

# Derived Algebraic Geometry I: Stable $\infty$ -Categories

October 8, 2009

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Stable <math>\infty</math>-Categories</b>	<b>3</b>
<b>3</b>	<b>The Homotopy Category of a Stable <math>\infty</math>-Category</b>	<b>6</b>
<b>4</b>	<b>Properties of Stable <math>\infty</math>-Categories</b>	<b>12</b>
<b>5</b>	<b>Exact Functors</b>	<b>14</b>
<b>6</b>	<b>t-Structures and Localizations</b>	<b>17</b>
<b>7</b>	<b>Boundedness and Completeness</b>	<b>22</b>
<b>8</b>	<b>Stabilization</b>	<b>25</b>
<b>9</b>	<b>The <math>\infty</math>-Category of Spectra</b>	<b>34</b>
<b>10</b>	<b>Excisive Functors</b>	<b>38</b>
<b>11</b>	<b>Filtered Objects and Spectral Sequences</b>	<b>42</b>
<b>12</b>	<b>The <math>\infty</math>-Categorical Dold-Kan Correspondence</b>	<b>48</b>
<b>13</b>	<b>Homological Algebra</b>	<b>53</b>
<b>14</b>	<b>The Universal Property of <math>\mathcal{D}^-(\mathcal{A})</math></b>	<b>56</b>
<b>15</b>	<b>Presentable Stable <math>\infty</math>-Categories</b>	<b>62</b>
<b>16</b>	<b>Accessible t-Structures</b>	<b>68</b>

# 1 Introduction

There is a very useful analogy between topological spaces and chain complexes with values in an abelian category. For example, it is customary to speak of *homotopies* between chain maps, *contractible* complexes, and so forth. The analogue of the homotopy category of topological spaces is the *derived category* of an abelian category  $\mathcal{A}$ , a triangulated category which provides a good setting for many constructions in homological algebra. However, it has long been recognized that for many purposes the derived category is too crude: it identifies homotopic morphisms of chain complexes without remembering *why* they are homotopic. It is possible to correct this defect by viewing the derived category as the homotopy category of an underlying  $\infty$ -category  $\mathcal{D}(\mathcal{A})$ . The  $\infty$ -categories which arise in this way have special features that reflect their “additive” origins: they are *stable*.

The goal of this paper is to provide an introduction to the theory of stable  $\infty$ -categories. We will begin in §2 by introducing the definition of stability and some other basic terminology. In many ways, an arbitrary stable  $\infty$ -category  $\mathcal{C}$  behaves like the derived category of an abelian category: in particular, we will see in §3 that for every stable  $\infty$ -category  $\mathcal{C}$ , the homotopy category  $\mathrm{h}\mathcal{C}$  is triangulated (Theorem 3.11). In §4 we will establish some other simple consequences of stability; for example, stable  $\infty$ -categories admit finite limits and colimits (Proposition 4.4).

The appropriate notion of *functor* between stable  $\infty$ -categories is an *exact* functor: that is, a functor which preserves finite colimits (or equivalently, finite limits: see Proposition 5.1). The collection of stable  $\infty$ -categories and exact functors between them can be organized into an  $\infty$ -category, which we will denote by  $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ . In §5, we will study the  $\infty$ -category  $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ ; in particular, we will show that it is stable under limits and filtered colimits in  $\mathrm{Cat}_\infty$ . The formation of limits in  $\mathrm{Cat}_\infty^{\mathrm{Ex}}$  provides a tool for addressing the classical problem of “gluing in the derived category”.

In §6, we will review the theory of t-structures on triangulated categories. We will see that, if  $\mathcal{C}$  is a stable  $\infty$ -category, there is a close relationship between t-structures on the homotopy category  $\mathrm{h}\mathcal{C}$  and localizations of  $\mathcal{C}$ . We will revisit this subject in §16, where we show that, under suitable set-theoretic hypotheses (to be described in §15), we can construct a t-structure “generated” by an arbitrary collection of objects of  $\mathcal{C}$ .

The most important example of a stable  $\infty$ -category is the  $\infty$ -category  $\mathrm{Sp}$  of *spectra*. The homotopy category of  $\mathrm{Sp}$  can be identified with the classical *stable homotopy category*. There are many approaches to the construction of  $\mathrm{Sp}$ . In §9 we will adopt the most classical perspective: we begin by constructing an  $\infty$ -category  $\mathcal{S}_\infty^{\mathrm{fin}}$  of *finite spectra*, obtained from the  $\infty$ -category of finite pointed spaces by formally inverting the suspension functor. The stability of  $\mathcal{S}_\infty^{\mathrm{fin}}$  follows from the classical *homotopy excision theorem*. We can then define the  $\infty$ -category  $\mathrm{Sp}$  as the  $\infty$ -category of Ind-objects of  $\mathcal{S}_\infty^{\mathrm{fin}}$ . The stability of  $\mathrm{Sp}$  follows from a general result on Ind-objects (Proposition 4.5).

There is another description of the  $\infty$ -category  $\mathrm{Sp}$  which is perhaps more familiar: it can be viewed as the  $\infty$ -category of *infinite loop spaces*, obtained from the  $\infty$ -category  $\mathcal{S}_*$  of pointed spaces by formally inverting the loop functor. More generally, one can begin with an arbitrary  $\infty$ -category  $\mathcal{C}$ , and construct a new  $\infty$ -category  $\mathrm{Stab}(\mathcal{C})$  of *infinite loop objects of  $\mathcal{C}$* . The  $\infty$ -category  $\mathrm{Stab}(\mathcal{C})$  can be regarded as universal among stable  $\infty$ -categories which admits a left exact functor to  $\mathcal{C}$  (Proposition 10.12). This leads to a characterization of  $\mathrm{Sp}$  by a mapping property: namely,  $\mathrm{Sp}$  is freely generated under colimits (as a stable  $\infty$ -category) by a single object, the *sphere spectrum* (Corollary 15.6).

A classical result of Dold and Kan asserts that, if  $\mathcal{A}$  is an abelian category, then the category of simplicial objects in  $\mathcal{A}$  is equivalent to the category of nonnegatively graded chain complexes in  $\mathcal{A}$ . In §12, we will formulate and prove an  $\infty$ -categorical version of this result, where the abelian category  $\mathcal{A}$  is replaced by a stable  $\infty$ -category. Here we must replace the notion of “chain complex” by the related notion “filtered object”. If  $\mathcal{C}$  is a stable  $\infty$ -category equipped with a t-structure, then every filtered object of  $\mathcal{C}$  determines a spectral sequence; we will give the details of this construction in §11.

In §13, we will return to the subject of homological algebra. We will explain how to pass from a suitable abelian category  $\mathcal{A}$  to a stable  $\infty$ -category  $\mathcal{D}^-(\mathcal{A})$ , which we will call the *derived  $\infty$ -category of  $\mathcal{A}$* . The homotopy category of  $\mathcal{D}^-(\mathcal{A})$  can be identified with the classical derived category of  $\mathcal{A}$ .

Our final goal in this paper is to characterize  $\mathcal{D}^-(\mathcal{A})$  by a universal mapping property. In §14, we will

show that  $\mathcal{D}^-(\mathcal{A})$  is universal among stable  $\infty$ -categories equipped with a suitable embedding of the ordinary category  $\mathcal{A}$  (Corollary 14.13).

The theory of stable  $\infty$ -categories is not really new: most of the results presented here are well-known to experts. There exists a sizable literature on the subject in the setting of *stable model categories* (see, for example, [27]). The theory of stable model categories is essentially equivalent to the notion of a *presentable* stable  $\infty$ -category, which we discuss in §15. For a brief account in the more flexible setting of Segal categories, we refer the reader to [72].

In this paper, we will use the language of  *$\infty$ -categories* (also called quasicategories or weak Kan complexes), as described in [40]. We will use the letter T to indicate references to [40]. For example, Theorem T.6.1.0.6 refers to Theorem 6.1.0.6 of [40].

## 2 Stable $\infty$ -Categories

In this section, we will introduce our main object of study: stable  $\infty$ -categories. We begin with a brief review of some ideas from §T.7.2.2.

**Definition 2.1.** Let  $\mathcal{C}$  be an  $\infty$ -category. A *zero object* of  $\mathcal{C}$  is an object which is both initial and final. We will say that  $\mathcal{C}$  is *pointed* if it contains a zero object.

In other words, an object  $0 \in \mathcal{C}$  is zero if the spaces  $\mathrm{Map}_{\mathcal{C}}(X, 0)$  and  $\mathrm{Map}_{\mathcal{C}}(0, X)$  are both contractible for every object  $X \in \mathcal{C}$ . Note that if  $\mathcal{C}$  contains a zero object, then that object is determined up to equivalence. More precisely, the full subcategory of  $\mathcal{C}$  spanned by the zero objects is a contractible Kan complex (Proposition T.1.2.12.9).

**Remark 2.2.** Let  $\mathcal{C}$  be an  $\infty$ -category. Then  $\mathcal{C}$  is pointed if and only if the following conditions are satisfied:

- (1) The  $\infty$ -category  $\mathcal{C}$  has an initial object  $\emptyset$ .
- (2) The  $\infty$ -category  $\mathcal{C}$  has a final object  $1$ .
- (3) There exists a morphism  $f : 1 \rightarrow \emptyset$  in  $\mathcal{C}$ .

The “only if” direction is obvious. For the converse, let us suppose that (1), (2), and (3) are satisfied. We invoke the assumption that  $\emptyset$  is initial to deduce the existence of a morphism  $g : \emptyset \rightarrow 1$ . Because  $\emptyset$  is initial,  $f \circ g \simeq \mathrm{id}_{\emptyset}$ , and because  $1$  is final,  $g \circ f \simeq \mathrm{id}_1$ . Thus  $g$  is a homotopy inverse to  $f$ , so that  $f$  is an equivalence. It follows that  $\emptyset$  is also a final object of  $\mathcal{C}$ , so that  $\mathcal{C}$  is pointed.

**Remark 2.3.** Let  $\mathcal{C}$  be an  $\infty$ -category with a zero object  $0$ . For any  $X, Y \in \mathcal{C}$ , the natural map

$$\mathrm{Map}_{\mathcal{C}}(X, 0) \times \mathrm{Map}_{\mathcal{C}}(0, Y) \rightarrow \mathrm{Map}_{\mathcal{C}}(X, Y)$$

has contractible source. We therefore obtain a well defined morphism  $X \rightarrow Y$  in the homotopy category  $\mathrm{h}\mathcal{C}$ , which we will refer to as the *zero morphism* and also denote by  $0$ .

**Definition 2.4.** Let  $\mathcal{C}$  be a pointed  $\infty$ -category. A *triangle* in  $\mathcal{C}$  is a diagram  $\Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$ , depicted as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

where  $0$  is a zero object of  $\mathcal{C}$ . We will say that a triangle in  $\mathcal{C}$  is *exact* if it is a pullback square, and *coexact* if it is a pushout square.

**Remark 2.5.** Let  $\mathcal{C}$  be a pointed  $\infty$ -category. A triangle in  $\mathcal{C}$  consists of the following data:

- (1) A pair of morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  in  $\mathcal{C}$ .
- (2) A 2-simplex in  $\mathcal{C}$  corresponding to a diagram

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow g \\ X & \xrightarrow{h} & Z \end{array}$$

in  $\mathcal{C}$ , which identifies  $h$  with the composition  $g \circ f$ .

- (3) A 2-simplex

$$\begin{array}{ccc} & 0 & \\ \nearrow & & \searrow \\ X & \xrightarrow{h} & Z \end{array}$$

in  $\mathcal{C}$ , which we may view as a *nullhomotopy* of  $h$ .

We will sometimes indicate a triangle by specifying only the pair of maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

with the data of (2) and (3) being implicitly assumed.

**Definition 2.6.** Let  $\mathcal{C}$  be a pointed  $\infty$ -category containing a morphism  $g : X \rightarrow Y$ . A *kernel* of  $g$  is an exact triangle

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Y. \end{array}$$

Dually, a *cokernel* for  $g$  is a coexact triangle

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z. \end{array}$$

We will sometimes abuse terminology by simply referring to  $W$  and  $Z$  as the kernel and cokernel of  $g$ . We will also write  $W = \ker(g)$  and  $Z = \operatorname{coker}(g)$ .

**Remark 2.7.** Let  $\mathcal{C}$  be a pointed  $\infty$ -category containing a morphism  $f : X \rightarrow Y$ . A cokernel of  $f$ , if it exists, is uniquely determined up to equivalence. More precisely, consider the full subcategory  $\mathcal{E} \subseteq \operatorname{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$  spanned by the coexact triangles. Let  $\theta : \mathcal{E} \rightarrow \operatorname{Fun}(\Delta^1, \mathcal{C})$  be the forgetful functor, which associates to a diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

the morphism  $g : X \rightarrow Y$ . Applying Proposition T.4.3.2.15 twice, we deduce that  $\theta$  is a Kan fibration, whose fibers are either empty or contractible (depending on whether or not a morphism  $g : X \rightarrow Y$  in  $\mathcal{C}$  admits a cokernel). In particular, if every morphism in  $\mathcal{C}$  admits a cokernel, then  $\theta$  is a trivial Kan fibration, and

therefore admits a section  $\text{coker} : \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \text{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ , which is well defined up to a contractible space of choices. We will often abuse notation by also letting  $\text{coker} : \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$  denote the composition

$$\text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \text{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}) \rightarrow \mathcal{C},$$

where the second map is given by evaluation at the final object of  $\Delta^1 \times \Delta^1$ .

**Remark 2.8.** The functor  $\text{coker} : \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$  can be identified with a left adjoint to the left Kan extension functor  $\mathcal{C} \simeq \text{Fun}(\{1\}, \mathcal{C}) \rightarrow \text{Fun}(\Delta^1, \mathcal{C})$ , which associates to each object  $X \in \mathcal{C}$  a zero morphism  $0 \rightarrow X$ . It follows that  $\text{coker}$  preserves all colimits which exist in  $\text{Fun}(\Delta^1, \mathcal{C})$  (Proposition T.5.2.3.5).

**Definition 2.9.** An  $\infty$ -category  $\mathcal{C}$  is *stable* if it satisfies the following conditions:

- (1) There exists a zero object  $0 \in \mathcal{C}$ .
- (2) Every morphism in  $\mathcal{C}$  admits a kernel and a cokernel.
- (3) A triangle in  $\mathcal{C}$  is exact if and only if it is coexact.

**Remark 2.10.** Condition (3) of Definition 2.9 is analogous to the axiom for abelian categories which requires that the image of a morphism be isomorphic to its coimage.

**Example 2.11.** Recall that a *spectrum* consists of an infinite sequence of pointed topological spaces  $\{X_i\}_{i \geq 0}$ , together with homeomorphisms  $X_i \simeq \Omega X_{i+1}$ , where  $\Omega$  denotes the loop space functor. The collection of spectra can be organized into a stable  $\infty$ -category  $\text{Sp}$ . Moreover,  $\text{Sp}$  is in some sense the universal example of a stable  $\infty$ -category. This motivates the terminology of Definition 2.9: an  $\infty$ -category  $\mathcal{C}$  is stable if it resembles the  $\infty$ -category  $\text{Sp}$ , whose homotopy category  $\text{hSp}$  can be identified with the classical *stable homotopy category*. We will return to the theory of spectra (using a slightly different definition) in §9.

**Example 2.12.** Let  $\mathcal{A}$  be an abelian category. Under mild hypotheses, we can construct a stable  $\infty$ -category  $\mathcal{D}(\mathcal{A})$  whose homotopy category  $\text{h}\mathcal{D}(\mathcal{A})$  can be identified with the *derived category of  $\mathcal{A}$* , in the sense of classical homological algebra. We will outline the construction of  $\mathcal{D}(\mathcal{A})$  in §13.

**Remark 2.13.** If  $\mathcal{C}$  is a stable  $\infty$ -category, then the opposite  $\infty$ -category  $\mathcal{C}^{op}$  is also stable.

**Remark 2.14.** One attractive feature of the theory of stable  $\infty$ -categories is that stability is a property of  $\infty$ -categories, rather than additional data. The situation for additive categories is similar. Although additive categories are often presented as categories equipped with additional structure (an abelian group structure on all  $\text{Hom}$ -sets), this additional structure is in fact determined by the underlying category. If a category  $\mathcal{C}$  has a zero object, finite sums, and finite products, then there always exists a unique map  $A \oplus B \rightarrow A \times B$  which can be described by the matrix

$$\begin{bmatrix} \text{id}_A & 0 \\ 0 & \text{id}_B \end{bmatrix}.$$

If this morphism has an inverse  $\phi_{A,B}$ , then we may define a sum of two morphisms  $f, g : X \rightarrow Y$  by defining  $f+g$  to be the composition  $X \rightarrow X \times X \xrightarrow{f,g} Y \times Y \xrightarrow{\phi_{Y,Y}} Y \oplus Y \rightarrow Y$ . This definition endows each morphism set  $\text{Hom}_{\mathcal{C}}(X, Y)$  with the structure of a commutative monoid. If each  $\text{Hom}_{\mathcal{C}}(X, Y)$  is actually a group (in other words, if every morphism  $f : X \rightarrow Y$  has an additive inverse), then  $\mathcal{C}$  is an additive category. This statement has an analogue in the setting of stable  $\infty$ -categories: any stable  $\infty$ -category  $\mathcal{C}$  is automatically enriched over the  $\infty$ -category of spectra. Since we do not wish to develop the language of enriched  $\infty$ -categories, we will not pursue this point further.

### 3 The Homotopy Category of a Stable $\infty$ -Category

Our goal in this section is to show that if  $\mathcal{C}$  is a stable  $\infty$ -category, then the homotopy category  $\mathrm{h}\mathcal{C}$  is triangulated (Theorem 3.11). We begin by reviewing the definition of a triangulated category.

**Definition 3.1** (Verdier). A *triangulated category* consists of the following data:

- (1) An additive category  $\mathcal{D}$ .
- (2) A translation functor

$$\begin{aligned} \mathcal{D} &\rightarrow \mathcal{D} \\ X &\mapsto X[1], \end{aligned}$$

which is an equivalence of categories.

- (3) A collection of *distinguished triangles*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

These data are required to satisfy the following axioms:

- (TR1) (a) Every morphism  $f : X \rightarrow Y$  in  $\mathcal{D}$  can be extended to distinguished triangle in  $\mathcal{D}$ .
- (b) The collection of distinguished triangles is stable under isomorphism.
- (c) Given an object  $X \in \mathcal{D}$ , the diagram

$$X \xrightarrow{\mathrm{id}_X} X \rightarrow 0 \rightarrow X[1]$$

is a distinguished triangle.

- (TR2) A diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle if and only if the rotated diagram

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is a distinguished triangle.

- (TR3) Given a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow f & & \downarrow & & \downarrow \text{ } & & \downarrow f[1] \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

in which both horizontal rows are distinguished triangles, there exists a dotted arrow rendering the entire diagram commutative.

- (TR4) Suppose given three distinguished triangles

$$\begin{aligned} X &\xrightarrow{f} Y \xrightarrow{u} Y/X \xrightarrow{d} X[1] \\ Y &\xrightarrow{g} Z \xrightarrow{v} Z/Y \xrightarrow{d'} Y[1] \end{aligned}$$



**Remark 3.5.** If the  $\infty$ -category  $\mathcal{C}$  is pointed but not necessarily stable, the suspension and loop space functors need not be homotopy inverses but are nevertheless *adjoint* to one another (provided that both functors are defined).

If  $\mathcal{C}$  is a pointed  $\infty$ -category containing a pair of objects  $X$  and  $Y$ , then the space  $\mathrm{Map}_{\mathcal{C}}(X, Y)$  has a natural base point, given by the zero map. Moreover, if  $\mathcal{C}$  admits cokernels, then the suspension functor  $\Sigma_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$  is essentially characterized by the existence of natural homotopy equivalences

$$\mathrm{Map}_{\mathcal{C}}(\Sigma(X), Y) \rightarrow \Omega \mathrm{Map}_{\mathcal{C}}(X, Y).$$

In particular, we conclude that  $\pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma(X), Y) \simeq \pi_1 \mathrm{Map}_{\mathcal{C}}(X, Y)$ , so that  $\pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma(X), Y)$  has the structure of a group (here the fundamental group of  $\mathrm{Map}_{\mathcal{C}}(X, Y)$  is taken with base point given by the zero map). Similarly,  $\pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma^2(X), Y) \simeq \pi_2 \mathrm{Map}_{\mathcal{C}}(X, Y)$  has the structure of an *abelian group*. If the suspension functor  $X \mapsto \Sigma(X)$  is an equivalence of  $\infty$ -categories, then for every  $Z \in \mathcal{C}$  we can choose  $X$  such that  $\Sigma^2(X) \simeq Z$  to deduce the existence of an abelian group structure on  $\mathrm{Map}_{\mathcal{C}}(Z, Y)$ . It is easy to see that this group structure depends functorially on  $Z, Y \in \mathrm{h}\mathcal{C}$ . We are therefore most of the way to proving the following result:

**Lemma 3.6.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits cokernels, and suppose that the suspension functor  $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$  is an equivalence. Then  $\mathrm{h}\mathcal{C}$  is an additive category.*

*Proof.* The argument sketched above shows that  $\mathrm{h}\mathcal{C}$  is (canonically) enriched over the category of abelian groups. It will therefore suffice to prove that  $\mathrm{h}\mathcal{C}$  admits finite coproducts. We will prove a slightly stronger statement: the  $\infty$ -category  $\mathcal{C}$  itself admits finite coproducts. Since  $\mathcal{C}$  has an initial object, it will suffice to treat the case of pairwise coproducts. Let  $X, Y \in \mathcal{C}$ , and let  $\mathrm{coker} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$  be a cokernel functor, so that we have equivalences  $X \simeq \mathrm{coker}(X[-1] \xrightarrow{u} 0)$  and  $Y \simeq \mathrm{coker}(0 \xrightarrow{v} Y)$ . Proposition T.5.1.2.2 implies that  $u$  and  $v$  admit a coproduct in  $\mathrm{Fun}(\Delta^1, \mathcal{C})$  (namely, the zero map  $X[-1] \xrightarrow{0} Y$ ). Since the functor  $\mathrm{coker}$  preserves coproducts (Remark 2.8), we conclude that  $X$  and  $Y$  admit a coproduct (which can be constructed as the cokernel of the zero map from  $X[-1]$  to  $Y$ ).  $\square$

Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits cokernels. By construction, any diagram

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & Y \end{array}$$

which belongs to  $\mathcal{M}$  determines a canonical isomorphism  $X[1] \rightarrow Y$  in the homotopy category  $\mathrm{h}\mathcal{C}$ . We will need the following observation:

**Lemma 3.7.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits cokernels, and let*

$$\begin{array}{ccc} X & \xrightarrow{f} & 0 \\ \downarrow f' & & \downarrow \\ 0' & \longrightarrow & Y \end{array}$$

*be a diagram in  $\mathcal{C}$ , classifying a morphism  $\theta \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[1], Y)$ . (Here  $0$  and  $0'$  are zero objects of  $\mathcal{C}$ .) Then the transposed diagram*

$$\begin{array}{ccc} X & \xrightarrow{f'} & 0' \\ \downarrow f & & \downarrow \\ 0 & \longrightarrow & Y \end{array}$$

*classifies the morphism  $-\theta \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[1], Y)$ . Here  $-\theta$  denotes the inverse of  $\theta$  with respect to the group structure on  $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[1], Y) \simeq \pi_1 \mathrm{Map}_{\mathcal{C}}(X, Y)$ .*



*Proof.* Without loss of generality, we may suppose that  $0 = 0'$  and  $f = f'$ . Let  $\sigma : \Lambda_0^2 \rightarrow \mathcal{C}$  be the diagram

$$0 \xleftarrow{f} X \xrightarrow{f} 0.$$

For every diagram  $p : K \rightarrow \mathcal{C}$ , let  $\mathcal{D}(p)$  denote the Kan complex  $\mathcal{C}_{p/} \times_{\mathcal{C}} \{Y\}$ . Then  $\text{Hom}_{\text{h}\mathcal{C}}(X[1], Y) \simeq \pi_0 \mathcal{D}(\sigma)$ . We note that

$$\mathcal{D}(\sigma) \simeq \mathcal{D}(f) \times_{\mathcal{D}(X)} \mathcal{D}(f).$$

Since  $0$  is an initial object of  $\mathcal{C}$ ,  $\mathcal{D}(f)$  is contractible. In particular, there exists a point  $q \in \mathcal{D}(f)$ . Let

$$\mathcal{D}' = \mathcal{D}(f) \times_{\text{Fun}(\{0\}, \mathcal{D}(X))} \text{Fun}(\Delta^1, \mathcal{D}(X)) \times_{\text{Fun}(\{1\}, \mathcal{D}(X))} \mathcal{D}(f)$$

$$\mathcal{D}'' = \{q\} \times_{\text{Fun}(\{0\}, \mathcal{D}(X))} \text{Fun}(\Delta^1, \mathcal{D}(X)) \times_{\text{Fun}(\{1\}, \mathcal{D}(X))} \{q\}$$

so that we have canonical inclusions

$$\mathcal{D}'' \hookrightarrow \mathcal{D}' \hookrightarrow \mathcal{D}(\sigma).$$

The left map is a homotopy equivalence because  $\mathcal{D}(f)$  is contractible, and the right map is a homotopy equivalence because the projection  $\mathcal{D}(f) \rightarrow \mathcal{D}(X)$  is a Kan fibration. We observe that  $\mathcal{D}''$  can be identified with the simplicial loop space of  $\text{Hom}_{\mathcal{C}}^L(X, Y)$  (taken with the base point determined by  $q$ , which we can identify with the zero map from  $X$  to  $Y$ ). Each of the Kan complexes  $\mathcal{D}(\sigma)$ ,  $\mathcal{D}'$ ,  $\mathcal{D}''$  is equipped with a canonical involution. On  $\mathcal{D}(\sigma)$ , this involution corresponds to the transposition of diagrams as in the statement of the lemma. On  $\mathcal{D}''$ , this involution corresponds to reversal of loops. The desired conclusion now follows from the observation that these involutions are compatible with the inclusions  $\mathcal{D}'', \mathcal{D}(\sigma) \subseteq \mathcal{D}'$ .  $\square$

**Definition 3.8.** Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits cokernels. Suppose given a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in the homotopy category  $\text{h}\mathcal{C}$ . We will say that this diagram is a *distinguished triangle* if there exists a diagram  $\Delta^1 \times \Delta^2 \rightarrow \mathcal{C}$  as shown

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{f}} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow \tilde{g} & & \downarrow \\ 0' & \longrightarrow & Z & \xrightarrow{\tilde{h}} & W, \end{array}$$

satisfying the following conditions:

- (i) The objects  $0, 0' \in \mathcal{C}$  are zero.
- (ii) Both squares are pushout diagrams in  $\mathcal{C}$ .
- (iii) The morphisms  $\tilde{f}$  and  $\tilde{g}$  represent  $f$  and  $g$ , respectively.
- (iv) The map  $h : Z \rightarrow X[1]$  is the composition of (the homotopy class of)  $\tilde{h}$  with the isomorphism  $W \simeq X[1]$  determined by the outer rectangle.

