let I denote the remaining weight of the sculpture. At an arbitrary height h, the amount of work required to move a small distance dh is

$$dW = I\,dh.$$

Furthermore, the weight is related to the height by the given rates. If the elevator has traveled h feet, then $h/20$ minutes have elapsed, and thus $-10 \cdot (h/20) = -h/2$ pounds of ice have melted. Hence $I = 1000 - h/2$.

Therefore the total work is

$$W = \int dW = \int_{h=0}^{200} \left(1000 - \frac{h}{2}\right) dh = \left(1000h - \frac{h^2}{4}\right)\Big|_0^{200}$$

$$= 200,000 - \frac{40,000}{4} = \boxed{\mathbf{190,000 ft\text{-}lbs}}\,.$$