18.01 Exam 1 Solutions

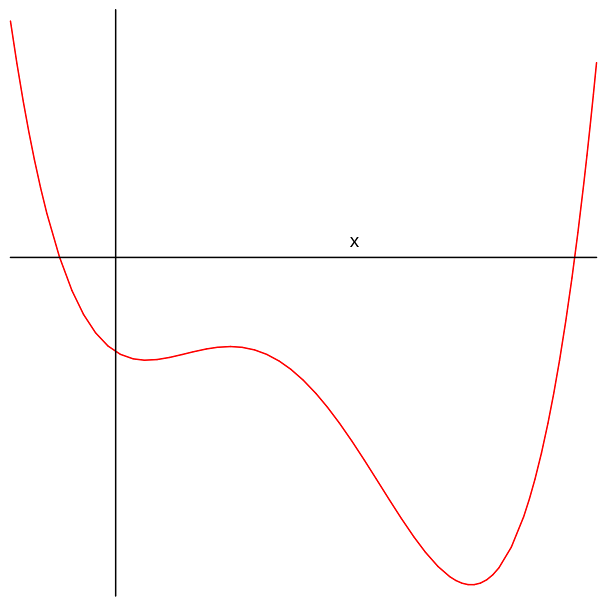
Problem 1. (*Short answer; 5 pts each*) Unless asked otherwise, you are not required to show detailed work for these questions, and need only give a brief explanation.

(a) Calculate the derivative of $f(x) = \dfrac{x^4 - x + 7}{x^2}$.

Solution. It's best to **not** use the quotient rule – instead, write each term as a power of x, so $f(x) = x^2 - x^{-1} + 7x^{-2}$. Then

$$f'(x) = 2x + x^{-2} - 14x^{-3} = \boxed{\mathbf{2x + \frac{1}{x^2} - \frac{14}{x^3}}}.$$

(b) Identify and **clearly** label the points on the following graph at which $f'(x) = 0$ and where $f''(x) = 0$.



Solution. Moving from left to right, there is a minima, an inflection point, a maxima, another inflection point, and a final minima.

(c) Evaluate $\lim\limits_{x \to 0} \dfrac{x+1}{\sqrt{x^2+1}}$.

Solution. This limit can be evaluated directly as $\dfrac{0+1}{\sqrt{0+1}} = \boxed{\mathbf{1}}$.

(d) Determine whether $\lim\limits_{x \to 1} \dfrac{x-1}{x^2-1}$ exists, and evaluate it if it does.

Solution. Direct substitution gives the indeterminate form $\frac{0}{0}$, but factorization gives

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = \lim_{x \to 1} \frac{1}{x+1} = \boxed{\mathbf{\frac{1}{2}}}.$$

Alternatively, L'Hospital's rule also gives $\lim\limits_{x \to 1} \dfrac{x-1}{x^2-1} = \lim\limits_{x \to 1} \dfrac{1}{2x} = \dfrac{1}{2}$.

(e) Calculate the derivative of $\cot(x)$ (if you do not make any calculations, then you must write down the derivatives of *all* trigonometric functions to receive full credit).

Solution.

$$\frac{d}{dx}\cot x = \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = -\frac{1}{\sin^2 x} = \boxed{-\csc^2 x}\,.$$

(f) A leaky faucet drips onto the floor and forms a circular puddle of constant thickness that grows at the rate of $10\text{cm}^2/\text{min}$. How quickly is the radius of the puddle growing when it has reached 5cm?

Solution. The area of the puddle is $A = \pi r^2$. Implicit differentiation then implies

$$\frac{dA}{dt} = 2\pi r\,\frac{dr}{dt},$$

so $\frac{dr}{dt} = \frac{1}{2\pi r}\cdot\frac{dA}{dt} = \frac{10}{2\pi r}$. Thus the desired rate is

$$\frac{dr}{dt} = \frac{10}{2\pi\cdot 5} = \boxed{\frac{1}{\pi}}\,.$$

(g) Calculate $\frac{d^3}{dx^3}\left(\sin(2x) - x^2\right)$.

Solution. Basic rules of differentiation give $\boxed{-8\cos 2x}$.

(h) Evaluate $\lim_{x\to\infty} x^{1/x}$.

Solution. The continuity of the exponential implies that

$$\lim_{x\to\infty} x^{1/x} = e^{\lim_{x\to\infty}\ln(x^{1/x})} = e^{\lim_{x\to\infty} 1/x\cdot\ln(x)},$$

and L'Hospital's rule gives $\lim_{x\to\infty}\frac{1}{x}\ln x = \lim_{x\to\infty}\frac{1/x}{1} = 0$, so the overall limit is $e^0 = \boxed{1}$.

Problem 2. (*15 pts: 5+5+5*)

(a) Calculate the derivative of $f(x) = \sqrt{x^2 + \cos^2(x)}$.

Solution.

$$f'(x) = \frac{1}{2\sqrt{x^2+\cos^2(x)}}\cdot(2x + 2\cos x\cdot(-\sin x)) = \boxed{\frac{x - \cos x\sin x}{\sqrt{x^2+\cos^2(x)}}}\,.$$

(b) Calculate the derivative of $f(x) = \frac{1}{\ln x}$.

Solution.