**Remark 3.9.** We will generally only use Definition 3.8 in the case where  $\mathcal{C}$  is a stable  $\infty$ -category. However, it will be convenient to have the terminology available in the case where  $\mathcal{C}$  is not yet known to be stable.

The following result is an immediate consequence of Lemma 3.7:

**Lemma 3.10.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category. Suppose given a diagram  $\Delta^2 \times \Delta^1 \rightarrow \mathcal{C}$ , depicted as*

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow f & & \downarrow \\ Y & \xrightarrow{g} & Z \\ \downarrow & & \downarrow h \\ 0' & \longrightarrow & W, \end{array}$$

where both squares are pushouts and the objects  $0, 0' \in \mathcal{C}$  are zero. Then the diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{-h'} X[1]$$

is a distinguished triangle in  $\mathfrak{h}\mathcal{C}$ , where  $h'$  denotes the composition of  $h$  with the isomorphism  $W \simeq X[1]$  determined by the outer square, and  $-h'$  denotes the composition of  $h'$  with the map  $-\text{id} \in \text{Hom}_{\mathfrak{h}\mathcal{C}}(X[1], X[1]) \simeq \pi_1 \text{Map}_{\mathcal{C}}(X, X[1])$ .

We can now state the main result of this section:

**Theorem 3.11.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits cokernels, and suppose that the suspension functor  $\Sigma$  is an equivalence. Then the translation functor of Notation 3.4 and the class of distinguished triangles of Definition 3.8 endow  $\mathfrak{h}\mathcal{C}$  with the structure of a triangulated category.*

**Remark 3.12.** The hypotheses of Theorem 3.11 hold whenever  $\mathcal{C}$  is stable. In fact, the hypotheses of Theorem 3.11 are *equivalent* to the stability of  $\mathcal{C}$ : see Corollary 8.28.

*Proof.* We must verify that Verdier's axioms (TR1) through (TR4) are satisfied.

(TR1) Let  $\mathcal{E} \subseteq \text{Fun}(\Delta^1 \times \Delta^2, \mathcal{C})$  be the full subcategory spanned by those diagrams

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0' & \longrightarrow & Z & \longrightarrow & W \end{array}$$

of the form considered in Definition 3.8, and let  $e : \mathcal{E} \rightarrow \text{Fun}(\Delta^1, \mathcal{C})$  be the restriction to the upper left horizontal arrow. Repeated use of Proposition T.4.3.2.15 implies  $e$  is a trivial fibration. In particular, every morphism  $f : X \rightarrow Y$  can be completed to a diagram belonging to  $\mathcal{E}$ . This proves (a). Part (b) is obvious, and (c) follows from the observation that if  $f = \text{id}_X$ , then the object  $Z$  in the above diagram is a zero object of  $\mathcal{C}$ .

(TR2) Suppose that

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle in  $\mathfrak{h}\mathcal{C}$ , corresponding to a diagram  $\sigma \in \mathcal{E}$  as depicted above. Extend  $\sigma$  to a diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0' & \longrightarrow & Z & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow u \\ 0'' & \longrightarrow & 0''' & \longrightarrow & V \end{array}$$

where the lower right square is a pushout, and the objects  $0'', 0''' \in \mathcal{C}$  are zero. We have a map between the squares

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & W \end{array} \quad \begin{array}{ccc} Y & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0''' & \longrightarrow & V \end{array}$$

which induces a commutative diagram in the homotopy category  $\mathrm{h}\mathcal{C}$

$$\begin{array}{ccc} W & \longrightarrow & X[1] \\ \downarrow u & & \downarrow f[1] \\ V & \longrightarrow & Y[1] \end{array}$$

where the horizontal arrows are isomorphisms. Applying Lemma 3.10 to the rectangle on the right of the large diagram, we conclude that

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is a distinguished triangle in  $\mathrm{h}\mathcal{C}$ .

Conversely, suppose that

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is a distinguished triangle in  $\mathrm{h}\mathcal{C}$ . Since the functor  $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$  is an equivalence, we conclude that the triangle

$$Y[-2] \xrightarrow{g[-2]} Z[-2] \xrightarrow{h[-2]} X[-1] \xrightarrow{-f[-1]} Y[-1]$$

is distinguished. Applying the preceding argument five times, we conclude that the triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is distinguished, as desired.

(TR3) Suppose distinguished triangles

$$X \xrightarrow{f} Y \rightarrow Z \rightarrow X[1]$$

$$X' \xrightarrow{f'} Y' \rightarrow Z' \rightarrow X'[1]$$

in  $\mathrm{h}\mathcal{C}$ . Without loss of generality, we may suppose that these triangles are induced by diagrams  $\sigma, \sigma' \in \mathcal{E}$ . Any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y' \end{array}$$

in the homotopy category  $\mathrm{h}\mathcal{C}$  can be lifted (nonuniquely) to a square in  $\mathcal{C}$ , which we may identify with a morphism  $\phi : e(\sigma) \rightarrow e(\sigma')$  in the  $\infty$ -category  $\mathrm{Fun}(\Delta^1, \mathcal{C})$ . Since  $e$  is a trivial fibration of simplicial sets,  $\phi$  can be lifted to a morphism  $\sigma \rightarrow \sigma'$  in  $\mathcal{E}$ , which determines a natural transformation of distinguished triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

(TR4) Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be morphisms in  $\mathcal{C}$ . In view of the fact that  $e : \mathcal{E} \rightarrow \text{Fun}(\Delta^1, \mathcal{C})$  is a trivial fibration, any distinguished triangle in  $\text{h}\mathcal{C}$  beginning with  $f$ ,  $g$ , or  $g \circ f$  is uniquely determined up to (nonunique) isomorphism. Consequently, it will suffice to prove that there exist *some* triple of distinguished triangles which satisfies the conclusions of (TR4). To prove this, we construct a diagram in  $\mathcal{C}$

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Y/X & \longrightarrow & Z/X & \longrightarrow & X' \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & \longrightarrow & Z/Y & \longrightarrow & Y' \longrightarrow (Y/X)'
\end{array}$$

where  $0$  is a zero object of  $\mathcal{C}$ , and each square in the diagram is a pushout (more precisely, we apply Proposition T.4.3.2.15 repeatedly to construct a map from the nerve of the appropriate partially ordered set into  $\mathcal{C}$ ). Restricting to appropriate rectangles contained in the diagram, we obtain isomorphisms  $X' \simeq X[1]$ ,  $Y' \simeq Y[1]$ ,  $(Y/X)' \simeq Y/X[1]$ , and four distinguished triangles

$$X \xrightarrow{f} Y \rightarrow Y/X \rightarrow X[1]$$

$$Y \xrightarrow{g} Z \rightarrow Z/Y \rightarrow Y[1]$$

$$X \xrightarrow{g \circ f} Z \rightarrow Z/X \rightarrow X[1]$$

$$Y/X \rightarrow Z/X \rightarrow Z/Y \rightarrow Y/X[1].$$

The commutativity in the homotopy category  $\text{h}\mathcal{C}$  required by (TR4) follows from the (stronger) commutativity of the above diagram in  $\mathcal{C}$  itself. □

**Remark 3.13.** The definition of a stable  $\infty$ -category is quite a bit simpler than that of a triangulated category. In particular, the octahedral axiom (TR4) is a consequence of  $\infty$ -categorical principles which are basic and easily motivated.

**Notation 3.14.** Let  $\mathcal{C}$  be a stable  $\infty$ -category containing a pair of objects  $X$  and  $Y$ . We let  $\text{Ext}_{\mathcal{C}}^n(X, Y)$  denote the abelian group  $\text{Hom}_{\text{h}\mathcal{C}}(X[n], Y)$ . If  $n$  is negative, this can be identified with the homotopy group  $\pi_{-n} \text{Map}_{\mathcal{C}}(X, Y)$ . More generally,  $\text{Ext}_{\mathcal{C}}^n(X, Y)$  can be identified with the  $(-n)$ th homotopy group of an appropriate *spectrum* of maps from  $X$  to  $Y$ .

## 4 Properties of Stable $\infty$ -Categories

According to Definition 2.9, a pointed  $\infty$ -category  $\mathcal{C}$  is stable if it admits certain pushout squares and certain pullback squares, which are required to coincide with one another. Our goal in this section is to prove that a stable  $\infty$ -category  $\mathcal{C}$  admits *all* finite limits and colimits, and that the pushout squares in  $\mathcal{C}$  coincide with the pullback squares in general (Proposition 4.4). To prove this, we will need the following easy observation (which is quite useful in its own right):

**Proposition 4.1.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category, and let  $K$  be a simplicial set. Then the  $\infty$ -category  $\text{Fun}(K, \mathcal{C})$  is stable.*

*Proof.* This follows immediately from the fact that kernels and cokernels in  $\text{Fun}(K, \mathcal{C})$  can be computed pointwise (Proposition T.5.1.2.2). □

**Definition 4.2.** If  $\mathcal{C}$  is stable  $\infty$ -category, and  $\mathcal{C}_0$  is a full subcategory containing a zero object and stable under the formation of kernels and cokernels, then  $\mathcal{C}_0$  is itself stable. In this case, we will say that  $\mathcal{C}_0$  is a *stable subcategory* of  $\mathcal{C}$ .

**Lemma 4.3.** Let  $\mathcal{C}$  be a stable  $\infty$ -category, and let  $\mathcal{C}' \subseteq \mathcal{C}$  be a full subcategory which is stable under cokernels and under translation. Then  $\mathcal{C}'$  is a stable subcategory of  $\mathcal{C}$ .

*Proof.* It will suffice to show that  $\mathcal{C}'$  is stable under kernels. Let  $f : X \rightarrow Y$  be a morphism in  $\mathcal{C}$ . Theorem 3.11 shows that there is a canonical equivalence  $\ker(f) \simeq \operatorname{coker}(f)[-1]$ .  $\square$

**Proposition 4.4.** Let  $\mathcal{C}$  be a pointed  $\infty$ -category. Then  $\mathcal{C}$  is stable if and only if the following conditions are satisfied:

- (1) The  $\infty$ -category  $\mathcal{C}$  admits finite limits and colimits.
- (2) A square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

in  $\mathcal{C}$  is a pushout if and only if it is a pullback.

*Proof.* Condition (1) implies the existence of kernels and cokernels in  $\mathcal{C}$ , and condition (2) implies that the exact triangles coincide with the coexact triangles. This proves the “if” direction.

Suppose now that  $\mathcal{C}$  is stable. We begin by proving (1). It will suffice to show that  $\mathcal{C}$  admits finite colimits; the dual argument will show that  $\mathcal{C}$  admits finite limits as well. According to Proposition T.4.4.3.2, it will suffice to show that  $\mathcal{C}$  admits coequalizers and finite coproducts. The existence of finite coproducts was established in Lemma 3.6. We now conclude by observing that a coequalizer for a diagram

$$X \begin{array}{c} \xrightarrow{f} \\ \rightrightarrows \\ \xrightarrow{f'} \end{array} Y$$

can be identified with  $\operatorname{coker}(f - f')$ .

We now show that every pushout square in  $\mathcal{C}$  is a pullback; the converse will follow by a dual argument. Let  $\mathcal{D} \subseteq \operatorname{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$  be the full subcategory spanned by the pullback squares. Then  $\mathcal{D}$  is stable under finite limits and under translations. It follows from Lemma 4.3 that  $\mathcal{D}$  is a stable subcategory of  $\operatorname{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ .

Let  $i : \Lambda_0^2 \hookrightarrow \Delta^1 \times \Delta^1$  be the inclusion, and let  $i_! : \operatorname{Fun}(\Lambda_0^2, \mathcal{C}) \rightarrow \operatorname{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$  be a functor of left Kan extension. Then  $i_!$  preserves finite colimits, and is therefore exact (Proposition 5.1). Let  $\mathcal{D}' = i_!^{-1} \mathcal{D}$ . Then  $\mathcal{D}'$  is a stable subcategory of  $\operatorname{Fun}(\Lambda_0^2, \mathcal{C})$ ; we wish to show that  $\mathcal{D}' = \operatorname{Fun}(\Lambda_0^2, \mathcal{C})$ . To prove this, we observe that any diagram

$$X' \leftarrow X \rightarrow X''$$

can be obtained as a (finite) colimit

$$e'_{X'} \coprod_{e'_X} e_X \coprod_{e''_X} e''_{X''}$$

where  $e_X \in \operatorname{Fun}(\Lambda_0^2, \mathcal{C})$  denotes the diagram  $X \leftarrow X \rightarrow X$ ,  $e'_Z \in \operatorname{Fun}(\Lambda_0^2, \mathcal{C})$  denotes the diagram  $Z \leftarrow 0 \rightarrow 0$ , and  $e''_Z \in \operatorname{Fun}(\Lambda_0^2, \mathcal{C})$  denotes the diagram  $0 \leftarrow 0 \rightarrow Z$ . It will therefore suffice to prove that pushout of any of these five diagrams is also a pullback. This follows immediately from the following more general observation: any pushout square

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow f & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

in an (arbitrary)  $\infty$ -category  $\mathcal{C}$  is also pullback square, provided that  $f$  is an equivalence.  $\square$

**Proposition 4.5.** *Let  $\mathcal{C}$  be a (small) stable  $\infty$ -category, and let  $\kappa$  be a regular cardinal. Then the  $\infty$ -category  $\text{Ind}_\kappa(\mathcal{C})$  is stable.*

*Proof.* The functor  $j$  preserves finite limits and colimits (Propositions T.5.1.3.2 and T.5.3.5.14). It follows that  $j(0)$  is a zero object of  $\text{Ind}_\kappa(\mathcal{C})$ , so that  $\text{Ind}_\kappa(\mathcal{C})$  is pointed.

We next show that every morphism  $f : X \rightarrow Y$  in  $\text{Ind}_\kappa(\mathcal{C})$  admits a kernel and a cokernel. According to Proposition T.5.3.5.15, we may assume that  $f$  is a  $\kappa$ -filtered colimit of morphisms  $f_\alpha : X_\alpha \rightarrow Y_\alpha$  which belong to the essential image  $\mathcal{C}'$  of  $j$ . Since  $j$  preserves kernels and cokernels, each of the maps  $f_\alpha$  has a kernel and a cokernel in  $\text{Ind}_\kappa$ . It follows immediately that  $f$  has a cokernel (which can be written as a colimit of the cokernels of the maps  $f_\alpha$ ). The existence of  $\ker(f)$  is slightly more difficult. Choose a  $\kappa$ -filtered diagram  $p : \mathcal{J} \rightarrow \text{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}')$ , where each  $p(\alpha)$  is a pullback square

$$\begin{array}{ccc} Z_\alpha & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha. \end{array}$$

Let  $\sigma$  be a colimit of the diagram  $p$ ; we wish to show that  $\sigma$  is a pullback diagram in  $\text{Ind}_\kappa(\mathcal{C})$ . Since  $\text{Ind}_\kappa(\mathcal{C})$  is stable under  $\kappa$ -small limits in  $\mathcal{P}(\mathcal{C})$ , it will suffice to show that  $\sigma$  is a pullback square in  $\mathcal{P}(\mathcal{C})$ . Since  $\mathcal{P}(\mathcal{C})$  is an  $\infty$ -topos, filtered colimits in  $\mathcal{P}(\mathcal{C})$  are left exact (Example T.7.3.4.7); it will therefore suffice to show that each  $p(\alpha)$  is a pullback diagram in  $\mathcal{P}(\mathcal{C})$ . This is obvious, since the inclusion  $\mathcal{C}' \subseteq \mathcal{P}(\mathcal{C})$  preserves all limits which exist in  $\mathcal{C}'$  (Proposition T.5.1.3.2).

To complete the proof, we must show that a triangle in  $\text{Ind}_\kappa(\mathcal{C})$  is exact if and only if it is coexact. Suppose given an exact triangle

$$\begin{array}{ccc} Z & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

in  $\text{Ind}_\kappa(\mathcal{C})$ . The above argument shows that we can write this triangle as a filtered colimit of exact triangles

$$\begin{array}{ccc} Z_\alpha & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ X_\alpha & \longrightarrow & Y_\alpha \end{array}$$

in  $\mathcal{C}'$ . Since  $\mathcal{C}'$  is stable, we conclude that these triangles are also coexact. The original triangle is therefore a filtered colimit of coexact triangles in  $\mathcal{C}'$ , hence coexact. The converse follows by the same argument.  $\square$

## 5 Exact Functors

Let  $F : \mathcal{C} \rightarrow \mathcal{C}'$  be a functor between stable  $\infty$ -categories. Suppose that  $F$  carries zero objects into zero objects. It follows immediately that  $F$  carries triangles into triangles. If, in addition,  $F$  carries exact triangles into exact triangles, then we will say that  $F$  is *exact*. The exactness of a functor  $F$  admits the following alternative characterizations:

**Proposition 5.1.** *Let  $F : \mathcal{C} \rightarrow \mathcal{C}'$  be a functor between stable  $\infty$ -categories. The following conditions are equivalent:*

- (1) *The functor  $F$  is left exact. That is,  $F$  commutes with finite limits.*

(2) The functor  $F$  is right exact. That is,  $F$  commutes with finite colimits.

(3) The functor  $F$  is exact.

*Proof.* We will prove that (2)  $\Leftrightarrow$  (3); the equivalence (1)  $\Leftrightarrow$  (3) will follow by a dual argument. The implication (2)  $\Rightarrow$  (3) is obvious. Conversely, suppose that  $F$  is exact. The proof of Proposition 4.4 shows that  $F$  preserves coequalizers, and the proof of Lemma 3.6 shows that  $F$  preserves finite coproducts. It follows that  $F$  preserves all finite colimits (see the proof of Proposition T.4.4.3.2).  $\square$

The identity functor from any stable  $\infty$ -category to itself is exact, and a composition of exact functors is exact. Consequently, there exists a subcategory  $\text{Cat}_\infty^{\text{Ex}} \subseteq \text{Cat}_\infty$  in which the objects are stable  $\infty$ -categories and the morphisms are the exact functors. Our next few results concern the stability properties of this subcategory.

**Proposition 5.2.** *Suppose given a homotopy Cartesian diagram of  $\infty$ -categories*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{G'} & \mathcal{C} \\ \downarrow F' & & \downarrow F \\ \mathcal{D}' & \xrightarrow{G} & \mathcal{D}. \end{array}$$

*Suppose further that  $\mathcal{C}$ ,  $\mathcal{D}'$ , and  $\mathcal{D}$  are stable, and that the functors  $F$  and  $G$  are exact. Then:*

(1) The  $\infty$ -category  $\mathcal{C}'$  is stable.

(2) The functors  $F'$  and  $G'$  are exact.

(3) If  $\mathcal{E}$  is a stable  $\infty$ -category, then a functor  $H : \mathcal{E} \rightarrow \mathcal{C}'$  is exact if and only if the functors  $F' \circ H$  and  $G' \circ H$  are exact.

*Proof.* Combine Proposition 4.4 with Lemma T.5.4.5.5.  $\square$

**Proposition 5.3.** *Let  $\{\mathcal{C}_\alpha\}_{\alpha \in A}$  be a collection of stable  $\infty$ -categories. Then the product*

$$\mathcal{C} = \prod_{\alpha \in A} \mathcal{C}_\alpha$$

*is stable. Moreover, for any stable  $\infty$ -category  $\mathcal{D}$ , a functor  $F : \mathcal{D} \rightarrow \mathcal{C}$  is exact if and only if each of the compositions*

$$\mathcal{D} \xrightarrow{F} \mathcal{C} \rightarrow \mathcal{C}_\alpha$$

*is an exact functor.*

*Proof.* This follows immediately from the fact that limits and colimits in  $\mathcal{C}$  are computed pointwise.  $\square$

**Theorem 5.4.** *The  $\infty$ -category  $\text{Cat}_\infty^{\text{Ex}}$  admits small limits, and the inclusion*

$$\text{Cat}_\infty^{\text{Ex}} \subseteq \text{Cat}_\infty$$

*preserves small limits.*

*Proof.* Using Propositions 5.2 and 5.3, one can repeat the argument used to prove Proposition T.5.4.7.3.  $\square$

We now prove an analogue of Theorem 5.4.

**Proposition 5.5.** *Let  $p : X \rightarrow S$  be an inner fibration of simplicial sets. Suppose that:*

- (i) For each vertex  $s$  of  $S$ , the fiber  $X_s = X \times_S \{s\}$  is a stable  $\infty$ -category.
- (ii) For every edge  $s \rightarrow s'$  in  $S$ , the restriction  $X \times_S \Delta^1 \rightarrow \Delta^1$  is a coCartesian fibration, associated to an exact functor  $X_s \rightarrow X_{s'}$ .

Then:

- (1) The  $\infty$ -category  $\text{Map}_S(S, X)$  of sections of  $p$  is stable.
- (2) If  $\mathcal{C}$  is an arbitrary stable  $\infty$ -category, and  $f : \mathcal{C} \rightarrow \text{Map}_S(S, X)$  induces an exact functor  $\mathcal{C} \xrightarrow{f} \text{Map}_S(S, X) \rightarrow X_s$  for every vertex  $s$  of  $S$ , then  $f$  is exact.
- (3) For every set  $\mathcal{E}$  of edges of  $S$ , let  $Y(\mathcal{E}) \subseteq \text{Map}_S(S, X)$  be the full subcategory spanned by those sections  $f : S \rightarrow X$  of  $p$  with the following property:
  - (\*) For every  $e \in \mathcal{E}$ ,  $f$  carries  $e$  to a  $p_e$ -coCartesian edge of the fiber product  $X \times_S \Delta^1$ , where  $p_e : X \times_S \Delta^1 \rightarrow \Delta^1$  denotes the projection.

Then each  $Y(\mathcal{E})$  is a stable subcategory of  $\text{Map}_S(S, X)$ .

*Proof.* Combine Proposition T.5.4.7.11, Theorem 5.4, and Proposition 4.1. □

**Proposition 5.6.** *The  $\infty$ -category  $\text{Cat}_\infty^{\text{Ex}}$  admits small filtered colimits, and the inclusion  $\text{Cat}_\infty^{\text{Ex}} \subseteq \text{Cat}_\infty$  preserves filtered colimits.*

*Proof.* Let  $\mathcal{J}$  be a filtered  $\infty$ -category,  $p : \mathcal{J} \rightarrow \text{Cat}_\infty^{\text{Ex}}$  a diagram, which we will indicate by  $\{\mathcal{C}_I\}_{I \in \mathcal{J}}$ , and  $\mathcal{C}$  a colimit of the induced diagram  $\mathcal{J} \rightarrow \text{Cat}_\infty$ . We must prove:

- (i) The  $\infty$ -category  $\mathcal{C}$  is stable.
- (ii) Each of the canonical functors  $\theta_I : \mathcal{C}_I \rightarrow \mathcal{C}$  is exact.
- (iii) Given an arbitrary stable  $\infty$ -category  $\mathcal{D}$ , a functor  $f : \mathcal{C} \rightarrow \mathcal{D}$  is exact if and only if each of the composite functors  $\mathcal{C}_I \xrightarrow{\theta_I} \mathcal{C} \rightarrow \mathcal{D}$  is exact.

In view of Proposition 5.1, (ii) and (iii) follow immediately from Proposition T.5.5.7.11. The same result implies that  $\mathcal{C}$  admits finite limits and colimits, and that each of the functors  $\theta_I$  preserves finite limits and colimits.

To prove that  $\mathcal{C}$  has a zero object, we select an object  $I \in \mathcal{J}$ . The functor  $\mathcal{J} \rightarrow \mathcal{C}$  preserves initial and final objects. Since  $\mathcal{C}_I$  has a zero object, so does  $\mathcal{C}$ .

We will complete the proof by showing that every exact triangle in  $\mathcal{C}$  is coexact (the converse follows by the same argument). Fix a morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ . Without loss of generality, we may suppose that there exists  $I \in \mathcal{J}$  and a morphism  $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$  in  $\mathcal{C}_I$  such that  $f = \theta_I(\tilde{f})$  (Proposition T.5.4.1.2). Form a pullback diagram  $\tilde{\sigma}$

$$\begin{array}{ccc} \tilde{W} & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \tilde{Y} \end{array}$$

in  $\mathcal{C}_I$ . Since  $\mathcal{C}_I$  is stable, this diagram is also a pushout. It follows that  $\theta_I(\sigma)$  is triangle  $W \rightarrow X \xrightarrow{f} Y$  which is both exact and coexact in  $\mathcal{C}$ . □



## 6 t-Structures and Localizations

Let  $\mathcal{C}$  be an  $\infty$ -category. Recall that we say a full subcategory  $\mathcal{C}' \subseteq \mathcal{C}$  is a *localization* of  $\mathcal{C}$  if the inclusion functor  $\mathcal{C}' \subseteq \mathcal{C}$  has a left adjoint (§T.5.2.7). In this section, we will introduce a special class of localizations, called *t-localizations*, in the case where  $\mathcal{C}$  is stable. We will further show that there is a bijective correspondence between t-localizations of  $\mathcal{C}$  and *t-structures* on the triangulated category  $\mathrm{h}\mathcal{C}$ . We begin with a review of the classical theory of t-structures; for a more thorough introduction we refer the reader to [6].

**Definition 6.1.** Let  $\mathcal{D}$  be a triangulated category. A *t-structure* on  $\mathcal{D}$  is defined to be a pair of full subcategories  $\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0}$  (always assumed to be stable under isomorphism) having the following properties:

- (1) For  $X \in \mathcal{D}_{\geq 0}$  and  $Y \in \mathcal{D}_{\leq -1}$ , we have  $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ .
- (2)  $\mathcal{D}_{\geq 0}[1] \subseteq \mathcal{D}_{\geq 0}$ ,  $\mathcal{D}_{\leq 0}[-1] \subseteq \mathcal{D}_{\leq 0}$ .
- (3) For any  $X \in \mathcal{D}$ , there exists a distinguished triangle  $X' \rightarrow X \rightarrow X'' \rightarrow X'[1]$  where  $X' \in \mathcal{D}_{\geq 0}$  and  $X'' \in \mathcal{D}_{\leq 0}[-1]$ .

**Notation 6.2.** If  $\mathcal{D}$  is a triangulated category equipped with a t-structure, we will write  $\mathcal{D}_{\geq n}$  for  $\mathcal{D}_{\geq 0}[n]$  and  $\mathcal{D}_{\leq n}$  for  $\mathcal{D}_{\leq 0}[n]$ . Observe that we use a *homological* indexing convention.

**Remark 6.3.** In Definition 6.1, either of the full subcategories  $\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0} \subseteq \mathcal{C}$  determines the other. For example, an object  $X \in \mathcal{D}$  belongs to  $\mathcal{D}_{\leq -1}$  if and only if  $\mathrm{Hom}_{\mathcal{D}}(Y, X)$  vanishes for all  $Y \in \mathcal{D}_{\geq 0}$ .

**Definition 6.4.** Let  $\mathcal{C}$  be a stable  $\infty$ -category. A *t-structure* on  $\mathcal{C}$  is a t-structure on the homotopy category  $\mathrm{h}\mathcal{C}$ . If  $\mathcal{C}$  is equipped with a t-structure, we let  $\mathcal{C}_{\geq n}$  and  $\mathcal{C}_{\leq n}$  denote the full subcategories of  $\mathcal{C}$  spanned by those objects which belong to  $(\mathrm{h}\mathcal{C})_{\geq n}$  and  $(\mathrm{h}\mathcal{C})_{\leq n}$ , respectively.

**Proposition 6.5.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a t-structure. For each  $n \in \mathbf{Z}$ , the full subcategory  $\mathcal{C}_{\leq n}$  is a localization of  $\mathcal{C}$ .*

*Proof.* Without loss of generality, we may suppose  $n = -1$ . According to Proposition T.5.2.7.8, it will suffice to prove that for each  $X \in \mathcal{C}$ , there exists a map  $f : X \rightarrow X''$ , where  $X'' \in \mathcal{C}_{\leq -1}$  and for each  $Y \in \mathcal{C}_{\leq -1}$ , the map

$$\mathrm{Map}_{\mathcal{C}}(X'', Y) \rightarrow \mathrm{Map}_{\mathcal{C}}(X, Y)$$

is a weak homotopy equivalence. Invoking part (3) of Definition 6.1, we can choose  $f$  to fit into a distinguished triangle

$$X' \rightarrow X \xrightarrow{f} X'' \rightarrow X'[1]$$

where  $X' \in \mathcal{C}_{\geq 0}$ . According to Whitehead's theorem, we need to show that for every  $k \leq 0$ , the map

$$\mathrm{Ext}_{\mathcal{C}}^k(X'', Y) \rightarrow \mathrm{Ext}_{\mathcal{C}}^k(X, Y)$$

is an isomorphism of abelian groups. Using the long exact sequence associated to the exact triangle above, we are reduced to proving that the groups  $\mathrm{Ext}_{\mathcal{C}}^k(X', Y)$  vanish for  $k \leq 0$ . We now use condition (2) of Definition 6.1 to conclude that  $X'[-k] \in \mathcal{C}_{\geq 0}$ . Condition (1) of Definition 6.1 now implies that

$$\mathrm{Ext}_{\mathcal{C}}^k(X', Y) \simeq \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X'[-k], Y) \simeq 0.$$

□

**Corollary 6.6.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a t-structure. The full subcategories  $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$  are stable under all limits which exist in  $\mathcal{C}$ . Dually, the full subcategories  $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$  are stable under all colimits which exist in  $\mathcal{C}$ .*

**Notation 6.7.** Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a t-structure. We will let  $\tau_{\leq n}$  denote a left adjoint to the inclusion  $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$ , and  $\tau_{\geq n}$  a right adjoint to the inclusion  $\mathcal{C}_{\geq n} \subseteq \mathcal{C}$ .

**Remark 6.8.** Fix  $n, m \in \mathbf{Z}$ , and let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a t-structure. Then the truncation functors  $\tau_{\leq n}, \tau_{\geq n}$  map the full subcategory  $\mathcal{C}_{\leq m}$  to itself. To prove this, we first observe that  $\tau_{\leq n}$  is equivalent to the identity on  $\mathcal{C}_{\leq m}$  if  $m \leq n$ , while if  $m \geq n$  the essential image of  $\tau_{\leq n}$  is contained in  $\mathcal{C}_{\leq n} \subseteq \mathcal{C}_{\leq m}$ . To prove the analogous result for  $\tau_{\geq n}$ , we observe that the proof of Proposition 6.5 implies that for each  $X$ , we have a distinguished triangle

$$\tau_{\geq n}X \rightarrow X \xrightarrow{f} \tau_{\leq n-1}X \rightarrow (\tau_{\geq n}X)[1].$$

If  $X \in \mathcal{C}_{\leq m}$ , then  $\tau_{\leq n-1}X$  also belongs to  $\mathcal{C}_{\leq m}$ , so that  $\tau_{\geq n}X \simeq \ker(f)$  belongs to  $\mathcal{C}_{\leq m}$  since  $\mathcal{C}_{\leq m}$  is stable under limits.

**Warning 6.9.** In §T.5.5.6, we introduced for every  $\infty$ -category  $\mathcal{C}$  a full subcategory  $\tau_{\leq n}\mathcal{C}$  of *n-truncated objects* of  $\mathcal{C}$ . In that context, we used the symbol  $\tau_{\leq n}$  to denote a left adjoint to the inclusion  $\tau_{\leq n}\mathcal{C} \subseteq \mathcal{C}$ . This is *not* compatible with Notation 6.7. In fact, if  $\mathcal{C}$  is a stable  $\infty$ -category, then it has no nonzero truncated objects at all: if  $X \in \mathcal{C}$  is nonzero, then the identity map from  $X$  to itself determines a nontrivial homotopy class in  $\pi_n \text{Map}_{\mathcal{C}}(X[-n], X)$ , for all  $n \geq 0$ . Nevertheless, the two notations are consistent when restricted to  $\mathcal{C}_{\geq 0}$ , in view of the following fact:

- Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a t-structure. An object  $X \in \mathcal{C}_{\geq 0}$  is *k-truncated* (as an object of  $\mathcal{C}_{\geq 0}$ ) if and only if  $X \in \mathcal{C}_{\leq k}$ .

In fact, we have the following more general statement: for any  $X \in \mathcal{C}$  and  $k \geq -1$ ,  $X$  belongs to  $\mathcal{C}_{\leq k}$  if and only if  $\text{Map}_{\mathcal{C}}(Y, X)$  is *k-truncated* for every  $Y \in \mathcal{C}_{\geq 0}$ . Because the latter condition is equivalent to the vanishing of  $\text{Ext}_{\mathcal{C}}^n(Y, X)$  for  $n < -k$ , we can use the shift functor to reduce to the case where  $n = 0$  and  $k = -1$ , which is covered by Remark 6.3.

Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a t-structure, and let  $n, m \in \mathbf{Z}$ . Remark 6.8 implies that we have a commutative diagram of simplicial sets

$$\begin{array}{ccc} \mathcal{C}_{\geq n} & \hookrightarrow & \mathcal{C} \\ \downarrow \tau_{\leq m} & & \downarrow \tau_{\leq m} \\ \mathcal{C}_{\geq n} \cap \mathcal{C}_{\leq m} & \hookrightarrow & \mathcal{C}_{\leq m} \end{array}$$

As explained in §T.7.3.1, we get an induced transformation of functors

$$\theta : \tau_{\leq m} \circ \tau_{\geq n} \rightarrow \tau_{\geq n} \circ \tau_{\leq m}.$$

**Proposition 6.10.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with t-structure. Then the natural transformation*

$$\theta : \tau_{\leq m} \circ \tau_{\geq n} \rightarrow \tau_{\geq n} \circ \tau_{\leq m}$$

*is an equivalence of functors  $\mathcal{C} \rightarrow \mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n}$ .*

*Proof.* This is a classical fact concerning triangulated categories; we include a proof for completeness. Fix  $X \in \mathcal{C}$ ; we wish to show that

$$\theta(X) : \tau_{\leq m}\tau_{\geq n}X \rightarrow \tau_{\geq n}\tau_{\leq m}X$$

is an isomorphism in the homotopy category of  $\mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n}$ . If  $m < n$ , then both sides are zero and there is nothing to prove; let us therefore assume that  $m \geq n$ . Fix  $Y \in \mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n}$ ; it will suffice to show that composition with  $\theta(X)$  induces an isomorphism

$$\text{Ext}^0(\tau_{\geq n}\tau_{\leq m}X, Y) \rightarrow \text{Ext}^0(\tau_{\leq m}\tau_{\geq n}X, Y) \simeq \text{Ext}^0(\tau_{\geq n}X, Y).$$

We have a map of long exact sequences

$$\begin{array}{ccc}
\mathrm{Ext}^0(\tau_{\leq n-1}\tau_{\leq m}X, Y) & \xrightarrow{f_0} & \mathrm{Ext}^0(\tau_{\leq n-1}X, Y) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^0(\tau_{\leq m}X, Y) & \xrightarrow{f_1} & \mathrm{Ext}^0(X, Y) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^0(\tau_{\geq n}\tau_{\leq m}X, Y) & \xrightarrow{f_2} & \mathrm{Ext}^0(\tau_{\geq n}X, Y) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1(\tau_{\leq n-1}\tau_{\leq m}X, Y) & \xrightarrow{f_3} & \mathrm{Ext}^1(\tau_{\leq n-1}X, Y) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1(\tau_{\leq m}X, Y) & \xrightarrow{f_4} & \mathrm{Ext}^1(X, Y).
\end{array}$$

Since  $m \geq n$ , the natural transformation  $\tau_{\leq n-1} \rightarrow \tau_{\leq n-1}\tau_{\leq m}$  is an equivalence of functors; this proves that  $f_0$  and  $f_3$  are bijective. Since  $Y \in \mathcal{C}_{\leq m}$ ,  $f_1$  is bijective and  $f_4$  is injective. It follows from the ‘‘five lemma’’ that  $f_2$  is bijective, as desired.  $\square$

**Definition 6.11.** Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a  $t$ -structure. The *heart*  $\mathcal{C}^\heartsuit$  of  $\mathcal{C}$  is the full subcategory  $\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ . For each  $n \in \mathbf{Z}$ , we let  $\pi_0 : \mathcal{C} \rightarrow \mathcal{C}^\heartsuit$  denote the functor  $\tau_{\leq 0} \circ \tau_{\geq 0} \simeq \tau_{\geq 0} \circ \tau_{\leq 0}$ , and we let  $\pi_n : \mathcal{C} \rightarrow \mathcal{C}^\heartsuit$  denote the composition of  $\pi_0$  with the shift functor  $X \mapsto X[-n]$ .

**Remark 6.12.** Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a  $t$ -structure, and let  $X, Y \in \mathcal{C}^\heartsuit$ . The homotopy group  $\pi_n \mathrm{Map}_{\mathcal{C}}(X, Y) \simeq \mathrm{Ext}_{\mathcal{C}}^{-n}(X, Y)$  vanishes for  $n > 0$ . It follows that  $\mathcal{C}^\heartsuit$  is equivalent to (the nerve of) its homotopy category  $\mathrm{h}\mathcal{C}^\heartsuit$ . Moreover, the category  $\mathrm{h}\mathcal{C}^\heartsuit$  is abelian ([6]).

Let  $\mathcal{C}$  be a stable  $\infty$ -category. In view of Remark 6.3,  $t$ -structures on  $\mathcal{C}$  are determined by the corresponding localizations  $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ . However, not every localization of  $\mathcal{C}$  arises in this way. Recall (see §T.5.5.4) that every localization of  $\mathcal{C}$  has the form  $S^{-1}\mathcal{C}$ , where  $S$  is an appropriate collection of morphisms of  $\mathcal{C}$ . Here  $S^{-1}\mathcal{C}$  denotes the full subcategory of  $\mathcal{C}$  spanned by  $S$ -local objects, where an object  $X \in \mathcal{C}$  is said to be  $S$ -local if and only if, for each  $f : Y' \rightarrow Y$  in  $S$ , composition with  $f$  induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(Y, X) \rightarrow \mathrm{Map}_{\mathcal{C}}(Y', X).$$

If  $\mathcal{C}$  is stable, then we extend the morphism  $f$  to a distinguished triangle

$$Y' \rightarrow Y \rightarrow Y'' \rightarrow Y'[1],$$

and we have an associated long exact sequence

$$\dots \rightarrow \mathrm{Ext}_{\mathcal{C}}^i(Y'', X) \rightarrow \mathrm{Ext}_{\mathcal{C}}^i(Y, X) \xrightarrow{\theta_i} \mathrm{Ext}_{\mathcal{C}}^i(Y', X) \rightarrow \mathrm{Ext}_{\mathcal{C}}^{i+1}(Y'', X) \rightarrow \dots$$

The requirement that  $X$  be  $\{f\}$ -local amounts to the condition that  $\theta_i$  be an isomorphism for  $i \leq 0$ . Using the long exact sequence, we see that if  $X$  is  $\{f\}$ -local, then  $\mathrm{Ext}_{\mathcal{C}}^i(Y'', X) = 0$  for  $i \leq 0$ . Conversely, if  $\mathrm{Ext}_{\mathcal{C}}^i(Y'', X) = 0$  for  $i \leq 1$ , then  $X$  is  $\{f\}$ -local. Experience suggests that it is usually more natural to require the vanishing of the groups  $\mathrm{Ext}_{\mathcal{C}}^i(Y'', X)$  than it is to require that the maps  $\theta_i$  to be isomorphisms. Of course, if  $Y'$  is a zero object of  $\mathcal{C}$ , then the distinction between these conditions disappears.

**Definition 6.13.** Let  $\mathcal{C}$  be an  $\infty$ -category which admits pushouts. We will say that a collection  $S$  of morphisms of  $\mathcal{C}$  is *quasisaturated* if it satisfies the following conditions:

- (1) Every equivalence in  $\mathcal{C}$  belongs to  $S$ .
- (2) Given a 2-simplex  $\Delta^2 \rightarrow \mathcal{C}$

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ & \searrow f & \nearrow g \\ & & Y, \end{array}$$

if any two of  $f$ ,  $g$ , and  $h$  belongs to  $S$ , then so does the third.

- (3) Given a pushout diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow f & & \downarrow f' \\ Y & \longrightarrow & Y', \end{array}$$

if  $f \in S$ , then  $f' \in S$ .

Any intersection of quasisaturated collections of morphisms is weakly saturated. Consequently, for any collection of morphisms  $S$  there is a smallest quasisaturated collection  $\overline{S}$  containing  $S$ . We will say that  $\overline{S}$  is the *quasisaturated collection of morphisms generated by  $S$* .

**Definition 6.14.** Let  $\mathcal{C}$  be a stable  $\infty$ -category. A full subcategory  $\mathcal{C}' \subseteq \mathcal{C}$  is *closed under extensions* if, for every distinguished triangle

$$X \rightarrow Y \rightarrow Z \rightarrow X[1]$$

such that  $X$  and  $Z$  belong to  $\mathcal{C}'$ , the object  $Y$  also belongs to  $\mathcal{C}'$ .

We observe that if  $\mathcal{C}$  is as in Definition 6.13 and  $L : \mathcal{C} \rightarrow \mathcal{C}$  is a localization functor, then the collection of all morphisms  $f$  of  $\mathcal{C}$  such that  $L(f)$  is an equivalence is quasisaturated.

**Proposition 6.15.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category, let  $L : \mathcal{C} \rightarrow \mathcal{C}$  be a localization functor, and let  $S$  be the collection of morphisms  $f$  in  $\mathcal{C}$  such that  $L(f)$  is an equivalence. The following conditions are equivalent:*

- (1) *There exists a collection of morphisms  $\{f : 0 \rightarrow X\}$  which generates  $S$  (as a quasisaturated collection of morphisms).*
- (2) *The collection of morphisms  $\{0 \rightarrow X : L(X) \simeq 0\}$  generates  $S$  (as a quasisaturated collection of morphisms).*
- (3) *The essential image of  $L$  is closed under extensions.*
- (4) *For any  $A \in \mathcal{C}$ ,  $B \in L\mathcal{C}$ , the natural map  $\text{Ext}^1(LA, B) \rightarrow \text{Ext}^1(A, B)$  is injective.*
- (5) *The full subcategories  $\mathcal{C}_{\geq 0} = \{A : LA \simeq 0\}$  and  $\mathcal{C}_{\leq -1} = \{A : LA \simeq A\}$  determine a  $t$ -structure on  $\mathcal{C}$ .*

*Proof.* The implication (1)  $\Rightarrow$  (2) is obvious. We next prove that (2)  $\Rightarrow$  (3). Suppose given an exact triangle

$$X \rightarrow Y \rightarrow Z$$

where  $X$  and  $Z$  are both  $S$ -local. We wish to prove that  $Y$  is  $S$ -local. In view of assumption (2), it will suffice to show that  $\text{Map}_{\mathcal{C}}(A, Y)$  is contractible, provided that  $L(A) \simeq 0$ . In other words, we must show that  $\text{Ext}_{\mathcal{C}}^i(A, Y) \simeq 0$  for  $i \leq 0$ . We now observe that there is an exact sequence

$$\text{Ext}_{\mathcal{C}}^i(A, X) \rightarrow \text{Ext}_{\mathcal{C}}^i(A, Y) \rightarrow \text{Ext}_{\mathcal{C}}^i(A, Z)$$

where the outer groups vanish, since  $X$  and  $Z$  are  $S$ -local and the map  $0 \rightarrow A$  belongs to  $S$ .

We next show that (3)  $\Rightarrow$  (4). Let  $B \in L\mathcal{C}$ , and let  $\eta \in \text{Ext}_{\mathcal{C}}^1(LA, B)$  classify a distinguished triangle

$$B \rightarrow C \xrightarrow{g} LA \xrightarrow{\eta} B[1].$$

Condition (3) implies that  $C \in L\mathcal{C}$ . If the image of  $\eta$  in  $\text{Ext}_{\mathcal{C}}^1(A, B)$  is trivial, then the localization map  $A \rightarrow LA$  factors as a composition

$$A \xrightarrow{f} C \xrightarrow{g} LA.$$

Applying  $L$  to this diagram (and using the fact that  $C$  is local) we conclude that the map  $g$  admits a section, so that  $\eta = 0$ .

We now claim that (4)  $\Rightarrow$  (5). Assume (4), and define  $\mathcal{C}_{\geq 0}$ ,  $\mathcal{C}_{\leq -1}$  as in (5). We will show that the axioms of Definition 6.1 are satisfied:

- If  $X \in \mathcal{C}_{\geq 0}$  and  $Y \in \mathcal{C}_{\leq -1}$ , then  $\text{Ext}_{\mathcal{C}}^0(X, Y) \simeq \text{Ext}_{\mathcal{C}}^0(LX, Y) \simeq \text{Ext}_{\mathcal{C}}^0(0, Y) \simeq 0$ .
- Since  $\mathcal{C}_{\leq -1}$  is a localization of  $\mathcal{C}$ , it is stable under limits, so that  $\mathcal{C}_{\leq -1}[-1] \subseteq \mathcal{C}_{\leq -1}$ . Similarly, since the functor  $L : \mathcal{C} \rightarrow \mathcal{C}_{\leq -1}$  preserves all colimits which exist in  $\mathcal{C}$ , the subcategory  $\mathcal{C}_{\geq 0}$  is stable under finite colimits, so that  $\mathcal{C}_{\geq 0}[1] \subseteq \mathcal{C}_{\geq 0}$ .
- Let  $X \in \mathcal{C}$ , and form a distinguished triangle

$$X' \rightarrow X \rightarrow LX \rightarrow X'[1].$$

We claim that  $X' \in \mathcal{C}_{\geq 0}$ ; in other words, that  $LX' = 0$ . For this, it suffices to show that for all  $Y \in L\mathcal{C}$ , the morphism space

$$\text{Ext}_{\mathcal{C}}^0(LX', Y) = 0.$$

Since  $Y$  is local, we have isomorphisms

$$\text{Ext}_{\mathcal{C}}^0(LX', Y) \simeq \text{Ext}_{\mathcal{C}}^0(X', Y) \simeq \text{Ext}_{\mathcal{C}}^1(X'[1], Y).$$

We now observe that there is a long exact sequence

$$\text{Ext}_{\mathcal{C}}^0(LX, Y) \xrightarrow{f} \text{Ext}_{\mathcal{C}}^0(X, Y) \rightarrow \text{Ext}_{\mathcal{C}}^1(X'[1], Y) \rightarrow \text{Ext}_{\mathcal{C}}^1(LX, Y) \xrightarrow{f'} \text{Ext}_{\mathcal{C}}^1(X, Y).$$

Here  $f$  is bijective (since  $Y$  is local) and  $f'$  is injective (in virtue of assumption (4)).