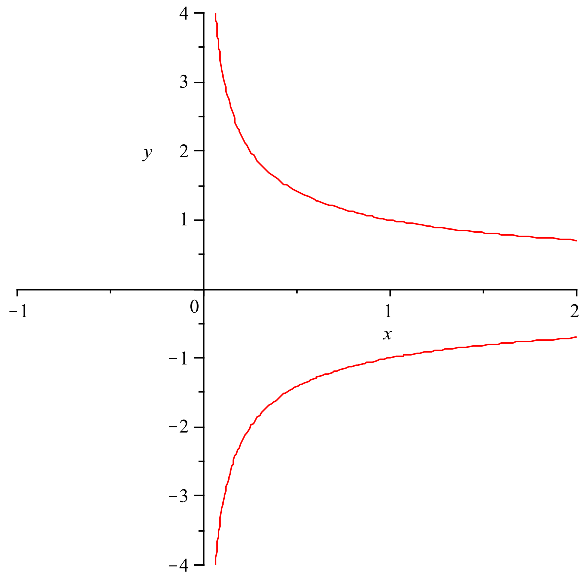
$$f'(x) = -\frac{1}{(\ln x)^2}\cdot\frac{1}{x} = \boxed{-\frac{1}{x(\ln x)^2}}\,.$$

(c) Calculate the derivative of $f(x) = 2^{3^x}$.

Solution.

$$f'(x) = \frac{d}{dx} e^{3^x \ln 2} = e^{3^x \ln 2} \cdot \frac{d}{dx}(\ln 2 \cdot 3^x) = \boxed{\ln(2) \cdot \ln(3) 3^x \cdot 2^{3^x}}.$$

Problem 3. (*10 pts: 2 + 8*) Consider the curve defined implicitly by $xy^2 = 1$.
(a) Plug in a few points and try to sketch the curve.



(b) Find all points where the tangent slope is $-1/2$.

Solution. Implicit differentiation yields $dx\, y^2 + 2xy\, dy = 0$, and thus the tangent slope is

$$\frac{dy}{dx} = -\frac{y^2}{2xy} = -\frac{y}{2x}.$$

Setting this equal to $-1/2$ gives $y = x$, and thus the corresponding points on the curve satisfy $x \cdot x^2 = 1$, so $x = 1$. Plugging this in to the curve equation gives $y = \pm 1$, but since the tangent slope is $-y/2x$, the only point with the correct slope is $\boxed{(1, 1)}$.

Problem 4. (*15 pts: 5+10*) It costs a box manufacturer 20 cents per linear inch to construct a box. In other words, a box of width w, length ℓ and height h (all in inches) costs $20(w + \ell + h)$. The manufacturer is then able to sell each box at 5 cents per cubic inch of volume.

(a) If the manufacturer makes a cubic box of side length x, how much money do they earn/lose with each sale?

Solution. The cost is $20 \cdot 3x = 60x$, and the sale price is $5x^3$, so the total loss is cost − sale = $\boxed{60x - 5x^3}$ (equivalently, the net earnings per box is $5x^3 - 60x$).

(b) What is the **worst** choice of x (the value that maximizes the loss per box)?

Solution. To find a minima, calculate the derivative of the net loss and find the roots; this means that $60 - 15x^2 = 0$, so $4 - x^2 = 0$, which has positive solution $x = 2$. At this value, the loss is $120 - 40 = 80$ cents. We also must check the boundary $x = 0$; at that price the loss is 0. So the global maximum loss occurs at $\boxed{\boldsymbol{x = 2}}$.

Problem 5. (*20 pts: 10+10*)

(a) Calculate $\lim_{x\to\infty} x^n e^{-x^2}$ for any integer n (positive or negative).

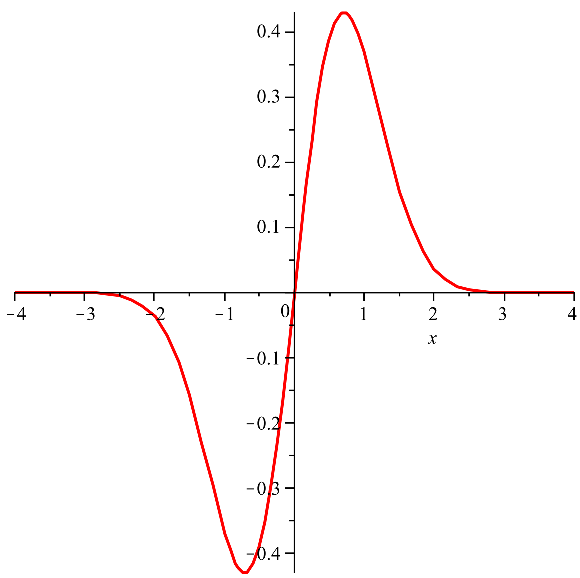
Solution. If $n \leq 0$, the limit may be evaluated directly as $\boxed{\mathbf{0}}$.

For positive n, L'Hospital's rule implies that

$$\lim_{x\to\infty} \frac{x^n}{e^{x^2}} = \lim_{x\to\infty} \frac{nx^{n-1}}{2xe^{x^2}} = \lim_{x\to\infty} \frac{nx^{n-2}}{2e^{x^2}}.$$

After repeating a finite number of times, the exponent on the numerator is less than equal to zero, so the limit is again $\boxed{\mathbf{0}}$.

(b) Sketch the graph of $f(x) = xe^{-x^2}$ using any techniques you like. For full credit, identify maxima and minima. If you think there are any inflection points, then label them, but you **do not** need to find them explicitly.



Solution. The derivative is $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2}$, so the critical points are $x = \pm 1/\sqrt{2}$. Furthermore, $f(x)$ is an odd function that is positive for positive x, and negative for negative x. There are also two inflection points, but these are much more difficult to find exactly.