We conclude by showing that (5)  $\Rightarrow$  (1). Let  $S'$  be the smallest quasisaturated collection of morphisms which contains the zero map  $0 \rightarrow A$ , for every  $A \in \mathcal{C}_{\geq 0}$ . We wish to prove that  $S = S'$ . For this, we choose an arbitrary morphism  $u : X \rightarrow Y$  belonging to  $S$ . Then  $Lu : LX \rightarrow LY$  is an equivalence, so we have a pushout diagram

$$\begin{array}{ccc} X' & \xrightarrow{u'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{u} & Y, \end{array}$$

where  $X'$  and  $Y'$  are kernels of the respective localization maps  $X \rightarrow LX$ ,  $Y \rightarrow LY$ . Consequently, it will suffice to prove that  $u' \in S'$ . Since  $X', Y' \in \mathcal{C}_{\geq 0}$ , this follows from the two-out-of-three property, applied to the diagram

$$\begin{array}{ccc} & X' & \\ & \nearrow & \searrow^{u'} \\ 0 & \longrightarrow & Y'. \end{array}$$

□

**Proposition 6.16.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a left-complete  $t$ -structure. Let  $P \in \mathcal{C}_{\geq 0}$ . The following conditions are equivalent:*

- (1) *The object  $P$  is projective in  $\mathcal{C}_{\geq 0}$ .*
- (2) *For every  $Q \in \mathcal{C}_{\leq 0}$ , the abelian group  $\mathrm{Ext}_{\mathcal{C}}^1(P, Q)$  vanishes.*
- (3) *Given a distinguished triangle*

$$N' \rightarrow N \rightarrow N'' \rightarrow N'[1],$$

*where  $N', N, N'' \in \mathcal{C}_{\geq 0}$ , the induced map  $\mathrm{Ext}_{\mathcal{C}}^0(P, N) \rightarrow \mathrm{Ext}_{\mathcal{C}}^0(P, N'')$  is surjective.*

*Proof.* It follows from Lemma 14.9 that  $\mathcal{C}_{\geq 0}$  admits geometric realizations for simplicial objects, so that condition (1) makes sense. We first show that (1)  $\Rightarrow$  (2). Let  $f : \mathcal{C} \rightarrow \mathcal{S}$  be the functor corepresented by  $P$ . Let  $M_{\bullet}$  be a Čech nerve for the morphism  $0 \rightarrow Q[1]$ , so that  $M_n \simeq Q^n \in \mathcal{C}_{\geq 0}$ . Then  $Q[1]$  can be identified with the geometric realization  $|M_{\bullet}|$ . Since  $P$  is projective,  $f(Q[1])$  is equivalent to the geometric realization  $|f(M_{\bullet})|$ . We have a surjective map  $* \simeq \pi_0 f(M_0) \rightarrow \pi_0 |f(M_{\bullet})|$ , so that  $\pi_0 f(Q[1]) = \mathrm{Ext}_{\mathcal{C}}^1(P, Q) = 0$ .

We now show that (2)  $\Rightarrow$  (1). Proposition 10.12 implies that  $f$  is homotopic to a composition

$$\mathcal{C} \xrightarrow{F} \mathrm{Sp} \xrightarrow{\Omega^{\infty}} \mathcal{S},$$

where  $F$  is an exact functor. Applying (2), we deduce that  $F$  is right  $t$ -exact (Definition 14.7). Lemma 14.9 implies that the induced map  $\mathcal{C}_{\geq 0} \rightarrow \mathrm{Sp}^{\mathrm{conn}}$  preserves geometric realizations of simplicial objects. Applying Proposition 9.11, we conclude that  $f|_{\mathcal{C}_{\geq 0}}$  preserves geometric realizations as well.

The implication (2)  $\Rightarrow$  (3) follows immediately from the exactness of the sequence

$$\mathrm{Ext}_{\mathcal{C}}^0(P, N) \rightarrow \mathrm{Ext}_{\mathcal{C}}^0(P, N'') \rightarrow \mathrm{Ext}_{\mathcal{C}}^1(P, N').$$

Conversely, suppose that (3) is satisfied, and let  $\eta \in \mathrm{Ext}_{\mathcal{C}}^1(P, Q)$ . Then  $\eta$  classifies a distinguished triangle

$$Q \rightarrow Q' \xrightarrow{g} P \rightarrow Q[1].$$

Since  $Q, P \in \mathcal{C}_{\geq 0}$ , we have  $Q' \in \mathcal{C}_{\geq 0}$  as well. Invoking (3), we deduce that  $g$  admits a section, so that  $\eta = 0$ .  $\square$

## 7 Boundedness and Completeness

Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a  $t$ -structure. We let  $\mathcal{C}^+ = \bigcup \mathcal{C}_{\leq n} \subseteq \mathcal{C}$ ,  $\mathcal{C}^- = \bigcup \mathcal{C}_{\geq -n}$ , and  $\mathcal{C}^b = \mathcal{C}^+ \cap \mathcal{C}^-$ . It is easy to see that  $\mathcal{C}^-$ ,  $\mathcal{C}^+$ , and  $\mathcal{C}^b$  are stable subcategories of  $\mathcal{C}$ . We will say that  $\mathcal{C}$  is *left bounded* if  $\mathcal{C} = \mathcal{C}^+$ , *right bounded* if  $\mathcal{C} = \mathcal{C}^-$ , and *bounded* if  $\mathcal{C} = \mathcal{C}^b$ .

At the other extreme, given a stable  $\infty$ -category  $\mathcal{C}$  equipped with a  $t$ -structure, we define the *left completion*  $\widehat{\mathcal{C}}$  of  $\mathcal{C}$  to be homotopy limit of the tower

$$\dots \rightarrow \mathcal{C}_{\leq 2} \xrightarrow{\tau_{\leq 1}} \mathcal{C}_{\leq 1} \xrightarrow{\tau_{\leq 0}} \mathcal{C}_{\leq 0} \xrightarrow{\tau_{\leq -1}} \dots$$

Using the results of §T.3.3.3, we can obtain a very concrete description of this inverse limit: it is the full subcategory of  $\mathrm{Fun}(\mathbf{N}(\mathbf{Z}), \mathcal{C})$  spanned by those functors  $F : \mathbf{N}(\mathbf{Z}) \rightarrow \mathcal{C}$  with the following properties:

- (1) For each  $n \in \mathbf{Z}$ ,  $F(n) \in \mathcal{C}_{\leq -n}$ .
- (2) For each  $m \leq n \in \mathbf{Z}$ , the associated map  $F(m) \rightarrow F(n)$  induces an equivalence  $\tau_{\leq -n} F(m) \rightarrow F(n)$ .

We will denote this inverse limit by  $\widehat{\mathcal{C}}$ , and refer to it as the *left completion* of  $\mathcal{C}$ .

**Proposition 7.1.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a  $t$ -structure. Then:*

(1) The left completion  $\widehat{\mathcal{C}}$  is also stable.

(2) Let  $\widehat{\mathcal{C}}_{\leq 0}$  and  $\widehat{\mathcal{C}}_{\geq 0}$  be the full subcategories of  $\widehat{\mathcal{C}}$  spanned by those functors  $F : \mathbf{N}(\mathbf{Z}) \rightarrow \mathcal{C}$  which factor through  $\mathcal{C}_{\leq 0}$  and  $\mathcal{C}_{\geq 0}$ , respectively. Then these subcategories determine a t-structure on  $\widehat{\mathcal{C}}$ .

(3) There is a canonical functor  $\mathcal{C} \rightarrow \widehat{\mathcal{C}}$ . This functor is exact, and induces an equivalence  $\mathcal{C}_{\leq 0} \rightarrow \widehat{\mathcal{C}}_{\leq 0}$ .

*Proof.* We observe that  $\widehat{\mathcal{C}}$  can be identified with the homotopy inverse limit of the tower

$$\dots \rightarrow \mathcal{C}_{\leq 0} \xrightarrow{\tau_{\leq 0}^{\Sigma}} \mathcal{C}_{\leq 0} \xrightarrow{\tau_{\leq 0}^{\Sigma}} \mathcal{C}_{\leq 0}.$$

In other words,  $\widehat{\mathcal{C}}^{op} \simeq \mathrm{Sp}(\mathcal{C}^{op})$  (see §13). Assertion (1) now follows from Proposition 8.27.

We next prove (2). We begin by observing that, if we identify  $\widehat{\mathcal{C}}$  with a full subcategory of  $\mathrm{Fun}(\mathbf{N}(\mathbf{Z}), \mathcal{C})$ , then the shift functors on  $\widehat{\mathcal{C}}$  can be defined by the formula

$$(F[n])(m) = F(m - n)[n].$$

This proves immediately that  $\widehat{\mathcal{C}}_{\geq 0}[1] \subseteq \widehat{\mathcal{C}}_{\geq 0}$  and  $\widehat{\mathcal{C}}_{\leq 0}[-1] \subseteq \widehat{\mathcal{C}}_{\leq 0}$ . Moreover, if  $X \in \widehat{\mathcal{C}}_{\geq 0}$  and  $Y \in \widehat{\mathcal{C}}_{\leq -1} = \widehat{\mathcal{C}}_{\leq 0}[-1]$ , then  $\mathrm{Map}_{\widehat{\mathcal{C}}}(X, Y)$  can be identified with the homotopy limit of a tower of spaces

$$\dots \rightarrow \mathrm{Map}_{\mathcal{C}}(X(n), Y(n)) \rightarrow \mathrm{Map}_{\mathcal{C}}(X(n-1), Y(n-1)) \rightarrow \dots$$

Since each of these spaces is contractible, we conclude that  $\mathrm{Map}_{\widehat{\mathcal{C}}}(X, Y) \simeq *$ ; in particular,  $\mathrm{Ext}_{\widehat{\mathcal{C}}}^0(X, Y) = 0$ .

Finally, we consider an arbitrary  $X \in \widehat{\mathcal{C}}$ . Let  $X'' = \tau_{\leq -1} \circ X : \mathbf{N}(\mathbf{Z}) \rightarrow \mathcal{C}$ , and let  $u : X \rightarrow X''$  be the induced map. It is easy to check that  $X'' \in \widehat{\mathcal{C}}_{\leq -1}$  and that  $\ker(u) \in \widehat{\mathcal{C}}_{\geq 0}$ . This completes the proof of (2).

To prove (3), we let  $\mathcal{D}$  denote the full subcategory of  $\mathbf{N}(\mathbf{Z}) \times \mathcal{C}$  spanned by pairs  $(n, C)$  such that  $C \in \mathcal{C}_{\leq -n}$ . Using Proposition T.5.2.7.8, we deduce that the inclusion  $\mathcal{D} \subseteq \mathbf{N}(\mathbf{Z}) \times \mathcal{C}$  admits a left adjoint  $L$ . The composition

$$\mathbf{N}(\mathbf{Z}) \times \mathcal{C} \xrightarrow{L} \mathcal{D} \subseteq \mathbf{N}(\mathbf{Z}) \times \mathcal{C} \rightarrow \mathcal{C}$$

can be identified with a functor  $\theta : \mathcal{C} \rightarrow \mathrm{Fun}(\mathbf{N}(\mathbf{Z}), \mathcal{C})$  which factors through  $\widehat{\mathcal{C}}$ . To prove that  $\theta$  is exact, it suffices to show that  $\theta$  is right exact (Proposition 5.1). Since the truncation functors  $\tau_{\leq n} : \mathcal{C}_{\leq n+1} \rightarrow \mathcal{C}_{\leq n}$  are right exact, finite colimits in  $\widehat{\mathcal{C}}$  are computed pointwise. Consequently, it suffices to prove that each of compositions

$$\mathcal{C} \xrightarrow{\theta} \widehat{\mathcal{C}} \rightarrow \tau_{\leq n} \mathcal{C}$$

is right exact. But this composition can be identified with the functor  $\tau_{\leq n}$ .

Finally, we observe that  $\widehat{\mathcal{C}}_{\leq 0}$  can be identified with a homotopy limit of the essentially constant tower

$$\dots \mathcal{C}_{\leq 0} \xrightarrow{\mathrm{id}} \mathcal{C}_{\leq 0} \xrightarrow{\mathrm{id}} \mathcal{C}_{\leq 0} \xrightarrow{\tau_{\leq -1}} \mathcal{C}_{\leq -1} \rightarrow \dots,$$

and that  $\theta$  induces an identification of this homotopy limit with  $\mathcal{C}_{\leq 0}$ .  $\square$

If  $\mathcal{C}$  is a stable  $\infty$ -category equipped with a t-structure, then we will say that  $\mathcal{C}$  is *left complete* if the functor  $\mathcal{C} \rightarrow \widehat{\mathcal{C}}$  described in Proposition 7.1 is an equivalence.

**Remark 7.2.** Let  $\mathcal{C}$  be as in Proposition 7.1. Then the inclusion  $\mathcal{C}^+ \subseteq \mathcal{C}$  induces an equivalence  $\widehat{\mathcal{C}}^+ \rightarrow \widehat{\mathcal{C}}$ , and the functor  $\mathcal{C} \rightarrow \widehat{\mathcal{C}}$  induces an equivalence  $\mathcal{C}^+ \rightarrow \widehat{\mathcal{C}}^+$ . Consequently, the constructions

$$\begin{aligned} \mathcal{C} &\mapsto \widehat{\mathcal{C}} \\ \mathcal{C} &\mapsto \mathcal{C}^+ \end{aligned}$$

furnish an equivalence between the theory of left bounded stable  $\infty$ -categories and the theory of left complete stable  $\infty$ -categories.

We conclude this section with a useful criterion for establishing left completeness.

**Proposition 7.3.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a  $t$ -structure. Suppose that  $\mathcal{C}$  admits countable products, and that  $\mathcal{C}_{\geq 0}$  is stable under countable products. The following conditions are equivalent:*

- (1) *The  $\infty$ -category  $\mathcal{C}$  is left complete.*
- (2) *The full subcategory  $\mathcal{C}_{\geq \infty} = \bigcap \mathcal{C}_{\geq n} \subseteq \mathcal{C}$  consists only of zero objects of  $\mathcal{C}$ .*

*Proof.* We first observe every tower of objects

$$\dots \rightarrow X_n \rightarrow X_{n-1} \rightarrow \dots$$

in  $\mathcal{C}$  admits a limit  $\varprojlim\{X_n\}$ : we can compute this limit as the kernel of an appropriate map

$$\prod X_n \rightarrow \prod X_n.$$

Moreover, if each  $X_n$  belongs to  $\mathcal{C}_{\geq 0}$ , then  $\varprojlim\{X_n\}$  belongs to  $\mathcal{C}_{\geq -1}$ .

The functor  $F : \mathcal{C} \rightarrow \widehat{\mathcal{C}}$  of Proposition 7.1 admits a right adjoint  $G$ , given by

$$f \in \widehat{\mathcal{C}} \subseteq \text{Fun}(\mathbf{N}(\mathbf{Z}), \mathcal{C}) \mapsto \varprojlim(f).$$

Assertion (1) is equivalent to the statement that the unit and counit maps

$$u : F \circ G \rightarrow \text{id}_{\widehat{\mathcal{C}}}$$

$$v : \text{id}_{\mathcal{C}} \rightarrow G \circ F$$

are equivalences. If  $v$  is an equivalence, then any object  $X \in \mathcal{C}$  can be recovered as the limit of the tower  $\{\tau_{\leq n} X\}$ . In particular, this implies that  $X = 0$  if  $X \in \mathcal{C}_{\geq \infty}$ , so that (1)  $\Rightarrow$  (2).

Now assume (2); we will prove that  $u$  and  $v$  are both equivalences. To prove that  $u$  is an equivalence, we must show that for every  $f \in \widehat{\mathcal{C}}$ , the natural map

$$\theta : \varprojlim(f) \rightarrow f(n)$$

induces an equivalence  $\tau_{\leq -n} \varprojlim(f) \rightarrow f(n)$ . In other words, we must show that the kernel of  $\theta$  belongs to  $\mathcal{C}_{\geq -n+1}$ . To prove this, we first observe that  $\theta$  factors as a composition

$$\varprojlim(f) \xrightarrow{\theta'} f(n-1) \xrightarrow{\theta''} f(n).$$

The octahedral axiom ((TR4) of Definition 3.1) implies the existence of an exact triangle

$$\ker(\theta') \rightarrow \ker(\theta) \rightarrow \ker(\theta'').$$

Since  $\ker(\theta'')$  clearly belongs to  $\mathcal{C}_{\geq -n+1}$ , it will suffice to show that  $\ker(\theta')$  belongs to  $\mathcal{C}_{\geq -n+1}$ . We observe that  $\ker(\theta')$  can be identified with the limit of a tower  $\{\ker(f(m) \rightarrow f(n-1))\}_{m < n}$ . It therefore suffices to show that each  $\ker(f(m) \rightarrow f(n-1))$  belongs to  $\mathcal{C}_{\geq -n+2}$ , which is clear.

We now show prove that  $v$  is an equivalence. Let  $X$  be an object of  $\mathcal{C}$ , and  $v_X : X \rightarrow (G \circ F)(X)$  the associated map. Since  $u$  is an equivalence of functors, we conclude that  $\tau_{\leq n}(v_X)$  is an equivalence for all  $n \in \mathbf{Z}$ . It follows that  $\text{coker}(v_X) \in \mathcal{C}_{\geq n+1}$  for all  $n \in \mathbf{Z}$ . Invoking assumption (2), we conclude that  $\text{coker}(v_X) \simeq 0$ , so that  $v_X$  is an equivalence as desired.  $\square$

**Remark 7.4.** The ideas introduced above can be dualized in an obvious way, so that we can speak of *right completions* and *right completeness* for a stable  $\infty$ -category equipped with a  $t$ -structure.



## 8 Stabilization

In this section, we will describe a method for constructing stable  $\infty$ -categories: namely, for any  $\infty$ -category  $\mathcal{C}$  which admits finite limits, one can consider an  $\infty$ -category  $\mathrm{Sp}(\mathcal{C})$  of *spectrum objects* of  $\mathcal{C}$ . In the case where  $\mathcal{C}$  is the  $\infty$ -category of spaces, we recover classical stable homotopy theory, which we will discuss in §9.

**Definition 8.1.** Let  $\mathcal{C}$  be an  $\infty$ -category. A *prespectrum object* of  $\mathcal{C}$  is a functor  $X : \mathbf{N}(\mathbf{Z} \times \mathbf{Z}) \rightarrow \mathcal{C}$  with the following property: for every pair of integers  $i \neq j$ , the value  $X(i, j)$  is a zero object of  $\mathcal{C}$ . We let  $\mathrm{PSp}(\mathcal{C})$  denote the full subcategory of  $\mathrm{Fun}(\mathbf{N}(\mathbf{Z} \times \mathbf{Z}), \mathcal{C})$  spanned by the prespectrum objects of  $\mathcal{C}$ .

For every integer  $n$ , evaluation at  $(n, n) \in \mathbf{Z} \times \mathbf{Z}$  induces a functor  $\mathrm{PSp}(\mathcal{C}) \rightarrow \mathcal{C}$ . We will refer to this functor as the  *$n$ th space functor* and denote it by  $\Omega_{\mathcal{C}}^{\infty-n}$ .

**Remark 8.2.** The partially ordered set  $\mathbf{Z} \times \mathbf{Z}$  is isomorphic to its opposite, via the map  $(i, j) \mapsto (-i, -j)$ . Composing with this map, we obtain an equivalence

$$\mathrm{PSp}(\mathcal{C})^{op} \simeq \mathrm{PSp}(\mathcal{C}^{op}).$$

**Remark 8.3.** Let  $X$  be a prespectrum object of an  $\infty$ -category  $\mathcal{C}$ . Since the objects  $X(i, j) \in \mathcal{C}$  are zero for  $i \neq j$ , it is customary to ignore them and instead emphasize the objects  $X(n, n) \in \mathcal{C}$  lying along the diagonal. We will often denote  $X(n, n) = \Omega_{\mathcal{C}}^{\infty-n} X$  by  $X[n]$ . For each  $n \geq 0$ , the diagram

$$\begin{array}{ccc} X(n, n) & \longrightarrow & X(n, n+1) \\ \downarrow & & \downarrow \\ X(n+1, n) & \longrightarrow & X(n+1, n+1) \end{array}$$

determines an (adjoint) pair of morphisms

$$\alpha : \Sigma_{\mathcal{C}} X[n] \rightarrow X[n+1] \quad \beta : X[n] \rightarrow \Omega_{\mathcal{C}} X[n+1].$$

**Definition 8.4.** Let  $X$  be a prespectrum object of a pointed  $\infty$ -category  $\mathcal{C}$ , and  $n$  an integer. We will say that  $X$  is a *spectrum below  $n$*  if the canonical map  $\beta : X[m-1] \rightarrow \Omega_{\mathcal{C}} X[m]$  is an equivalence for each  $m \leq n$ . We say that  $X$  is a *suspension prespectrum above  $n$*  if the canonical map  $\alpha : \Sigma_{\mathcal{C}} X[m] \rightarrow X[m+1]$  is an equivalence for all  $m \geq n$ . We say that  $X$  is an  *$n$ -suspension prespectrum* if it is a suspension prespectrum above  $n$  and a spectrum below  $n$ . We say that  $X$  is a *spectrum object* if it is a spectrum object below  $n$  for all integers  $n$ . We let  $\mathrm{Sp}(\mathcal{C})$  denote the full subcategory of  $\mathrm{PSp}(\mathcal{C})$  spanned by the spectrum objects of  $\mathcal{C}$ .

If  $\mathcal{C}$  is an arbitrary  $\infty$ -category, we let  $\mathrm{Stab}(\mathcal{C}) = \mathrm{Sp}(\mathcal{C}_*)$ . Here  $\mathcal{C}_*$  denotes the  $\infty$ -category of pointed objects of  $\mathcal{C}$ . We will refer to  $\mathrm{Stab}(\mathcal{C})$  as the *stabilization* of  $\mathcal{C}$ .

**Remark 8.5.** Suppose that  $\mathcal{C}$  is a pointed  $\infty$ -category. Then the forgetful functor  $\mathcal{C}_* \rightarrow \mathcal{C}$  is a trivial Kan fibration, which induces a trivial Kan fibration  $\mathrm{Stab}(\mathcal{C}) \rightarrow \mathrm{Sp}(\mathcal{C})$ .

**Example 8.6.** Let  $\mathbf{Q}$  be the ring of rational numbers, let  $\mathbf{A}$  be the category of simplicial commutative  $\mathbf{Q}$ -algebras, viewed as simplicial model category (see Proposition T.5.5.9.1), and let  $\mathcal{C} = \mathbf{N}(\mathbf{A}^{\circ})$  be the underlying  $\infty$ -category. Suppose that  $R$  is a commutative  $\mathbf{Q}$ -algebra, regarded as an object of  $\mathcal{C}$ . Then  $\mathrm{Stab}(\mathcal{C}/R)$  is a stable  $\infty$ -category, whose homotopy category is equivalent to the (unbounded) derived category of  $R$ -modules. The loop functor  $\Omega^{\infty} : \mathrm{Stab}(\mathcal{C}/R) \rightarrow \mathcal{C}/R$  admits a left adjoint  $\Sigma^{\infty} : \mathcal{C}/R \rightarrow \mathrm{Stab}(\mathcal{C}/R)$  (Proposition 15.4). This left adjoint assigns to each morphism of commutative rings  $S \xrightarrow{\phi} R$  an object  $\Sigma^{\infty}(\phi) \in \mathrm{Stab}(\mathcal{C}/R)$ , which can be identified with  $L_S \otimes_S R$ , where  $L_S$  denotes the (absolute) *cotangent complex* of  $S$ . We will discuss this example in greater detail in [44]; see also [59] for discussion.

**Remark 8.7.** Let  $\mathcal{C}$  be a pointed presentable  $\infty$ -category. Using Lemmas T.5.5.4.17, T.5.5.4.18, and T.5.5.4.19, we deduce that  $\mathrm{P}\mathrm{Sp}(\mathcal{C})$  and  $\mathrm{Sp}(\mathcal{C})$  are accessible localizations of  $\mathrm{Fun}(\mathbf{N}(\mathbf{Z} \times \mathbf{Z}), \mathcal{C})$ . It follows that  $\mathrm{P}\mathrm{Sp}(\mathcal{C})$  and  $\mathrm{Sp}(\mathcal{C})$  are themselves presentable  $\infty$ -categories. Moreover, the inclusion  $\mathrm{Sp}(\mathcal{C}) \subseteq \mathrm{P}\mathrm{Sp}(\mathcal{C})$  admits an accessible left adjoint  $L_{\mathcal{C}}$ , which we will refer to as the *spectrification functor*. We will give a more direct construction of  $L_{\mathcal{C}}$  below in the case where  $\mathcal{C}$  satisfies some mild hypotheses.

**Remark 8.8.** Suppose that  $\mathcal{C}$  is a pointed  $\infty$ -category which admits finite limits and countable colimits, and that the loop functor  $\Omega_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$  preserves sequential colimits. Then the collection of prespectrum objects of  $\mathcal{C}$  which are spectra below  $n$  is closed under sequential colimits.

**Remark 8.9.** The hypotheses of Remark 8.8 are always satisfied in any of the following cases:

- (1) The  $\infty$ -category  $\mathcal{C}$  is pointed and compactly generated.
- (2) The  $\infty$ -category  $\mathcal{C}$  is an  $\infty$ -topos (Example T.7.3.4.7).
- (3) The  $\infty$ -category  $\mathcal{C}$  is stable and admits countable coproducts. In this case, Proposition T.4.4.3.2 guarantees that  $\mathcal{C}$  admits all countable colimits, and the functor  $\Omega_{\mathcal{C}}$  is an equivalence and therefore preserves all colimits which exist in  $\mathcal{C}$ .

In order to work effectively with prespectrum objects, it is convenient to introduce a bit of additional terminology.

**Notation 8.10.** For  $-\infty \leq a \leq b \leq \infty$ , we let  $Q(a, b) = \{(i, j) \in \mathbf{Z} \times \mathbf{Z} : (i \neq j) \vee (a \leq i = j \leq b)\}$ . If  $\mathcal{C}$  is an  $\infty$ -category, we let  $\mathrm{P}\mathrm{Sp}_a^b(\mathcal{C})$  denote the full subcategory of  $\mathrm{Fun}(\mathbf{N}(Q(a, b)), \mathcal{C})$  spanned by those functors  $X$  such that  $X(i, j)$  is a zero object of  $\mathcal{C}$  for  $i \neq j$ .

**Lemma 8.11.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits finite limits, and suppose given  $\infty < a \leq b \leq \infty$ . Let  $X_0 \in \mathrm{P}\mathrm{Sp}_a^b(\mathcal{C})$ . Then:*

- (1) *There exists an object  $X \in \mathrm{P}\mathrm{Sp}_{a-1}^b(\mathcal{C})$  which is a right Kan extension of  $X_0$ .*
- (2) *An arbitrary object  $X \in \mathrm{P}\mathrm{Sp}_{a-1}^b(\mathcal{C})$  which extends  $X_0$  is a right Kan extension of  $X_0$  if and only if the induced map  $X[a-1] \rightarrow X[a]$ .*

*Proof.* Note that  $Q(a-1, b)$  is obtained from  $Q(a, b)$  by adjoining a single additional object  $(a-1, a-1)$ . It now suffices to observe that the inclusion of  $\infty$ -categories

$$\mathbf{N}(\{(a-1, a), (a, a), (a, a-1)\})^{op} \subseteq \mathbf{N}(\{(i, j) \in Q(a, b) \mid (a-1 \leq i, j)\})^{op}$$

is cofinal, which follows immediately from the criterion of Theorem T.4.1.3.1. □

**Lemma 8.12.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits finite limits and suppose given  $\infty < a \leq b \leq \infty$ . Let  $X_0 \in \mathrm{P}\mathrm{Sp}_a^b(\mathcal{C})$ . Then:*

- (1) *There exists an object  $X \in \mathrm{P}\mathrm{Sp}_{-\infty}^b(\mathcal{C})$  which is a right Kan extension of  $X_0$ .*
- (2) *An arbitrary object  $X \in \mathrm{P}\mathrm{Sp}_{-\infty}^b(\mathcal{C})$  extending  $X_0$  is a right Kan extension of  $X_0$  if and only if  $X$  is a spectrum object below  $a$ .*

*Proof.* Combine Lemma 8.11 with Proposition T.4.3.2.8. □

**Lemma 8.13.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category and  $n$  an integer. Then evaluation at  $(n, n)$  induces a trivial Kan fibration  $\mathrm{P}\mathrm{Sp}_n^n(\mathcal{C}) \rightarrow \mathcal{C}$ .*

*Proof.* Let  $Q' = \{(i, j) \in \mathbf{Z} \times \mathbf{Z} : (i < j \leq n) \vee (j < i \leq n) \vee (i = j = n)\}$ , and let  $\mathcal{D} \subseteq \mathrm{Fun}(\mathbf{N}(Q'), \mathcal{C})$  denote the full subcategory spanned by those functors  $X$  such that  $X(i, j)$  is a zero object of  $\mathcal{C}$  for  $i \neq j$ . We observe the following:

- (a) A functor  $X : \mathbf{N}(Q') \rightarrow \mathcal{C}$  belongs to  $\mathcal{D}$  if and only if  $X$  is a left Kan extension of  $X|_{\{(n, n)\}}$ .
- (b) A functor  $X : \mathbf{N}(Q(n, n)) \rightarrow \mathcal{C}$  belongs to  $\mathbf{PSp}_n^n(\mathcal{C})$  if and only if  $X|_{\mathbf{N}(Q')} \in \mathcal{D}$  and  $X$  is a right Kan extension of  $X|_{\mathbf{N}(Q')}$ .

It now follows from Proposition T.4.3.2.15 that the restriction functors

$$\mathbf{PSp}_n^n(\mathcal{C}) \rightarrow \mathcal{D} \rightarrow \mathcal{C}$$

are trivial Kan fibrations, and the composition is given by evaluation at  $(n, n)$ .  $\square$

We can now describe the stabilization  $\mathbf{Sp}(\mathcal{C})$  of a pointed  $\infty$ -category  $\mathcal{C}$  in more conceptual terms:

**Proposition 8.14.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits finite limits. Then the  $\infty$ -category  $\mathbf{Sp}(\mathcal{C})$  can be identified with the homotopy inverse limit of the tower*

$$\dots \rightarrow \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}.$$

*Proof.* For every nonnegative integer  $n$ , let  $\mathcal{D}_n$  denote the full subcategory of  $\mathbf{PSp}_{-\infty}^n(\mathcal{C})$  spanned by those functors  $X$  such that the diagram

$$\begin{array}{ccc} X(m, m) & \longrightarrow & X(m, m+1) \\ \downarrow & & \downarrow \\ X(m+1, m) & \longrightarrow & X(m+1, m+1) \end{array}$$

is a pullback square for each  $m < n$ . We note that Lemma 8.12 and Proposition T.4.3.2.15 imply that the composition

$$\mathcal{D}_n \subseteq \mathbf{PSp}_{-\infty}^n(\mathcal{C}) \rightarrow \mathbf{PSp}_n^n(\mathcal{C})$$

is a trivial Kan fibration. Combining this with Lemma 8.13, we deduce that evaluation at  $(n, n)$  induces a trivial Kan fibration  $\psi_n : \mathcal{D}_n \rightarrow \mathcal{C}$ . Let  $s_n$  denote a section to  $\psi_n$ . We observe that the composite functor

$$\mathcal{C} \xrightarrow{s_n} \mathcal{D}_n \rightarrow \mathcal{D}_{n-1} \xrightarrow{\psi_n} \mathcal{C}$$

can be identified with the loop functor  $\Omega_{\mathcal{C}}$ . It follows that the tower

$$\dots \rightarrow \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}.$$

is equivalent to the tower of restriction maps

$$\dots \rightarrow \mathcal{D}_2 \rightarrow \mathcal{D}_1 \rightarrow \mathcal{D}_0.$$

This tower consists of categorical fibrations between  $\infty$ -categories, so its homotopy inverse limit coincides with the actual inverse limit  $\varprojlim \{\mathcal{D}_n\}_{n \geq 0} \simeq \mathbf{Sp}(\mathcal{C})$ .  $\square$

We now study the ‘‘spectrification functor’’  $\mathbf{PSp}(\mathcal{C}) \rightarrow \mathbf{Sp}(\mathcal{C})$  in the case where  $\mathcal{C}$  is well-behaved.

**Lemma 8.15.** *Let  $P = \{(i, j, k) \in \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \mid (i \neq j) \vee (i = j \geq k)\}$ . Let  $X : \mathbf{N}(P) \rightarrow \mathcal{C}$  be a functor, where  $\mathcal{C}$  is a pointed  $\infty$ -category which admits finite limits. Suppose that  $X(i, j, k)$  is a zero object of  $\mathcal{C}$  for all  $i \neq j$ . Then:*

- (1) Let  $\overline{X} : \mathbf{N}(\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}) \rightarrow \mathcal{C}$  be an extension of  $X$ . The following conditions are equivalent:

- (i) The functor  $\overline{X}$  is a right Kan extension of  $X$ .

(ii) For each  $k \geq 0$ , the induced functor  $\overline{X}|N(\mathbf{Z} \times \mathbf{Z} \times \{k\})$  is a right Kan extension of  $X|N(Q(k, \infty) \times \{k\})$ .

(iii) For each  $k \geq 0$ , the functor  $\overline{X}|N(\mathbf{Z} \times \mathbf{Z} \times \{k\})$  is a spectrum object below  $k$ .

(2) There exists a functor  $\overline{X} : N(\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}) \rightarrow \mathcal{C}$  satisfying the equivalent conditions of (1).

*Proof.* Let  $\overline{X}$  be as in the statement of (1). To prove the equivalence of conditions (i) and (ii), it will suffice to prove the following more precise claim: for every triple of nonnegative integers  $(i, j, k)$ , the functor  $\overline{X}$  is a right Kan extension of  $X$  at  $(i, j, k)$  if and only if  $\overline{X}|N(\mathbf{Z} \times \mathbf{Z} \times \{k\})$  is a right Kan extension of  $X|N(Q(k, \infty) \times \{k\})$  at  $(i, j, k)$ . This follows from the observation that the inclusion

$$N(\{(i', j', k) \in P | (i' \geq i) \wedge (j' \geq j)\})^{op} \subseteq N(\{(i', j', k') \in P | (i' \geq i) \wedge (j' \geq j) \wedge (k' \geq k)\})^{op}$$

is cofinal (since it admits left adjoint, given by  $(i', j', k') \mapsto (i', j', k)$ ). The equivalence of (ii) and (iii) follows from Lemma 8.12.

To prove (2), we define a sequence of subsets

$$P = P(0) \subseteq P(1) \subseteq P(2) \subseteq \dots \subseteq \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$$

by the formula  $P(m) = \{(i, j, k) \in \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} | (i \neq j) \vee (i = j \geq k - m)\}$ . Using Proposition T.4.3.2.8, we deduce that if  $\overline{X} : N(\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}) \rightarrow \mathcal{C}$  is an extension of  $X$ , then  $\overline{X}$  is a right Kan extension of  $X$  if and only if each restriction  $\overline{X}|N(P(m+1))$  is a right Kan extension of  $\overline{X}|N(P(m))$ . Consequently, (2) is a consequence of the following more precise assertion:

(2') Every functor  $Y_0 : N(P(m)) \rightarrow \mathcal{C}$  extending  $X$  admits a right Kan extension  $Y : N(P(m+1)) \rightarrow \mathcal{C}$  satisfying (1').

To prove these assertions, we note that every element of  $P(m+1)$  which does not belong to  $P(m)$  has the form  $(n-1, n-1, n+m)$  for some integer  $n$ . It now suffices to observe that the inclusion  $N(P'_0)^{op} \subseteq N(P')^{op}$  is cofinal, where

$$P' = \{(i, j, k) \in P(m) : (i, j \geq n-1) \wedge (k \geq n+m)\}$$

$$P'_0 = \{(n-1, n, n+m), (n, n-1, n+m), (n, n, n+m)\} \subseteq P'.$$

This follows immediately from the criterion of Theorem T.4.1.3.1. □

**Corollary 8.16.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits finite limits. Then there exists a sequence of functors*

$$\text{id} \rightarrow L_0 \rightarrow L_1 \rightarrow L_2 \rightarrow \dots$$

*from  $\text{PSp}(\mathcal{C})$  to itself such that the following conditions are satisfied for every prespectrum object  $X$  of  $\mathcal{C}$  and every  $n \geq 0$ :*

- (1) *The prespectrum  $L_n(X)$  is a spectrum below  $n$ .*
- (2) *The map  $\alpha : X \rightarrow L_n(X)$  induces an equivalence  $X[m] \rightarrow L_n(X)[m]$  for  $m \geq n$ .*
- (3) *Suppose that  $X$  is a spectrum below  $n$ . Then the map  $\alpha : X \rightarrow L_n(X)$  is an equivalence.*
- (4) *Let  $Y$  be any prespectrum object of  $\mathcal{C}$  which is a spectrum below  $n$ . Then composition with  $\alpha$  induces a homotopy equivalence*

$$\text{Map}_{\text{PSp}(\mathcal{C})}(L_n(X), Y) \rightarrow \text{Map}_{\text{PSp}(\mathcal{C})}(X, Y).$$

*Proof.* Let  $P$  be defined as in Lemma 8.15, let  $\mathcal{D}$  denote the full subcategory of  $\text{Fun}(\mathbf{N}(P), \mathcal{C})$  spanned by those functors  $F$  such that  $F(i, j, k)$  is a zero object of  $\mathcal{C}$  for  $i \neq j$ , and let  $\mathcal{D}' \subseteq \text{Fun}(\mathbf{N}(\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}), \mathcal{C})$  denote the full subcategory spanned by those objects  $F$  such that  $F$  is a right Kan extension of  $F|_{\mathbf{N}(P)} \in \mathcal{D}$ . Using Lemma 8.15 and Proposition T.4.3.2.15, we deduce that the restriction map  $\mathcal{D}' \rightarrow \mathcal{D}$  is a trivial Kan fibration. Let  $p$  denote a section of this restriction map.

Let  $q : \text{PSP}(\mathcal{C}) \rightarrow \mathcal{D}$  denote the functor induced by the map of partially ordered sets  $P \rightarrow \mathbf{Z} \times \mathbf{Z}$   $(i, j, k) \mapsto (i, j)$ . The composition  $p \circ q : \text{PSP}(\mathcal{C}) \rightarrow \mathcal{D}'$  determines a map  $\text{PSP}(\mathcal{C}) \times \mathbf{N}(\mathbf{Z}_{\geq 0}) \rightarrow \text{PSP}(\mathcal{C})$ , which we can identify with a sequence of functors  $\{L_n\}_{n \in \mathbf{Z}}$  from  $\text{PSP}(\mathcal{C})$  to itself. By construction, we also have a canonical map  $\text{id} \rightarrow L_0$ . Assertions (1) and (2) are immediate consequences of the construction, and assertion (3) follows from (1) and (2). To prove (4), it will suffice (by virtue of (3)) to show that composition with  $\alpha$  induces a homotopy equivalence

$$\text{Map}_{\text{PSP}(\mathcal{C})}(L_n(X), L_n(Y)) \rightarrow \text{Map}_{\text{PSP}(\mathcal{C})}(X, L_n(Y)).$$

Since  $\alpha$  induces an equivalence  $X|_{\mathbf{N}(Q(n, \infty))} \simeq L_n(X)|_{\mathbf{N}(Q(n, \infty))}$ , it will suffice to show that  $L_n(Y)$  is a right Kan extension of  $L_n(Y)|_{\mathbf{N}(Q(n, \infty))}$ , which follows from the equivalence of (i) and (ii) in Lemma 8.12.  $\square$

**Corollary 8.17.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category satisfying the hypotheses of Remark 8.8. Let  $\{L_n : \text{PSP}(\mathcal{C}) \rightarrow \text{PSP}(\mathcal{C})\}_{n \geq 0}$  be the localization functors of Corollary 8.16. Then  $L = \varinjlim_n L_n$  is a localization functor from  $\text{PSP}(\mathcal{C})$  to itself, whose essential image is the collection of spectrum objects of  $\mathcal{C}$ .*

*Proof.* It will suffice to prove the following assertions for every prespectrum object  $X$  of  $\mathcal{C}$ :

- (1) The prespectrum  $LX$  is a spectrum object of  $\mathcal{C}$ .
- (2) For every spectrum object  $Y$  of  $\mathcal{C}$ , composition with the map  $\alpha : X = L_0X \rightarrow LX$  induces a homotopy equivalence

$$\text{Map}_{\text{PSP}(\mathcal{C})}(LX, Y) \rightarrow \text{Map}_{\text{PSP}(\mathcal{C})}(X, Y).$$

To prove (1), it suffices to show that  $LX$  is a prespectrum below  $n$ , for each  $n \geq 0$ . Since  $LX$  is a colimit of the sequence of prespectra  $\{L_m X\}_{m \geq n}$ , each of which is a spectrum below  $n$ , this follows from Remark 8.8.

To prove (2), we note that  $\text{Map}_{\text{PSP}(\mathcal{C})}(LX, Y)$  is given by the homotopy inverse limit of a tower of spaces  $\{\text{Map}_{\text{PSP}(\mathcal{C})}(L_n X, Y)\}_{n \geq 0}$ . Consequently, it will suffice to prove that each of the canonical maps  $\text{Map}_{\text{PSP}(\mathcal{C})}(L_n X, Y) \rightarrow \text{Map}_{\text{PSP}(\mathcal{C})}(X, Y)$  is a homotopy equivalence. This follows from Corollary 8.16, since  $Y$  is a spectrum below  $n$ .  $\square$

**Remark 8.18.** Let  $\mathcal{C}$  be a pointed  $\infty$ -category satisfying the hypotheses of Remark 8.8, and let  $f : X \rightarrow Y$  be a morphism between prespectrum objects of  $\mathcal{C}$ . Suppose that there exists an integer  $n \geq 0$  such that  $f$  induces an equivalence  $X[m] \rightarrow Y[m]$  for  $m \geq n$ . It follows that  $L_m(f) : L_m X \rightarrow L_m Y$  is an equivalence for  $m \geq n$ , so that  $L(f) = \varinjlim_m L_m(f)$  is an equivalence in  $\text{Sp}(\mathcal{C})$ .

**Remark 8.19.** Let  $\mathcal{C}$  be a presentable pointed  $\infty$ -category satisfying the hypotheses of Remark 8.8. Corollary 8.17 asserts that if  $X$  is a prespectrum object of  $\mathcal{C}$ , then the associated spectrum  $X'$  is computed in the usual way: the  $n$ th space  $X'[n]$  is given as a colimit  $\text{colim}_{m \geq 0} \Omega_{\mathcal{C}}^m X[n+m]$ .

Suppose that  $\mathcal{C}$  is a presentable pointed  $\infty$ -category. We note that  $\text{Sp}(\mathcal{C})$  is closed under small limits in  $\text{Fun}(\mathbf{N}(\mathbf{Z} \times \mathbf{Z}), \mathcal{C})$ , so that limits in  $\text{Sp}(\mathcal{C})$  are computed pointwise. It follows that the evaluation functors  $\Omega_{\mathcal{C}}^{\infty-n} : \text{Sp}(\mathcal{C}) \rightarrow \mathcal{C}$  preserve small limits. Since these functors are also accessible, Corollary T.5.5.2.9 guarantees that  $\Omega_{\mathcal{C}}^{\infty-n}$  admits a left adjoint, which we will denote by  $\Sigma_{\mathcal{C}}^{\infty-n} : \mathcal{C} \rightarrow \text{Sp}(\mathcal{C})$ . Our next goal is to describe this functor in more explicit terms.

**Lemma 8.20.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits finite limits and colimits. Then:*

- (1) *A prespectrum object  $X$  of  $\mathcal{C}$  is a suspension prespectrum above  $n$  if and only if  $X$  is a left Kan extension of  $X|_{Q(-\infty, n)}$ .*

- (2) For every  $X_0 \in \mathrm{PSP}_{-\infty}^n(\mathcal{C})$ , there exists an extension  $X \in \mathrm{PSP}(\mathcal{C})$  of  $X_0$  which satisfies the equivalent conditions of (1).
- (3) Let  $\mathcal{D}$  denote the full subcategory of  $\mathrm{PSP}(\mathcal{C})$  spanned by those prespectrum objects of  $\mathcal{C}$  which are suspension prespectra above  $n$ . Then  $\mathcal{D}$  is a colocalization of  $\mathrm{PSP}(\mathcal{C})$ . Moreover, a morphism of prespectra  $X \rightarrow Y$  exhibits  $X$  as a  $\mathcal{D}$ -colocalization of  $Y$  if and only if  $X \in \mathcal{D}$  and the induced map  $X[k] \rightarrow Y[k]$  is an equivalence for  $k \leq n$ .
- (4) Let  $\mathcal{D}_0$  denote the full subcategory of  $\mathrm{PSP}(\mathcal{C})$  spanned by the  $n$ -suspension prespectrum objects, and let  $\mathcal{E}$  denote the full subcategory of  $\mathrm{PSP}_{-\infty}^n(\mathcal{C})$  spanned by those functors  $X$  such that the induced map  $X[k] \rightarrow \Omega_{\mathcal{C}}X[k+1]$  is an equivalence for  $k < n$ . Then the restriction maps  $\mathcal{D} \rightarrow \mathrm{PSP}(\mathcal{C})$  and  $\mathcal{D}_0 \rightarrow \mathcal{E}$  are trivial Kan fibrations.
- (5) The  $\infty$ -category  $\mathcal{E}$  is a localization of  $\mathrm{PSP}_{-\infty}^n(\mathcal{C})$ . Moreover, a morphism  $X \rightarrow Y$  in  $\mathrm{PSP}_{-\infty}^n(\mathcal{C})$  exhibits  $Y$  as an  $\mathcal{E}$ -localization of  $X$  if and only if  $Y \in \mathcal{E}$  and the map  $X[n] \rightarrow Y[n]$  is an equivalence.
- (6) The  $\infty$ -category  $\mathcal{D}_0$  is a localization of  $\mathcal{D}$ . Moreover, a morphism  $X \rightarrow Y$  in  $\mathcal{D}$  exhibits  $Y$  as an  $\mathcal{D}_0$ -localization of  $X$  if and only if  $Y \in \mathcal{D}_0$  and the map  $X[n] \rightarrow Y[n]$  is an equivalence.

*Proof.* Assertions (1) and (2) follow by applying Lemma 8.12 to the opposite  $\infty$ -category  $\mathcal{C}^{op}$ . Assertion (4) follows from (1) and (2) together with Proposition T.4.3.2.15, and assertion (6) follows immediately from (5) and (4). We will give the proof of (3); the proof of (5) is similar.

Consider an arbitrary object  $Y \in \mathrm{PSP}(\mathcal{C})$ . Let  $Y_0 = Y|N(Q(-\infty, n))$ , and let  $X \in \mathrm{PSP}(\mathcal{C})$  be a left Kan extension of  $Y_0$  (whose existence is guaranteed by (2)). Then  $X \in \mathcal{D}$  and we have a canonical map  $\alpha : X \rightarrow Y$ . We claim that  $\alpha$  exhibits  $X$  as a  $\mathcal{D}$ -colocalization of  $Y$ . To prove this, let us consider an arbitrary object  $W \in \mathcal{D}$ . We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{PSP}(\mathcal{C})}(W, X) & \longrightarrow & \mathrm{Map}_{\mathrm{PSP}_{-\infty}^n(\mathcal{C})}(W|N(Q(-\infty, n)), X|N(Q(-\infty, n))) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{PSP}(\mathcal{C})}(W, Y) & \longrightarrow & \mathrm{Map}_{\mathrm{PSP}_{-\infty}^n(\mathcal{C})}(W|N(Q(-\infty, n)), Y|N(Q(-\infty, n))). \end{array}$$

Since  $X$  and  $Y$  have the same restriction to  $N(Q(-\infty, n))$ , the right vertical map is a homotopy equivalence. The horizontal maps are homotopy equivalences since  $W$  is a left Kan extension of  $W|N(Q(-\infty, n))$ , by virtue of (1). This completes the proof that  $\alpha$  exhibits  $X$  as a  $\mathcal{D}$ -colocalization of  $Y$ , and the proof that  $\mathcal{D}$  is a colocalization of  $\mathrm{PSP}(\mathcal{C})$ .

To complete the proof of (3), let us consider an arbitrary map  $\beta : X' \rightarrow Y$ , where  $X' \in \mathcal{D}$ . We wish to show that  $\beta$  exhibits  $X'$  as a  $\mathcal{D}$ -colocalization of  $Y$  if and only if the induced map  $X'[m] \rightarrow Y[m]$  is an equivalence for  $m \leq n$ . The above argument shows that  $\beta$  fits into a commutative triangle

$$\begin{array}{ccc} & X & \\ \gamma \nearrow & & \searrow \alpha \\ X' & \xrightarrow{\beta} & Y. \end{array}$$

Since  $\alpha$  exhibits  $X$  as a  $\mathcal{D}$ -colocalization of  $Y$ , and  $\alpha$  induces equivalences  $X[m] \rightarrow Y[m]$  for  $m \leq n$ , we can restate the desired assertion as follows: the map  $\gamma$  is an equivalence if and only if  $\gamma$  induces equivalences  $X[m] \rightarrow X'[m]$  for  $m \leq n$ . The “only if” part of the assertion is obvious, and the converse follows from the fact that both  $X$  and  $X'$  are left Kan extensions of their restrictions to  $N(Q(-\infty, n))$  (by virtue of (1)).  $\square$

**Proposition 8.21.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits finite limits and colimits, and let  $\mathcal{D}_0 \subseteq \mathrm{PSP}(\mathcal{C})$  denote the full subcategory spanned by the  $n$ -suspension prespectra. Then evaluation at  $(n, n)$  induces a trivial Kan fibration  $\mathcal{D}_0 \rightarrow \mathcal{C}$ .*

*Proof.* Let  $\mathcal{E} \subseteq \mathrm{PSp}_{-\infty}^n(\mathcal{C})$  be defined as in Lemma 8.20. The evaluation functor factors as a composition

$$\mathcal{D}_0 \xrightarrow{\phi_0} \mathcal{E} \xrightarrow{\phi_1} \mathrm{PSp}_n^n(\mathcal{C}) \xrightarrow{\phi_2} \mathcal{C}.$$

Here  $\phi_0$  is a trivial fibration by Lemma 8.20, the map  $\phi_1$  is a trivial fibration by virtue of Lemma 8.12 and Proposition T.4.3.2.15, and the map  $\phi_2$  is a trivial fibration by Lemma 8.13.  $\square$

**Notation 8.22.** Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits finite limits and colimits. We let  $\tilde{\Sigma}_{\mathcal{C}}^{\infty-n} : \mathcal{C} \rightarrow \mathrm{PSp}(\mathcal{C})$  denote a section of the trivial Kan fibration  $\mathcal{D}_0 \rightarrow \mathcal{C}$  of Proposition 8.21.

**Remark 8.23.** Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits finite limits and colimits, let  $C \in \mathcal{C}$  be an object and let  $X \in \mathrm{PSp}(\mathcal{C})$  be a spectrum below  $n$ . Then the canonical map

$$e : \mathrm{Map}_{\mathrm{PSp}(\mathcal{C})}(\tilde{\Sigma}_{\mathcal{C}}^{\infty-n}C, X) \rightarrow \mathrm{Map}_{\mathcal{C}}(C, \Omega_{\mathcal{C}}^{\infty-n}X)$$

is a homotopy equivalence. To prove this, we observe that  $e$  factors as a composition (using the conventions of Notation 8.10)

$$\begin{aligned} \mathrm{Map}_{\mathrm{PSp}(\mathcal{C})}(\tilde{\Sigma}_{\mathcal{C}}^{\infty-n}C, X) &\xrightarrow{\phi_0} \mathrm{Map}_{\mathrm{PSp}_{-\infty}^n(\mathcal{C})}((\tilde{\Sigma}_{\mathcal{C}}^{\infty-n}C)|N(Q(-\infty, n)), X|N(Q(-\infty, n))) \\ &\xrightarrow{\phi_1} \mathrm{Map}_{\mathrm{PSp}_n^n(\mathcal{C})}((\tilde{\Sigma}_{\mathcal{C}}^{\infty-n}C)|N(Q(n, n)), X|N(Q(n, n))) \\ &\xrightarrow{\phi_2} \mathrm{Map}_{\mathcal{C}}(C, X[n]). \end{aligned}$$

It will therefore suffice to prove that the maps  $\phi_0$ ,  $\phi_1$ , and  $\phi_2$  are homotopy equivalences. For  $\phi_0$ , the desired result follows from our assumption that  $\tilde{\Sigma}_{\mathcal{C}}^{\infty-n}C$  is a left Kan extension of its restriction to  $N(Q(-\infty, n))$ . For  $\phi_1$ , we invoke the fact that  $X|N(Q(-\infty, n))$  is a right Kan extension of its restriction to  $N(Q(n, n))$ . For  $\phi_2$ , we apply Lemma 8.13.

**Proposition 8.24.** *Let  $\mathcal{C}$  be a presentable pointed  $\infty$ -category, and let  $L : \mathrm{PSp}(\mathcal{C}) \rightarrow \mathrm{Sp}(\mathcal{C})$  denote a left adjoint to the inclusion. Then the evaluation functor  $\Omega_{\mathcal{C}}^{\infty-n} : \mathrm{Sp}(\mathcal{C}) \rightarrow \mathcal{C}$  admits a left adjoint, given by the composition*

$$\mathcal{C} \xrightarrow{\tilde{\Sigma}_{\mathcal{C}}^{\infty-n}} \mathrm{PSp}(\mathcal{C}) \xrightarrow{L} \mathrm{Sp}(\mathcal{C}).$$

*Proof.* The canonical natural transformation  $\mathrm{id}_{\mathrm{PSp}(\mathcal{C})} \rightarrow L$  induces a transformation

$$\alpha : \mathrm{id}_{\mathcal{C}} = \Omega_{\mathcal{C}}^{\infty-n} \circ \tilde{\Sigma}_{\mathcal{C}}^{\infty-n} \rightarrow \Omega_{\mathcal{C}}^{\infty-n} \circ (L \circ \tilde{\Sigma}_{\mathcal{C}}^{\infty-n}).$$

We claim that  $\alpha$  is the unit of an adjunction. To prove this, it suffices to show that for every object  $C \in \mathcal{C}$  and every spectrum object  $X \in \mathrm{Sp}(\mathcal{C})$ , the composite map

$$\mathrm{Map}_{\mathrm{Sp}(\mathcal{C})}(L\tilde{\Sigma}_{\mathcal{C}}^{\infty-n}C, X) \xrightarrow{\phi} \mathrm{Map}_{\mathrm{PSp}(\mathcal{C})}(\tilde{\Sigma}_{\mathcal{C}}^{\infty-n}C, X) \xrightarrow{\psi} \mathrm{Map}_{\mathcal{C}}(C, X[n])$$

is a homotopy equivalence. It now suffices to observe that  $\phi$  is a homotopy equivalence because  $X$  is a spectrum object, and  $\psi$  is a homotopy equivalence by virtue of Remark 8.23.  $\square$

We close this section by discussing the *shift* functor on prespectrum objects of an  $\infty$ -category  $\mathcal{C}$ . We observe that precomposition with the map  $(i, j) \mapsto (i+1, j+1)$  determines a functor  $S : \mathrm{PSp}(\mathcal{C}) \rightarrow \mathrm{PSp}(\mathcal{C})$ , which restricts to a functor  $\mathrm{Sp}(\mathcal{C}) \rightarrow \mathrm{Sp}(\mathcal{C})$  which we will also denote by  $S$ . By construction, this functor is an equivalence (in fact, an isomorphism of simplicial sets). We observe that if  $X$  is a spectrum object of  $\mathcal{C}$ , then we have canonical equivalences  $\Omega_e S(X)[n] = \Omega_e X[n+1] \simeq X$ ; this strongly suggests that  $S$  is a homotopy inverse to the loop functor  $\Omega_{\mathrm{Sp}(\mathcal{C})}$  given by pointwise composition with  $\Omega_{\mathcal{C}}$ . To prove this (in a slightly stronger form), we need to introduce a bit of notation.

**Notation 8.25.** Consider the order-preserving maps  $s_+, s_- : \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z} \times \mathbf{Z}$  defined by the formulae

$$s_+(i, j) = \begin{cases} (i, j) & \text{if } i \neq j \\ (i+1, j) & \text{if } i = j. \end{cases} \quad s_-(i, j) = \begin{cases} (i, j) & \text{if } i \neq j \\ (i, j+1) & \text{if } i = j. \end{cases}$$

For every  $\infty$ -category  $\mathcal{C}$ , composition with  $s_+$  and  $s_-$  induces functors  $S_+, S_- : \text{PSP}(\mathcal{C}) \rightarrow \text{PSP}(\mathcal{C})$ , fitting into a commutative diagram

$$\begin{array}{ccc} \text{id} & \longrightarrow & S_+ \\ \downarrow & & \downarrow \\ S_- & \longrightarrow & S. \end{array}$$

Note that the images of  $s_+$  and  $s_-$  are disjoint from the diagonal  $\{(n, n)\}_{n \geq 0} \subseteq \mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$ , so that  $S_+(X)$  and  $S_-(X)$  are zero objects of  $\text{PSP}(\mathcal{C})$  for every  $X \in \text{PSP}(\mathcal{C})$ . If  $\mathcal{C}$  admits finite limits, then the above diagram determines a morphism  $\alpha : X \rightarrow \Omega_{\text{PSP}(\mathcal{C})} S(X)$ . If  $\mathcal{C}$  also admits finite colimits, then  $\alpha$  admits an adjoint  $\beta : \Sigma_{\text{PSP}(\mathcal{C})} X \rightarrow S(X)$ .

**Lemma 8.26.** *Let  $\mathcal{C}$  be a small pointed  $\infty$ -category, and let  $\mathcal{P}_*(\mathcal{C})$  denote the full subcategory of  $\mathcal{P}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{op}, \mathcal{S})$  spanned by those functors which carry zero objects of  $\mathcal{C}$  to final objects of  $\mathcal{S}$ . Then:*

- (1) *Let  $S$  denote the set consisting of a single morphism from an initial object of  $\mathcal{P}(\mathcal{C})$  to a final object of  $\mathcal{P}(\mathcal{C})$ . Then  $\mathcal{P}_*(\mathcal{C}) = S^{-1} \mathcal{P}(\mathcal{C})$ .*
- (2) *The  $\infty$ -category  $\mathcal{P}_*(\mathcal{C})$  is an accessible localization of  $\mathcal{P}(\mathcal{C})$ . In particular,  $\mathcal{P}_*(\mathcal{C})$  is presentable.*
- (3) *The Yoneda embedding  $\mathcal{C} \rightarrow \mathcal{P}(\mathcal{C})$  factors through  $\mathcal{P}_*(\mathcal{C})$ , and the induced embedding  $j : \mathcal{C} \rightarrow \mathcal{P}_*(\mathcal{C})$  preserves zero objects.*
- (4) *Let  $\mathcal{D}$  be an  $\infty$ -category which admits small colimits, and let  $\text{Fun}^L(\mathcal{P}_*(\mathcal{C}), \mathcal{D})$  denote the full subcategory of  $\text{Fun}(\mathcal{P}_*(\mathcal{C}), \mathcal{D})$  spanned by those functors which preserve small colimits. Then composition with  $j$  induces an equivalence of  $\infty$ -categories  $\text{Fun}^L(\mathcal{P}_*(\mathcal{C}), \mathcal{D}) \rightarrow \text{Fun}_0(\mathcal{C}, \mathcal{D})$ , where  $\text{Fun}_0(\mathcal{C}, \mathcal{D})$  denotes the full subcategory of  $\text{Fun}(\mathcal{C}, \mathcal{D})$  spanned by those functors which carry zero objects of  $\mathcal{C}$  to initial objects of  $\mathcal{D}$ .*
- (5) *The  $\infty$ -category  $\mathcal{P}_*(\mathcal{C})$  is pointed.*
- (6) *The full subcategory  $\mathcal{P}_*(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$  is closed under small limits and under small colimits parametrized by weakly contractible simplicial sets. In particular,  $\mathcal{P}_*(\mathcal{C})$  is stable under small filtered colimits in  $\mathcal{P}(\mathcal{C})$ .*
- (7) *The functor  $j : \mathcal{C} \rightarrow \mathcal{P}_*(\mathcal{C})$  preserves all small limits which exist in  $\mathcal{C}$ .*
- (8) *The  $\infty$ -category  $\mathcal{P}_*(\mathcal{C})$  is compactly generated.*

*Proof.* For every object  $X \in \mathcal{S}$ , let  $F_X \in \mathcal{P}(\mathcal{C})$  denote the constant functor taking the value  $X$ . Then  $F_X$  is a left Kan extension of  $F_X|_{\{0\}}$ , where  $0$  denotes a zero object of  $\mathcal{C}$ . It follows that for any object  $G \in \mathcal{P}(\mathcal{C})$ , evaluation at  $0$  induces a homotopy equivalence

$$\text{Map}_{\mathcal{P}(\mathcal{C})}(F_X, G) \rightarrow \text{Map}_{\mathcal{S}}(F_X(0), G(0)) = \text{Map}_{\mathcal{S}}(X, G(0)).$$

We observe that the inclusion  $\emptyset \subseteq \Delta^0$  induces a map  $F_\emptyset \rightarrow F_{\Delta^0}$  from an initial object of  $\mathcal{P}(\mathcal{C})$  to a final object of  $\mathcal{P}(\mathcal{C})$ . It follows that an object  $G$  of  $\mathcal{P}(\mathcal{C})$  is  $S$ -local if and only if the induced map

$$G(0) \simeq \text{Map}_{\mathcal{S}}(\Delta^0, G(0)) \rightarrow \text{Map}_{\mathcal{S}}(\emptyset, G(0)) \simeq \Delta^0$$

is a homotopy equivalence: that is, if and only if  $G \in \mathcal{P}_*(\mathcal{C})$ . This proves (1).



Assertion (2) follows immediately from (1), and assertion (3) is obvious. Assertion (4) follows from (1), Theorem T.5.1.5.6, and Proposition T.5.5.4.20. To prove (5), we observe that  $F_{\Delta^0}$  is a final object of  $\mathcal{P}(\mathcal{C})$ , and therefore a final object of  $\mathcal{P}_*(\mathcal{C})$ . It therefore suffices to show that  $F_{\Delta^0}$  is an initial object of  $\mathcal{P}_*(\mathcal{C})$ . This follows from the observation that for every  $G \in \mathcal{P}(\mathcal{C})$ , we have homotopy equivalences  $\mathrm{Map}_{\mathcal{P}(\mathcal{C})}(F_{\Delta^0}, G) \simeq \mathrm{Map}_{\mathcal{S}}(\Delta^0, G(0)) \simeq G(0)$  so that the mapping space  $\mathrm{Map}_{\mathcal{P}(\mathcal{C})}(F_{\Delta^0}, G)$  is contractible if  $G \in \mathcal{P}_*(\mathcal{C})$ .

Assertion (6) is obvious, and (7) follows from (6) together with Proposition T.5.1.3.2. We deduce (8) from (6) together with Corollary T.5.5.7.3.  $\square$

**Proposition 8.27.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits finite limits. Then:*

- (1) *For every object  $X \in \mathrm{Sp}(\mathcal{C})$ , the canonical map  $X \rightarrow \Omega_{\mathrm{PSp}(\mathcal{C})}S(X)$  is an equivalence.*
- (2) *The shift functor  $S : \mathrm{Sp}(\mathcal{C}) \rightarrow \mathrm{Sp}(\mathcal{C})$  is a homotopy inverse to the loop functor  $\Omega_{\mathrm{Sp}(\mathcal{C})}$ .*
- (3) *The  $\infty$ -category  $\mathrm{Sp}(\mathcal{C})$  is stable.*

*Proof.* Assertion (1) is an immediate consequence of the definitions. We note that (1) implies that  $S$  is a right homotopy inverse to  $\Omega_{\mathrm{Sp}(\mathcal{C})}$ . Since  $S$  is invertible, it follows that  $S$  is also a left homotopy inverse to  $\Omega_{\mathrm{Sp}(\mathcal{C})}$ . In particular,  $\Omega_{\mathrm{Sp}(\mathcal{C})}$  is invertible.

To prove (3), we may assume without loss of generality that  $\mathcal{C}$  is small. Lemma 8.26 implies that there exists a fully faithful left exact functor  $j : \mathcal{C} \rightarrow \mathcal{D}$ , where  $\mathcal{D}$  is a compactly generated pointed  $\infty$ -category (this that the functor  $\Omega_{\mathcal{D}}$  preserves sequential colimits; see Remark 8.9). Then  $\mathrm{Sp}(\mathcal{C})$  is equivalent to a full subcategory of  $\mathrm{Sp}(\mathcal{D})$ , which is closed under finite limits and shifts. Consequently, it will suffice to show that  $\mathrm{Sp}(\mathcal{D})$  is stable, which is a consequence of Corollary 10.10 (proven in §10).  $\square$

**Corollary 8.28.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category. The following conditions are equivalent:*

- (1) *The  $\infty$ -category  $\mathcal{C}$  is stable.*
- (2) *The  $\infty$ -category  $\mathcal{C}$  admits finite colimits and the suspension functor  $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$  is an equivalence.*
- (3) *The  $\infty$ -category  $\mathcal{C}$  admits finite limits and the loop functor  $\Omega : \mathcal{C} \rightarrow \mathcal{C}$  is an equivalence.*

*Proof.* We will show that (1)  $\Leftrightarrow$  (3); the dual argument will prove that (1)  $\Leftrightarrow$  (2). The implication (1)  $\Rightarrow$  (3) is clear. Conversely, suppose that  $\mathcal{C}$  admits finite limits and that  $\Omega$  is an equivalence. Lemma T.7.2.2.9 asserts that the forgetful functor  $\mathcal{C}_* \rightarrow \mathcal{C}$  is a trivial fibration. Consequently,  $\mathrm{Sp}(\mathcal{C})$  can be identified with the homotopy inverse limit of the tower

$$\dots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}.$$

By assumption, the loop functor  $\Omega$  is an equivalence, so this tower is essentially constant. It follows that  $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \rightarrow \mathcal{C}$  is an equivalence of  $\infty$ -categories. Since  $\mathrm{Sp}(\mathcal{C})$  is stable (Proposition 8.27), so is  $\mathcal{C}$ .  $\square$

For later use, we record also the following result:

**Proposition 8.29.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category satisfying the hypotheses of Remark 8.8, and let  $L : \mathrm{PSp}(\mathcal{C}) \rightarrow \mathrm{Sp}(\mathcal{C})$  denote a left adjoint to the inclusion. Let  $X$  be a prespectrum object of  $\mathcal{C}$ , and let  $\beta : \Sigma_{\mathrm{PSp}(\mathcal{C})}X \rightarrow S(X)$  denote the map described in Notation 8.25. Then  $L(\beta)$  is an equivalence in  $\mathrm{Sp}(\mathcal{C})$ .*

*Proof.* Since  $L$  is a left adjoint, it commutes with suspensions. It will therefore suffice to show that  $\beta$  induces an equivalence  $\Sigma_{\mathrm{Sp}(\mathcal{C})}LX \rightarrow LS(X)$ : in other words, we must show that the diagram  $\sigma$  :

$$\begin{array}{ccc} LX & \longrightarrow & LS_+(X) \\ \downarrow & & \downarrow \\ LS_-(X) & \longrightarrow & LS(X) \end{array}$$

is a pushout square in the  $\infty$ -category  $\mathrm{Sp}(\mathcal{C})$ . Since  $\mathrm{Sp}(\mathcal{C})$  is stable, it suffices to show that  $\sigma$  is a pullback square. In other words, we must show that for each  $n \geq 0$ , the diagram

$$\begin{array}{ccc} LX[n] & \longrightarrow & LS_+(X)[n] \\ \downarrow & & \downarrow \\ LS_-(X)[n] & \longrightarrow & LS(X)[n] \end{array}$$

is a pullback square in  $\mathcal{C}$ .

Let  $P$  be defined as in Lemma 8.15, let  $\overline{X}_0 : N(P) \rightarrow \mathcal{C}$  be given by the composition

$$N(P) \rightarrow N(\mathbf{Z} \times \mathbf{Z}) \xrightarrow{X} \mathcal{C},$$

and let  $\overline{X} : N(\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}) \rightarrow \mathcal{C}$  be a right Kan extension of  $\overline{X}_0$ . Let  $\overline{S}(\overline{X})$ ,  $\overline{S}_+(\overline{X})$ , and  $\overline{S}_-(\overline{X})$  be obtained from  $\overline{X}$  by composing with the maps  $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$  given by  $(i, j, k) \mapsto (i+1, j+1, k)$ ,  $s_+ \times \mathrm{id}$ , and  $s_- \times \mathrm{id}$ . We have a commutative diagram

$$\begin{array}{ccc} \overline{X} & \longrightarrow & \overline{S}_+(\overline{X}) \\ \downarrow & & \downarrow \\ \overline{S}_-(\overline{X}) & \longrightarrow & \overline{S}(\overline{X}) \end{array}$$

which we can think of as encoding a sequence of commutative squares  $\{\sigma_k : \Delta^1 \times \Delta^1 \rightarrow \mathrm{P}\mathrm{Sp}(\mathcal{C})\}_{k \geq 0}$ . We can identify  $\sigma$  with the colimit of this sequence. Consequently, it will suffice to prove that for every integer  $n$ , the diagram

$$\begin{array}{ccc} \overline{X}(n, n, k) & \longrightarrow & \overline{S}_+(\overline{X})(n, n, k) \\ \downarrow & & \downarrow \\ \overline{S}_-(\overline{X})(n, n, k) & \longrightarrow & \overline{S}(\overline{X})(n, n, k) \end{array}$$

is a pullback square in  $\mathcal{C}$  for all sufficiently large  $k$ . In fact, this is true for all  $k > n$ , by virtue of Lemma 8.15.  $\square$

## 9 The $\infty$ -Category of Spectra

In this section, we will discuss what is perhaps the most important example of a stable  $\infty$ -category: the  $\infty$ -category of spectra. In classical homotopy theory, one defines a spectrum to be a sequence of pointed spaces  $\{X_n\}_{n \geq 0}$ , equipped with homotopy equivalences (or homeomorphisms, depending on the author)  $X_n \rightarrow \Omega(X_{n+1})$  for all  $n \geq 0$ . This admits an immediate  $\infty$ -categorical translation:

**Definition 9.1.** A *spectrum* is a spectrum object of the  $\infty$ -category  $\mathcal{S}_*$  of pointed spaces. We let  $\mathrm{Sp} = \mathrm{Sp}(\mathcal{S}_*) = \mathrm{Stab}(\mathcal{S})$  denote the  $\infty$ -category of spectra.

**Proposition 9.2.**

- (1) *The  $\infty$ -category  $\mathrm{Sp}$  is stable.*
- (2) *Let  $(\mathrm{Sp})_{\leq -1}$  denote the full subcategory of  $\mathrm{Sp}$  spanned by those objects  $X$  such that  $\Omega^\infty(X) \in \mathcal{S}$  is contractible. Then  $(\mathrm{Sp})_{\leq -1}$  determines an accessible  $t$ -structure on  $\mathrm{Sp}$  (see §16).*
- (3) *The  $t$ -structure on  $\mathrm{Sp}$  is both left complete and right complete, and the heart  $\mathcal{S}_\infty^\heartsuit$  is canonically equivalent to the (nerve of the) category of abelian groups.*

*Proof.* Assertion (1) follows immediately from Proposition 8.27. Assertion (2) is a special case of Proposition 16.4, which will be established in §16. We will prove (3). Note that a spectrum  $X$  can be identified with a sequence of pointed spaces  $\{X(n)\}$ , equipped with equivalences  $X(n) \simeq \Omega X(n+1)$  for all  $n \geq 0$ . We observe that  $X \in (\mathrm{Sp})_{\leq m}$  if and only if each  $X(n)$  is  $(n+m)$ -truncated. In general, the sequence  $\{\tau_{\leq n+m} X(n)\}$  itself determines a spectrum, which we can identify with the truncation  $\tau_{\leq m} X$ . It follows that  $X \in (\mathrm{Sp})_{\geq m+1}$  if and only if each  $X(n)$  is  $(n+m+1)$ -connective. In particular,  $X$  lies in the heart of  $\mathrm{Sp}$  if and only if each  $X(n)$  is an Eilenberg-MacLane object of  $\mathcal{S}$  of degree  $n$  (see Definition T.7.2.2.1). It follows that the heart of  $\mathrm{Sp}$  can be identified with the homotopy inverse limit of the tower of  $\infty$ -categories

$$\dots \xrightarrow{\Omega} \mathcal{EM}_1(\mathcal{S}) \xrightarrow{\Omega} \mathcal{EM}_0(\mathcal{S}),$$

where  $\mathcal{EM}_n(\mathcal{S})$  denotes the full subcategory of  $\mathcal{S}_*$  spanned by the Eilenberg-MacLane objects of degree  $n$ . Proposition T.7.2.2.12 asserts that after the second term, this tower is equivalent to the constant diagram taking the value  $\mathrm{N}(\mathcal{A}b)$ , where  $\mathcal{A}b$  is category of abelian groups.

It remains to prove that  $\mathrm{Sp}$  is both right and left complete. We begin by observing that if  $X \in \mathrm{Sp}$  is such that  $\pi_n X \simeq 0$  for all  $n \in \mathbf{Z}$ , then  $X$  is a zero object of  $\mathrm{Sp}$  (since each  $X(n) \in \mathcal{S}$  has vanishing homotopy groups, and is therefore contractible by Whitehead's theorem). Consequently, both  $\bigcap (\mathrm{Sp})_{\leq -n}$  and  $\bigcap (\mathrm{Sp})_{\geq n}$  coincide with the collection of zero objects of  $\mathrm{Sp}$ . It follows that

$$(\mathrm{Sp})_{\geq 0} = \{X \in \mathrm{Sp} : (\forall n < 0)[\pi_n X \simeq 0]\}$$

$$(\mathrm{Sp})_{\leq 0} = \{X \in \mathrm{Sp} : (\forall n > 0)[\pi_n X \simeq 0]\}.$$

According to Proposition 7.3, to prove that  $\mathrm{Sp}$  is left and right complete it will suffice to show that the subcategories  $(\mathrm{Sp})_{\geq 0}$  and  $(\mathrm{Sp})_{\leq 0}$  are stable under products and coproducts. In view of the above formulas, it will suffice to show that the homotopy group functors  $\pi_n : \mathrm{Sp} \rightarrow \mathrm{N}(\mathcal{A}b)$  preserve products and coproducts. Since  $\pi_n$  obviously commutes with finite coproducts, it will suffice to show that  $\pi_n$  commutes with products and filtered colimits. Shifting if necessary, we may reduce to the case  $n = 0$ . Since products and filtered colimits in the category of abelian groups can be computed at the level of the underlying sets, we are reduced to proving that the composition

$$\mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S} \xrightarrow{\pi_0} \mathrm{N}(\mathrm{Set})$$

preserves products and filtered colimits. This is clear, since each of the factors individually preserves products and filtered colimits.  $\square$

Our next goal is to prove that the  $\infty$ -category  $\mathrm{Sp}$  is compactly generated. To prove this, we need to review a bit of the theory of finite spaces.

**Notation 9.3.** Let  $\mathcal{S}_*$  denote the  $\infty$ -category of pointed objects of  $\mathcal{S}$ . That is,  $\mathcal{S}_*$  denotes the full subcategory of  $\mathrm{Fun}(\Delta^1, \mathcal{S})$  spanned by those morphisms  $f : X \rightarrow Y$  for which  $X$  is a final object of  $\mathcal{S}$  (Definition T.7.2.2.1). Let  $\mathcal{S}^{\mathrm{fin}}$  denote the smallest full subcategory of  $\mathcal{S}$  which contains the final object  $*$  and is stable under finite colimits. We will refer to  $\mathcal{S}^{\mathrm{fin}}$  as the  *$\infty$ -category of finite spaces*. We let  $\mathcal{S}_*^{\mathrm{fin}} \subseteq \mathcal{S}_*$  denote the  $\infty$ -category of pointed objects of  $\mathcal{S}^{\mathrm{fin}}$ . We observe that the suspension functor  $\Sigma : \mathcal{S}_* \rightarrow \mathcal{S}_*$  carries  $\mathcal{S}_*^{\mathrm{fin}}$  to itself. For each  $n \geq 0$ , we let  $S^n \in \mathcal{S}_*$  denote a representative for the (pointed)  $n$ -sphere.

**Remark 9.4.** It follows from Remark T.5.3.5.9 and Proposition T.4.3.2.15 that  $\mathcal{S}^{\mathrm{fin}}$  is characterized by the following universal property: for every  $\infty$ -category  $\mathcal{D}$  which admits finite colimits, evaluation at  $*$  induces an equivalence of  $\infty$ -categories  $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{S}^{\mathrm{fin}}, \mathcal{D}) \rightarrow \mathcal{D}$ . Here  $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{S}^{\mathrm{fin}}, \mathcal{D})$  denotes the full subcategory of  $\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}, \mathcal{D})$  spanned by the right exact functors.

**Lemma 9.5.** (1) *Each object of  $\mathcal{S}_*^{\mathrm{fin}}$  is compact in  $\mathcal{S}_*$ .*

(2) *The inclusion  $\mathcal{S}_*^{\mathrm{fin}} \subseteq \mathcal{S}_*$  induces an equivalence  $\mathrm{Ind}(\mathcal{S}_*^{\mathrm{fin}}) \rightarrow \mathcal{S}_*$ . In particular,  $\mathcal{S}_*$  is compactly generated.*

(3) The subcategory  $\mathcal{S}_*^{\text{fin}} \subseteq \mathcal{S}_*$  is the smallest full subcategory which contains  $S^0$  and is stable under finite colimits.

*Proof.* Since  $\mathcal{S}^{\text{fin}}$  consists of compact objects of  $\mathcal{S}$ , Proposition T.5.4.5.15 implies that  $\mathcal{S}_*^{\text{fin}}$  consists of compact objects of  $\mathcal{S}_*$ . This proves (1).

We next observe that  $\mathcal{S}_*^{\text{fin}}$  is stable under finite colimits in  $\mathcal{S}_*$ . Using the proof of Corollary T.4.4.2.4, we may reduce to showing that  $\mathcal{S}_*^{\text{fin}}$  is stable under pushouts and contains an initial object of  $\mathcal{S}_*$ . The second assertion is obvious, and the first follows from the fact that the forgetful functor  $\mathcal{S}_* \rightarrow \mathcal{S}$  commutes with pushouts (Proposition T.4.4.2.9).

We now prove (3). Let  $\mathcal{S}'_*$  be the smallest full subcategory which contains  $S^0$  and is stable under finite colimits. The above argument shows that  $\mathcal{S}'_* \subseteq \mathcal{S}_*^{\text{fin}}$ . To prove the converse, we let  $f : \mathcal{S} \rightarrow \mathcal{S}_*$  be a left adjoint to the forgetful functor, so that  $f(X) \simeq X \amalg *$ . Then  $f$  preserves small colimits. Since  $f(*) \simeq S^0 \in \mathcal{S}'_*$ , we conclude that  $f$  carries  $\mathcal{S}^{\text{fin}}$  into  $\mathcal{S}'_*$ . If  $x : * \rightarrow X$  is a pointed object of  $\mathcal{S}$ , then  $x$  can be written as a coproduct  $f(X) \amalg_{S^0} *$ . In particular, if  $x \in \mathcal{S}_*^{\text{fin}}$ , then  $X \in \mathcal{S}^{\text{fin}}$ , so that  $f(X), S^0, * \in \mathcal{S}'_*$ . Since  $\mathcal{S}'_*$  is stable under pushouts, we conclude that  $x \in \mathcal{S}'_*$ ; this completes the proof of (3).

We now prove (2). Part (1) and Proposition T.5.3.5.11 imply that we have a fully faithful functor  $\theta : \text{Ind}(\mathcal{S}_*^{\text{fin}}) \subseteq \mathcal{S}_*$ . Let  $\mathcal{S}''_*$  be the essential image of  $\theta$ . Proposition T.5.5.1.9 implies that  $\mathcal{S}''_*$  is stable under small colimits. Since  $S^0 \in \mathcal{S}''_*$  and  $f$  preserves small colimits, we conclude that  $f(X) \in \mathcal{S}''_*$  for all  $X \in \mathcal{S}$ . Since  $\mathcal{S}''_*$  is stable under pushouts, we conclude that  $\mathcal{S}''_* = \mathcal{S}_*$ , as desired.  $\square$

**Warning 9.6.** The  $\infty$ -category  $\mathcal{S}^{\text{fin}}$  does not coincide with the  $\infty$ -category of compact objects  $\mathcal{S}^\omega \subseteq \mathcal{S}$ . Instead, there is an inclusion  $\mathcal{S}^{\text{fin}} \subseteq \mathcal{S}^\omega$ , which realizes  $\mathcal{S}^\omega$  as an idempotent completion of  $\mathcal{S}^{\text{fin}}$ . An object of  $X \in \mathcal{S}^\omega$  belongs to  $\mathcal{S}^{\text{fin}}$  if and only if its *Wall finiteness obstruction* vanishes.

**Proposition 9.7.** *The  $\infty$ -category of spectra is compactly generated. Moreover, an object  $X \in \text{Sp}$  is compact if and only if it is a retract of  $\Sigma^{\infty-n}Y$ , for some  $Y \in \mathcal{S}_*^{\text{fin}}$  and some integer  $n$ .*

*Proof.* Let  $\text{Pr}_\omega^{\text{L}}$ ,  $\text{Pr}_\omega^{\text{R}}$ , and  $\text{Cat}_\infty^{\text{Rex}}$  be defined as §T.5.5.7. According to Proposition T.5.5.7.10, we can view the construction of Ind-categories as determining a localization functor  $\text{Ind} : \text{Cat}_\infty^{\text{Rex}} \rightarrow \text{Pr}_\omega^{\text{L}}$ . Let  $\mathcal{S}_\infty^{\text{fin}}$  denote the colimit of the sequence

$$\mathcal{S}_*^{\text{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\text{fin}} \xrightarrow{\Sigma} \dots$$

in  $\text{Cat}_\infty^{\text{Rex}}$ . Since  $\mathcal{S}_* \simeq \text{Ind}(\mathcal{S}_*^{\text{fin}})$  (Lemma 9.5) and the functor  $\text{Ind}$  preserves colimits, we conclude that  $\text{Ind}(\mathcal{S}_\infty^{\text{fin}})$  can be identified with the colimit of the sequence

$$\mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \dots$$

in  $\text{Pr}_\omega^{\text{L}}$ . Invoking the equivalence between  $\text{Pr}_\omega^{\text{L}}$  and  $(\text{Pr}_\omega^{\text{R}})^{\text{op}}$  (see Notation T.5.5.7.7), we can identify  $\text{Ind}(\mathcal{S}_\infty^{\text{fin}})$  with the *limit* of the tower

$$\dots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*$$

in  $\text{Pr}_\omega^{\text{R}}$ . Since the inclusion functor  $\text{Pr}_\omega^{\text{R}} \subseteq \widehat{\text{Cat}}_\infty$  preserves limits (Proposition T.5.5.7.6), we conclude that there is an equivalence  $F : \text{Ind}(\mathcal{S}_\infty^{\text{fin}}) \simeq \text{Sp}$  (Proposition 8.14). This proves that  $\text{Sp}$  is compactly generated, and that the compact objects of  $\text{Sp}$  are precisely those which appear as retracts of  $F(Y)$ , for some  $Y \in \text{Ind}(\mathcal{S}_\infty^{\text{fin}})$ . To complete the proof, we observe that  $Y$  itself lies in the image of one of the maps  $\mathcal{S}_*^{\text{fin}} \rightarrow \mathcal{S}_\infty^{\text{fin}}$ , and that the composite maps

$$\mathcal{S}_*^{\text{fin}} \rightarrow \mathcal{S}_\infty^{\text{fin}} \rightarrow \text{Ind}(\mathcal{S}_\infty^{\text{fin}}) \xrightarrow{F} \text{Sp}$$

are given by restricting the suspension spectrum functors  $\Sigma^{\infty-n} : \mathcal{S}_*^{\text{fin}} \rightarrow \text{Sp}$ .  $\square$

**Remark 9.8.** The proof of Proposition 9.7 implies that we can identify  $\mathcal{S}_\infty^{\text{fin}}$  with a full subcategory of the compact objects of  $\text{Sp}$ . In fact, every compact object of  $\text{Sp}$  belongs to this full subcategory. The proof of this is not completely formal (especially in view of Warning 9.6); it relies on the fact that the ring of integers  $\mathbf{Z}$  is a principal ideal domain, so that every finitely generated projective  $\mathbf{Z}$ -module is free.

**Remark 9.9.** It is possible to use the proof of Proposition 9.7 to prove directly that the  $\infty$ -category  $\mathrm{Sp}$  is stable, without appealing to the general results on stabilization proved in §8. Indeed, by virtue of Proposition 4.5, it suffices to show that the  $\infty$ -category  $\mathcal{S}_\infty^{\mathrm{fin}}$  of finite spectra is stable. The essence of the matter is now to show that every pushout square in  $\mathcal{S}_\infty^{\mathrm{fin}}$  is also a pullback square. Every pushout square is obtained from a pushout diagram

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

in the  $\infty$ -category  $\mathcal{S}_*^{\mathrm{fin}}$  of finite pointed spaces. This pushout square will typically not be homotopy Cartesian  $\mathcal{S}_*^{\mathrm{fin}}$ , but will be *approximately* homotopy Cartesian if the spaces involved are highly connected: this follows from the Blakers-Massey homotopy excision theorem (see for example [25], p. 360). Using the fact that the approximation gets better and better as we iterate the suspension functor  $\Sigma$  (which increases the connectivity of spaces), one can deduce that the image of the above square is a pullback in  $\mathcal{S}_*^{\mathrm{fin}}$ .

**Remark 9.10.** Let  $\mathcal{A}b$  denote the category of abelian groups. For each  $n \in \mathbf{Z}$ , we let  $\pi_n : \mathrm{Sp} \rightarrow \mathcal{N}(\mathcal{A}b)$  be the composition of the shift functor  $X \mapsto X[-n]$  with the equivalence  $\mathcal{S}_\infty^\heartsuit \simeq \mathcal{N}(\mathcal{A}b)$ . Note that if  $n \geq 2$ , then  $\pi_n$  can be identified with the composition

$$\mathrm{Sp} \xrightarrow{\Omega_\infty^\heartsuit} \mathcal{S}_* \xrightarrow{\pi_n} \mathcal{N}(\mathcal{A}b)$$

where the second map is the usual homotopy group functor. Since  $\mathrm{Sp}$  is both left and right complete, we conclude that a map  $f : X \rightarrow Y$  of spectra is an equivalence if and only if it induces isomorphisms  $\pi_n X \rightarrow \pi_n Y$  for all  $n \in \mathbf{Z}$ .

**Proposition 9.11.** *The functor  $\Omega^\infty : (\mathrm{Sp})_{\geq 0} \rightarrow \mathcal{S}$  preserves geometric realizations of simplicial objects.*

*Proof.* Since the simplicial set  $\mathcal{N}(\mathbf{\Delta})$  is weakly contractible, the forgetful functor  $\mathcal{S}_* \rightarrow \mathcal{S}$  preserves geometric realizations of simplicial objects (Proposition T.4.4.2.9). It will therefore suffice to prove that the functor  $\Omega_*^\infty |_{(\mathrm{Sp})_{\geq 0}} : \mathcal{S}_* \rightarrow \mathcal{S}_*$  preserves geometric realizations of simplicial objects.

For each  $n \geq 0$ , let  $\mathcal{S}^{\geq n}$  denote the full subcategory of  $\mathcal{S}$  spanned by the  $n$ -connective objects, and let  $\mathcal{S}_*^{\geq n}$  be the  $\infty$ -category of pointed objects of  $\mathcal{S}^{\geq n}$ . We observe that  $(\mathrm{Sp})_{\geq 0}$  can be identified with the homotopy inverse limit of the tower

$$\dots \xrightarrow{\Omega} \mathcal{S}_*^{\geq 1} \xrightarrow{\Omega} \mathcal{S}_*^{\geq 0}.$$

It will therefore suffice to prove that for every  $n \geq 0$ , the loop functor  $\Omega : \mathcal{S}_*^{\geq n+1} \rightarrow \mathcal{S}_*^{\geq n}$  preserves geometric realizations of simplicial objects.

The  $\infty$ -category  $\mathcal{S}^{\geq n}$  is the preimage (under  $\tau_{\leq n-1}$ ) of the full subcategory of  $\tau_{\leq n-1} \mathcal{S}$  spanned by the final objects. Since this full subcategory is stable under geometric realizations of simplicial objects and since  $\tau_{\leq n-1}$  commutes with all colimits, we conclude that  $\mathcal{S}^{\geq n} \subseteq \mathcal{S}$  is stable under geometric realizations of simplicial objects.

According to Lemmas T.7.2.2.11 and T.7.2.2.10, there is an equivalence of  $\mathcal{S}_*^{\geq 1}$  with the  $\infty$ -category of group objects  $\mathrm{Grp}(\mathcal{S}_*)$ . This restricts to an equivalence of  $\mathcal{S}_*^{\geq n+1}$  with  $\mathrm{Grp}(\mathcal{S}_*^{\geq n})$  for all  $n \geq 0$ . Moreover, under this equivalence, the loop functor  $\Omega$  can be identified with the composition

$$\mathrm{Grp}(\mathcal{S}_*^{\geq n}) \subseteq \mathrm{Fun}(\mathcal{N}(\mathbf{\Delta})^{op}, \mathcal{S}_*^{\geq n}) \rightarrow \mathcal{S}_*^{\geq n},$$

where the second map is given by evaluation at the object  $[1] \in \mathbf{\Delta}$ . This evaluation map commutes with geometric realizations of simplicial objects (Proposition T.5.1.2.2). Consequently, it will suffice to show that  $\mathrm{Grp}(\mathcal{S}_*^{\geq n}) \subseteq \mathrm{Fun}(\mathcal{N}(\mathbf{\Delta})^{op}, \mathcal{S}_*^{\geq n})$  is stable under geometric realizations of simplicial objects.

Without loss of generality, we may suppose  $n = 0$ ; now we are reduced to showing that  $\mathrm{Grp}(\mathcal{S}_*) \subseteq \mathrm{Fun}(\mathcal{N}(\mathbf{\Delta})^{op}, \mathcal{S}_*)$  is stable under geometric realizations of simplicial objects. In view of Lemma T.7.2.2.10, it will suffice to show that  $\mathrm{Grp}(\mathcal{S}) \subseteq \mathcal{S}_\Delta$  is stable under geometric realizations of simplicial objects. Invoking Proposition T.7.2.2.4, we are reduced to proving that the formation of geometric realizations in  $\mathcal{S}$  commutes with finite products, which follows from Lemma T.5.5.8.11.  $\square$

## 10 Excisive Functors

In order to study the relationship between an  $\infty$ -category  $\mathcal{C}$  and its stabilization  $\text{Stab}(\mathcal{C})$ , we need to introduce a bit of terminology.

**Definition 10.1.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between  $\infty$ -categories.

- (i) If  $\mathcal{C}$  has an initial object  $\emptyset$ , then we will say that  $F$  is *weakly excisive* if  $F(\emptyset)$  is a final object of  $\mathcal{D}$ . We let  $\text{Fun}_*(\mathcal{C}, \mathcal{D})$  denote the full subcategory of  $\text{Fun}(\mathcal{C}, \mathcal{D})$  spanned by the weakly excisive functors.
- (ii) If  $\mathcal{C}$  admits finite colimits, then we will say that  $F$  is *excisive* if it is weakly excisive, and  $F$  carries pushout squares in  $\mathcal{C}$  to pullback squares in  $\mathcal{D}$ . We let  $\text{Exc}(\mathcal{C}, \mathcal{D})$  denote the full subcategory of  $\text{Fun}(\mathcal{C}, \mathcal{D})$  spanned by the excisive functors.

**Warning 10.2.** Definition 10.1 is somewhat nonstandard: most authors do not require the property the preservation of zero objects in the definition of excisive functors.

**Remark 10.3.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between  $\infty$ -categories, and suppose that  $\mathcal{C}$  admits finite colimits. If  $\mathcal{C}$  is stable, then  $F$  is excisive if and only if it is left exact (Proposition 4.4). If instead  $\mathcal{D}$  is stable, then  $F$  is excisive if and only if it is right exact. In particular, if both  $\mathcal{C}$  and  $\mathcal{D}$  are stable, then  $F$  is excisive if and only if it is exact (Proposition 5.1).

**Lemma 10.4.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be  $\infty$ -categories, and assume that  $\mathcal{C}$  has an initial object. Then:

- (1) The forgetful functor  $\theta : \text{Fun}_*(\mathcal{C}, \mathcal{D}_*) \rightarrow \text{Fun}_*(\mathcal{C}, \mathcal{D})$  is a trivial fibration of simplicial sets.
- (2) If  $\mathcal{C}$  admits finite colimits, then the forgetful functor  $\theta' : \text{Exc}(\mathcal{C}, \mathcal{D}_*) \rightarrow \text{Exc}(\mathcal{C}, \mathcal{D})$  is a trivial fibration of simplicial sets.

**Remark 10.5.** If the  $\infty$ -category  $\mathcal{D}$  does not have a final object, then the conclusion of Lemma 10.4 is valid, but degenerate: both of the relevant  $\infty$ -categories of functors are empty.

*Proof.* To prove (1), we first observe that objects of  $\text{Fun}_*(\mathcal{C}, \mathcal{D}_*)$  can be identified with maps  $F : \mathcal{C} \times \Delta^1 \rightarrow \mathcal{D}$  with the following properties:

- (a) For every initial object  $C \in \mathcal{C}$ ,  $F(C, 1)$  is a final object of  $\mathcal{D}$ .
- (b) For every object  $C \in \mathcal{C}$ ,  $F(C, 0)$  is a final object of  $\mathcal{D}$ .

Assume for the moment that (a) is satisfied, and let  $\mathcal{C}' \subseteq \mathcal{C} \times \Delta^1$  be the full subcategory spanned by those objects  $(C, i)$  for which either  $i = 1$ , or  $C$  is an initial object of  $\mathcal{C}$ . We observe that (b) is equivalent to the following pair of conditions:

- (b') The functor  $F|_{\mathcal{C}'}$  is a right Kan extension of  $F|_{\mathcal{C} \times \{1\}}$ .
- (b'') The functor  $F$  is a left Kan extension of  $F|_{\mathcal{C}'}$ .

Let  $\mathcal{E}$  be the full subcategory of  $\text{Fun}(\mathcal{C} \times \Delta^1, \mathcal{D})$  spanned by those functors which satisfy conditions (b') and (b''). Using Proposition T.4.3.2.15, we deduce that the projection  $\bar{\theta} : \mathcal{E} \rightarrow \text{Fun}(\mathcal{C} \times \{1\}, \mathcal{D})$  is a trivial Kan fibration. Since  $\theta$  is a pullback of  $\bar{\theta}$ , we conclude that  $\theta$  is a trivial Kan fibration. This completes the proof of (1).

To prove (2), we observe that  $\theta'$  is a pullback of  $\theta$  (since Proposition T.1.2.13.8 asserts that a square in  $\mathcal{D}_*$  is a pullback if and only if the underlying square in  $\mathcal{D}$  is a pullback).  $\square$

**Remark 10.6.** Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits finite colimits, and  $\mathcal{D}$  a pointed  $\infty$ -category which admits finite limits. Let  $F : \text{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})$  be given by composition with the suspension functor  $\mathcal{C} \rightarrow \mathcal{C}$ , and let  $G : \text{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})$  be given by composition with the loop functor  $\Omega : \mathcal{D} \rightarrow \mathcal{D}$ . Then  $F$  and  $G$  restrict to give homotopy inverse equivalences

$$\text{Exc}(\mathcal{C}, \mathcal{D}) \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \text{Exc}(\mathcal{C}, \mathcal{D}).$$

**Notation 10.7.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between  $\infty$ -categories, and assume that  $\mathcal{D}$  admits finite limits. For every commutative square  $\tau$ :

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

in  $\mathcal{C}$ , we obtain a commutative square  $F(\tau)$ :

$$\begin{array}{ccc} F(W) & \longrightarrow & F(X) \\ \downarrow & & \downarrow \\ F(Y) & \longrightarrow & F(Z) \end{array}$$

in  $\mathcal{D}$ . This diagram determines a map  $\eta_\tau : F(W) \rightarrow F(X) \times_{F(Z)} F(Y)$  in the  $\infty$ -category  $\mathcal{D}$ , which is well-defined up to homotopy. If we suppose further that  $Y$  and  $Z$  are zero objects of  $\mathcal{C}$ , that  $F(Y)$  and  $F(Z)$  are zero objects of  $\mathcal{D}$ , and that  $\tau$  is a pushout diagram, then we obtain a map  $F(W) \rightarrow \Omega F(\Sigma W)$ , which we will denote simply by  $\eta_W$ .

**Proposition 10.8.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits finite colimits,  $\mathcal{D}$  a pointed  $\infty$ -category which admits finite limits, and let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor which carries zero objects of  $\mathcal{C}$  to zero objects of  $\mathcal{D}$ . The following conditions are equivalent:*

- (1) *The functor  $F$  is excisive: that is,  $F$  carries pushout squares in  $\mathcal{C}$  to pullback squares in  $\mathcal{D}$ .*
- (2) *For every object  $X \in \mathcal{C}$ , the canonical map  $\eta_X : F(X) \rightarrow \Omega F(\Sigma X)$  is an equivalence in  $\mathcal{D}$  (see Notation 10.7).*

**Corollary 10.9.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between stable  $\infty$ -categories. Then  $F$  is exact if and only if the following conditions are satisfied:*

- (1) *The functor  $F$  carries zero objects of  $\mathcal{C}$  to zero objects of  $\mathcal{D}$ .*
- (2) *For every object  $X \in \mathcal{C}$ , the canonical map  $\Sigma F(X) \rightarrow F(\Sigma X)$  is an equivalence in  $\mathcal{D}$ .*

**Corollary 10.10.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits finite limits and colimits. Then:*

- (1) *If the suspension functor  $\Sigma_{\mathcal{C}}$  is fully faithful, then every pushout square in  $\mathcal{C}$  is a pullback square.*
- (2) *If the loop functor  $\Omega_{\mathcal{C}}$  is fully faithful, then every pullback square in  $\mathcal{C}$  is a pushout square.*
- (3) *If the loop functor  $\Omega_{\mathcal{C}}$  is an equivalence of  $\infty$ -categories, then  $\mathcal{C}$  is stable.*

*Proof.* Assertion (1) follows by applying Proposition 10.8 to the identity functor  $\text{id}_{\mathcal{C}}$ , and assertion (2) follows from (1) by passing to the opposite  $\infty$ -category. Assertion (3) is an immediate consequence of (1) and (2).  $\square$

The proof of Proposition 10.8 makes use of the following lemma:

**Lemma 10.11.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits finite colimits,  $\mathcal{D}$  a pointed  $\infty$ -category which admits finite limits, and  $F : \mathcal{C} \rightarrow \mathcal{D}$  a functor which carries zero objects of  $\mathcal{C}$  to zero objects of  $\mathcal{D}$ . Suppose given a pushout diagram  $\tau$ :*

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

in  $\mathcal{C}$ . Then there exists a map  $\theta_\tau : F(X) \times_{F(Z)} F(Y) \rightarrow \Omega F(\Sigma W)$  with the following properties:

- (1) The composition  $\theta_\tau \circ \eta_\tau$  is homotopic to  $\eta_W$ . Here  $\eta_\tau$  and  $\eta_W$  are defined as in Notation 10.7.  
(2) Let  $\Sigma(\tau)$  denote the induced diagram

$$\begin{array}{ccc} \Sigma W & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow \\ \Sigma Y & \longrightarrow & \Sigma Z. \end{array}$$

Then there is a pullback square

$$\begin{array}{ccc} \eta_{\Sigma(\tau)} \circ \theta_\tau & \longrightarrow & \eta_X \\ \downarrow & & \downarrow \\ \eta_Y & \longrightarrow & \eta_Z \end{array}$$

in the  $\infty$ -category  $\text{Fun}(\Delta^1, \mathcal{D})$  of morphisms in  $\mathcal{D}$ .

*Proof.* In the  $\infty$ -category  $\mathcal{C}$ , we have the following commutative diagram (in which every square is a pushout):

$$\begin{array}{ccccccc} W & \longrightarrow & X & \longrightarrow & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ Y & \longrightarrow & X \amalg_W Y & \longrightarrow & 0 \amalg_W Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X \amalg_W 0 & \longrightarrow & \Sigma W & \longrightarrow & \Sigma Y \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X & \longrightarrow & \Sigma(X \amalg_W Y). & & \end{array}$$

Applying the functor  $F$ , and replacing the upper left square by a pullback, we obtain a new diagram

$$\begin{array}{ccccccc} F(X) \times_{F(Z)} F(Y) & \longrightarrow & F(X) & \longrightarrow & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ F(Y) & \longrightarrow & F(Z) & \longrightarrow & F(0 \amalg_W Y) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F(X \amalg_W 0) & \longrightarrow & F(\Sigma(W)) & \longrightarrow & F(\Sigma Y) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F(\Sigma X) & \longrightarrow & F(\Sigma Z). & & \end{array}$$

Restricting attention to the large square in the upper left, we obtain the desired map  $\theta_\tau : F(X) \times_{F(Z)} F(Y) \rightarrow \Omega F(\Sigma W)$ . It is easy to verify that  $\theta_\tau$  has the desired properties.  $\square$

*Proof of Proposition 10.8.* The implication (1)  $\Rightarrow$  (2) is obvious. Conversely, suppose that (2) is satisfied. We must show that for every pushout square  $\tau$ :

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \amalg_X Z \end{array}$$



in the  $\infty$ -category  $\mathcal{C}$ , the induced map  $\eta_\tau$  is an equivalence in  $\mathcal{D}$ . Let  $\theta_\tau$  be as in the statement of Lemma 10.11. Then  $\theta_\tau \circ \eta_\tau$  is homotopic to  $\eta_X$ , and is therefore an equivalence (in virtue of assumption (2)). It will therefore suffice to show that  $\eta_\tau$  is an equivalence. The preceding argument shows that  $\theta_\tau$  has a right homotopy inverse. To show that  $\theta_\tau$  admits a left homotopy inverse, it will suffice to show that  $\eta_{\Sigma\tau} \circ \theta_\tau$  is an equivalence. This follows from the second assertion of Lemma 10.11, since the maps  $\eta_Y$ ,  $\eta_Z$ , and  $\eta_Y \amalg_X Z$  are equivalences (by assumption (2), again).  $\square$

Let  $\mathcal{C}$  be a (small) pointed  $\infty$ -category. Let  $\mathcal{P}_*(\mathcal{C})$  be defined as in Lemma 8.26. Lemma 10.4 implies that the canonical map  $\text{Fun}_*(\mathcal{C}^{op}, \mathcal{S}_*) \rightarrow \mathcal{P}_*(\mathcal{C})$  is a trivial fibration. Consequently, the Yoneda embedding lifts to a fully faithful functor  $j' : \mathcal{C} \rightarrow \text{Fun}_*(\mathcal{C}^{op}, \mathcal{S}_*)$ , which we will refer to as the *pointed Yoneda embedding*. Our terminology is slightly abusive: the functor  $j'$  is only well-defined up to a contractible space of choices; we will ignore this ambiguity.

**Proposition 10.12.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits finite colimits and  $\mathcal{D}$  an  $\infty$ -category which admits finite limits. Then composition with the canonical map  $\text{Stab}(\mathcal{D}) \rightarrow \mathcal{D}$  induces an equivalence of  $\infty$ -categories*

$$\theta : \text{Exc}(\mathcal{C}, \text{Stab}(\mathcal{D})) \rightarrow \text{Exc}(\mathcal{C}, \mathcal{D}).$$

*Proof.* Since the loop functor  $\Omega_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D}$  is left exact, the domain of  $\theta$  can be identified with a homotopy limit of the tower

$$\dots \xrightarrow{\circ\Omega_{\mathcal{D}}} \text{Exc}(\mathcal{C}, \mathcal{D}_*) \xrightarrow{\circ\Omega_{\mathcal{D}}} \text{Exc}(\mathcal{C}, \mathcal{D}_*).$$

Remark 10.6 implies that this tower is essentially constant. Consequently, it will suffice to show that the canonical map  $\text{Exc}(\mathcal{C}, \mathcal{D}_*) \rightarrow \text{Exc}(\mathcal{C}, \mathcal{D})$  is a trivial fibration of simplicial sets, which follows from Lemma 10.4.  $\square$

**Example 10.13.** Let  $\mathcal{C}$  be an  $\infty$ -category which admits finite limits, and  $K$  an arbitrary simplicial set. Then  $\text{Fun}(K, \mathcal{C})$  admits finite limits (Proposition T.5.1.2.2). We have a canonical isomorphism  $\text{Fun}(K, \mathcal{C})_* \simeq \text{Fun}(K, \mathcal{C}_*)$ , and the loop functor on  $\text{Fun}(K, \mathcal{C})_*$  can be identified with the functor given by composition with  $\Omega : \mathcal{C}_* \rightarrow \mathcal{C}_*$ . It follows that there is a canonical equivalence of  $\infty$ -categories

$$\text{Stab}(\text{Fun}(K, \mathcal{C})) \simeq \text{Fun}(K, \text{Stab}(\mathcal{C})).$$

In particular,  $\text{Stab}(\mathcal{P}(K))$  can be identified with  $\text{Fun}(K, \text{Sp})$ .

We can apply Proposition 10.12 to give another description of the  $\infty$ -category  $\text{Stab}(\mathcal{C})$ .

**Lemma 10.14.** *Let  $\mathcal{C}$  be an  $\infty$ -category which admits finite colimits, let  $f : \mathcal{C} \rightarrow \mathcal{C}_*$  be a left adjoint to the forgetful functor, and let  $\mathcal{D}$  be a stable  $\infty$ -category. Then composition with  $f$  induces an equivalence of  $\infty$ -categories  $\phi : \text{Exc}(\mathcal{C}_*, \mathcal{D}) \rightarrow \text{Exc}(\mathcal{C}, \mathcal{D})$ .*

*Proof.* Consider the composition

$$\theta : \text{Fun}(\mathcal{C}, \mathcal{D}) \times \mathcal{C}_* \subseteq \text{Fun}(\mathcal{C}, \mathcal{D}) \times \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \text{Fun}(\Delta^1, \mathcal{D}) \xrightarrow{\text{coker}} \mathcal{D}.$$

We can identify  $\theta$  with a map  $\text{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}_*, \mathcal{D})$ . Since the collection of pullback squares in  $\mathcal{D}$  is a stable subcategory of  $\text{Fun}(\Delta^1 \times \Delta^1, \mathcal{D})$ , we conclude  $\theta$  restricts to a map  $\psi : \text{Exc}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Exc}(\mathcal{C}_*, \mathcal{D})$ . It is not difficult to verify that  $\psi$  is a homotopy inverse to  $\phi$ .  $\square$

**Proposition 10.15.** *Let  $\mathcal{C}$  be an  $\infty$ -category which admits finite colimits, and let  $\mathcal{D}$  be an  $\infty$ -category which admits finite limits. Then there is a canonical isomorphism  $\text{Exc}(\mathcal{C}_*, \mathcal{D}) \simeq \text{Exc}(\mathcal{C}, \text{Stab}(\mathcal{D}))$  in the homotopy category of  $\infty$ -categories.*

*Proof.* Combining Lemma 10.14 and Proposition 10.12, we obtain a diagram of equivalences

$$\text{Exc}(\mathcal{C}_*, \mathcal{D}) \leftarrow \text{Exc}(\mathcal{C}_*, \text{Stab}(\mathcal{D})) \rightarrow \text{Exc}(\mathcal{C}, \text{Stab}(\mathcal{D})).$$

$\square$

**Corollary 10.16.** *Let  $\mathcal{D}$  be an  $\infty$ -category which admits finite limits. Then there is a canonical equivalence  $\text{Stab}(\mathcal{D}) \simeq \text{Exc}(\mathcal{S}_*^{\text{fin}}, \mathcal{D})$  in the homotopy category of  $\infty$ -categories.*

*Proof.* Combine Proposition 10.15, Remark 9.4, and Remark 10.3. □

**Corollary 10.17.** *The  $\infty$ -category of spectra is equivalent to the  $\infty$ -category  $\text{Exc}(\mathcal{S}_*^{\text{fin}}, \mathcal{S})$ .*

**Remark 10.18.** Corollary 10.17 provides a very explicit model for spectra. Namely, we can identify a spectrum with an excisive functor  $F : \mathcal{S}_*^{\text{fin}} \rightarrow \mathcal{S}$ . We should think of  $F$  as a *homology theory*  $A$ . More precisely, given a pair of finite spaces  $X_0 \subseteq X$ , we can define the relative homology group  $A_n(X, X_0)$  to be  $\pi_n F(X/X_0)$ , where  $X/X_0$  denotes the pointed space obtained from  $X$  by collapsing  $X_0$  to a point (here the homotopy group is taken with base point provided by the map  $* \simeq F(*) \rightarrow F(X/X_0)$ ). The assumption that  $F$  is excisive is precisely what is needed to guarantee the existence of the usual excision exact sequences for the homology theory  $A$ .

## 11 Filtered Objects and Spectral Sequences

Suppose given a sequence of objects

$$\dots \rightarrow X(-1) \xrightarrow{f^0} X(0) \xrightarrow{f^1} X(1) \rightarrow \dots$$

in a stable  $\infty$ -category  $\mathcal{C}$ . Suppose further that  $\mathcal{C}$  is equipped with a t-structure, and that the heart of  $\mathcal{C}$  is equivalent to the nerve of an abelian category  $\mathcal{A}$ . In this section, we will construct a spectral sequence taking values in the abelian category  $\mathcal{A}$ , with the  $E_1$ -page described by the formula

$$E_1^{p,q} = \pi_{p+q} \text{coker}(f^p) \in \mathcal{A}.$$

Under appropriate hypotheses, we will see that this spectral sequence converges to the homotopy groups of the colimit  $\varinjlim(X(i))$ .

Our first step is to construct some auxiliary objects in  $\mathcal{C}$ .

**Definition 11.1.** Let  $\mathcal{C}$  be a pointed  $\infty$ -category, and let  $\mathcal{J}$  be a linearly ordered set. We let  $\mathcal{J}^{[1]}$  denote the partially ordered set of pairs of elements  $i \leq j$  of  $\mathcal{J}$ , where  $(i, j) \leq (i', j')$  if  $i \leq j'$  and  $i' \leq j$ . An  $\mathcal{J}$ -complex in  $\mathcal{C}$  is a functor  $F : \mathcal{N}(\mathcal{J}^{[1]}) \rightarrow \mathcal{C}$  with the following properties:

- (1) For each  $i \in \mathcal{J}$ ,  $F(i, i)$  is a zero object of  $\mathcal{C}$ .
- (2) For every  $i \leq j \leq k$ , the associated diagram

$$\begin{array}{ccc} F(i, j) & \longrightarrow & F(i, k) \\ \downarrow & & \downarrow \\ F(j, j) & \longrightarrow & F(j, k) \end{array}$$

is a pushout square in  $\mathcal{C}$ .

We let  $\text{Gap}(\mathcal{J}, \mathcal{C})$  denote the full subcategory of  $\text{Fun}(\mathcal{N}(\mathcal{J}^{[1]}), \mathcal{C})$  spanned by the  $\mathcal{J}$ -complexes in  $\mathcal{C}$ .

**Remark 11.2.** Let  $F \in \text{Gap}(\mathbf{Z}, \mathcal{C})$  be a  $\mathbf{Z}$ -complex in a stable  $\infty$ -category  $\mathcal{C}$ . For each  $n \in \mathbf{Z}$ , the functor  $F$  determines pushout square

$$\begin{array}{ccc} F(n-1, n) & \longrightarrow & F(n-1, n+1) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & F(n, n+1), \end{array}$$

hence a boundary  $\delta : F(n, n+1) \rightarrow F(n-1, n)[1]$ . If we set  $C_n = F(n-1, n)[-n]$ , then we obtain a sequence of maps

$$\dots \rightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \rightarrow \dots$$

in the homotopy category  $\mathbf{h}\mathcal{C}$ . The commutative diagram

$$\begin{array}{ccccc} F(n, n+1) & \xrightarrow{\delta} & F(n-2, n)[1] & \longrightarrow & F(n-1, n)[1] \\ & & \downarrow & & \downarrow \delta \\ 0 & \xrightarrow{\sim} & F(n-1, n-1)[1] & \longrightarrow & F(n-2, n-1)[2]. \end{array}$$

shows that  $d_{n-1} \circ d_n \simeq 0$ , so that  $(C_\bullet, d_\bullet)$  can be viewed as a chain complex in the triangulated category  $\mathbf{h}\mathcal{C}$ . This motivates the terminology of Definition 11.1.

**Lemma 11.3.** *Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits pushouts. Let  $\mathcal{J} = \mathcal{J}_0 \cup \{-\infty\}$  be a linearly ordered set containing a least element  $-\infty$ . We regard  $\mathcal{J}_0$  as a linearly ordered subset of  $\mathcal{J}^{[1]}$  via the embedding*

$$i \mapsto (-\infty, i).$$

*Then the restriction map  $\text{Gap}(\mathcal{J}, \mathcal{C}) \rightarrow \text{Fun}(\mathbf{N}(\mathcal{J}_0), \mathcal{C})$  is an equivalence of  $\infty$ -categories.*

*Proof.* Let  $\mathcal{J} = \{(i, j) \in \mathcal{J}^{[1]} : (i = -\infty) \vee (i = j)\}$ . We now make the following observations:

- (1) A functor  $F : \mathbf{N}(\mathcal{J}^{[1]}) \rightarrow \mathcal{C}$  is a complex if and only if  $F$  is a left Kan extension of  $F|_{\mathbf{N}(\mathcal{J})}$ , and  $F(i, i)$  is a zero object of  $\mathcal{C}$  for all  $i \in \mathcal{J}$ .
- (2) Any functor  $F_0 : \mathbf{N}(\mathcal{J}) \rightarrow \mathcal{C}$  admits a left Kan extension to  $\mathbf{N}(\mathcal{J}^{[1]})$  (use Lemma T.4.3.2.13 and the fact that  $\mathcal{C}$  admits pushouts).
- (3) A functor  $F_0 : \mathbf{N}(\mathcal{J}) \rightarrow \mathcal{C}$  has the property that  $F_0(i, i)$  is a zero object, for every  $i \in \mathcal{J}$ , if and only if  $F_0$  is a right Kan extension of  $F_0|_{\mathbf{N}(\mathcal{J}_0)}$ .
- (4) Any functor  $F_0 : \mathbf{N}(\mathcal{J}_0) \rightarrow \mathcal{C}$  admits a right Kan extension to  $\mathbf{N}(\mathcal{J})$  (use Lemma T.4.3.2.13 and the fact that  $\mathcal{C}$  has a final object).

The desired conclusion now follows immediately from Proposition T.4.3.2.15. □

**Remark 11.4.** Let  $\mathcal{C}$  be a pointed  $\infty$ -category which admits pushouts (for example, a stable  $\infty$ -category). For each  $n \geq 0$ , let  $\text{Gap}^0([n], \mathcal{C})$  be the largest Kan complex contained in  $\text{Gap}([n], \mathcal{C})$ . Then the assignment

$$[n] \mapsto \text{Gap}([n], \mathcal{C})$$

determines a simplicial object in the category  $\mathcal{K}\text{an}$  of Kan complexes. We can then define the *Waldhausen  $K$ -theory* of  $\mathcal{C}$  to be a geometric realization of this bisimplicial set (for example, the associated diagonal simplicial set). In the special case where  $A$  is an  $A_\infty$ -ring and  $\mathcal{C}$  is the smallest stable subcategory of  $\text{Mod}_A$  which contains  $A$ , this definition recovers the usual  $K$ -theory of  $A$ . We refer the reader to [72] for a related construction.

**Construction 11.5.** Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a t-structure, such that the heart of  $\mathcal{C}$  is equivalent to the nerve of an abelian category  $\mathcal{A}$ . Let  $X \in \text{Gap}(\mathbf{Z}, \mathcal{C})$ . We observe that for every triple of integers  $i \leq j \leq k$ , there is a long exact sequence

$$\dots \rightarrow \pi_n X(i, j) \rightarrow \pi_n X(i, k) \rightarrow \pi_n X(j, k) \xrightarrow{\delta} \pi_{n-1} X(i, j) \rightarrow \dots$$

in the abelian category  $\mathcal{A}$ . For every  $p, q \in \mathbf{Z}$  and every  $r \geq 1$ , we define the object  $E_r^{p,q} \in \mathcal{A}$  by the formula

$$E_r^{p,q} = \text{im}(\pi_{p+q}X(p-r, p) \rightarrow \pi_{p+q}X(p-1, p+r-1)).$$

There is a differential  $d_r : E_r^{p,q} \rightarrow E_r^{p-r, q+r-1}$ , uniquely determined by the requirement that the diagram

$$\begin{array}{ccccc} \pi_{p+q}X(p-r, p) & \longrightarrow & E_r^{p,q} & \longrightarrow & \pi_{p+q}X(p-1, p+r-1) \\ \downarrow \delta & & \downarrow d_r & & \downarrow \delta \\ \pi_{p+q-1}X(p-2r, p-r) & \longrightarrow & E_r^{p-r, q+r-1} & \longrightarrow & \pi_{p+q-1}X(p-r-1, p-1) \end{array}$$

be commutative.

**Proposition 11.6.** *Let  $X \in \text{Gap}(\mathbf{Z}, \mathcal{C})$  be as in Construction 11.5. Then:*

- (1) *For each  $r \geq 1$ , the composition  $d_r \circ d_r$  is zero.*
- (2) *There are canonical isomorphisms*

$$E_{r+1}^{p,q} \simeq \ker(d_r : E_r^{p,q} \rightarrow E_r^{p-r, q+r-1}) / \text{im}(d_r : E_r^{p+r, q-r+1} \rightarrow E_r^{p,q}).$$

Consequently,  $\{E_r^{p,q}, d_r\}$  is a spectral sequence (with values in the abelian category  $\mathcal{A}$ ).

**Remark 11.7.** For fixed  $q \in \mathbf{Z}$ , the complex  $(E_1^{\bullet, q}, d_1)$  in  $\mathcal{A}$  can be obtained from the  $\text{h}\mathcal{C}$ -valued chain complex  $C_\bullet$  described in Remark 11.2 by applying the cohomological functor  $\pi_q$ .

*Proof.* We have a commutative diagram

$$\begin{array}{ccccc} & & \pi_{p+q}X(p-r-1, p) & & \\ & & \downarrow & & \\ \pi_{p+q+1}X(p, p+r) & \xrightarrow{\delta} & \pi_{p+q}X(p-r, p) & \xrightarrow{\delta} & \pi_{p+q-1}X(p-2r, p-r) \\ \downarrow & & \downarrow & & \downarrow \\ E_r^{p+r, q-r+1} & \xrightarrow{d_r} & E_r^{p,q} & \xrightarrow{d_r} & E_r^{p-r, q+r-1} \\ \downarrow & & \downarrow & & \downarrow \\ \pi_{p+q+1}X(p+r-1, p+2r-1) & \xrightarrow{\delta} & \pi_{p+q}X(p-1, p+r-1) & \xrightarrow{\delta} & \pi_{p+q-1}X(p-r-1, p-1) \\ & & \downarrow & & \\ & & \pi_{p+q}X(p-1, p+r) & & \end{array}$$

Since the upper left vertical map is an epimorphism, (1) will follow provided that we can show that the composition

$$\pi_{p+q+1}X(p, p+r) \xrightarrow{\delta} \pi_{p+q}X(p-r, p) \xrightarrow{\delta} \pi_{p+q-1}X(p-2r, p-r)$$

is zero. This follows immediately from the commutativity of the diagram

$$\begin{array}{ccccc} X(p, p+r) & \xrightarrow{\delta} & X(p-2r, p)[1] & \longrightarrow & X(p-r, p)[1] \\ & & \downarrow & & \downarrow \delta \\ 0 & \xrightarrow{\sim} & X(p-r, p-r)[1] & \longrightarrow & X(p-2r, p-r)[2]. \end{array}$$

We next claim that the composite map

$$\phi : \pi_{p+q}X(p-r-1, p) \rightarrow E_r^{p,q} \xrightarrow{d_r} E_r^{p-r, q+r-1}$$

is zero. Because  $E_r^{p-r, q+r-1} \rightarrow \pi_{p+q-1}X(p-r-1, p-1)$  is a monomorphism, this follows from the commutativity of the diagram

$$\begin{array}{ccc} \pi_{p+q}X(p-r-1, p) & \longrightarrow & \pi_{p+q-1}X(p-2r, p-r-1) \\ \downarrow & & \downarrow \\ \pi_{p+q}X(p-r, p) & \longrightarrow & \pi_{p+q-1}X(p-2r, p-r) \\ \downarrow & & \downarrow \\ E_r^{p,q} & \xrightarrow{d_r} & E_r^{p-r, q+r-1} \\ & & \downarrow \\ & & \pi_{p+q-1}X(p-r-1, p-1), \end{array}$$

since the composition of the left vertical line factors through  $\pi_{p+q-1}X(p-r-1, p-r-1) \simeq 0$ . A dual argument shows that the composition

$$E_r^{p+r, q+r-1} \xrightarrow{d_r} E_r^{p,q} \rightarrow \pi_{p+q}X(p-1, p+r)$$

is zero as well.

Let  $Z = \ker(d_r : E_r^{p,q} \rightarrow E_r^{p-r, q+r-1})$  and  $B = \text{im}(d_r : E_r^{p+r, q+r-1} \rightarrow E_r^{p,q})$ . The above arguments yield a sequence of morphisms

$$\pi_{p+q}X(p-r-1, p) \xrightarrow{\phi} Z \xrightarrow{\phi'} Z/B \xrightarrow{\psi'} E_r^{p,q}/B \xrightarrow{\psi} \pi_{p+q}X(p-1, p+r).$$

To complete the proof of (2), it will suffice to show that  $\phi' \circ \phi$  is an epimorphism and that  $\psi \circ \psi'$  is a monomorphism. By symmetry, it will suffice to prove the first assertion. Since  $\phi'$  is evidently an epimorphism, we are reduced to showing that  $\phi$  is an epimorphism.

Let  $K$  denote the kernel of the composite map

$$\pi_{p+q}X(p-r, p) \rightarrow E_r^{p,q} \xrightarrow{d_r} E_r^{p-r, q+r-1} \rightarrow \pi_{p+q-1}X(p-r-1, p-1),$$

so that the canonical map  $K \rightarrow Z$  is an epimorphism. Choose a diagram

$$\begin{array}{ccccc} \tilde{K} & \xrightarrow{f} & K & \longrightarrow & 0 \\ \downarrow g & & \downarrow & & \downarrow \\ \pi_{p+q}X(p-r, p-1) & \longrightarrow & \pi_{p+q-1}X(p-r-1, p-r) & \longrightarrow & \pi_{p+q-1}X(p-r-1, p-1) \end{array}$$

where the square on the left is a pullback. The exactness of the bottom row implies that  $f$  is an epimorphism. Let  $f'$  denote the composition

$$\tilde{K} \xrightarrow{g} \pi_{p+q}X(p-r, p-1) \rightarrow \pi_{p+q}X(p-r, p).$$

The composition

$$\tilde{K} \xrightarrow{f'} \pi_{p+q}X(p-r, p) \rightarrow \pi_{p+q}X(p-1, p+r)$$

factors through  $\pi_{p+q}X(p-1, p-1) \simeq 0$ . Since  $E_r^{p,q} \rightarrow \pi_{p+q}X(p-1, p+r)$  is a monomorphism, we conclude that the composition  $\tilde{K} \xrightarrow{f'} \pi_{p+q}X(p-r, p) \rightarrow E_r^{p,q}$  is the zero map. It follows that the composition

$$\tilde{K} \xrightarrow{f-f'} \pi_{p+q}X(p-r, p) \rightarrow Z$$

coincides with the composition  $\tilde{K} \xrightarrow{f} K \rightarrow Z$ , and is therefore an epimorphism.

Form a diagram

$$\begin{array}{ccccc} \bar{K} & \xrightarrow{f''} & \tilde{K} & \longrightarrow & 0 \\ \downarrow & & \downarrow f-f' & & \downarrow \\ \pi_{p+q}X(p-r-1, p) & \longrightarrow & \pi_{p+q}X(p-r, p) & \longrightarrow & \pi_{n-1}X(p-r-1, p-r) \end{array}$$

where the left square is a pullback. Since the bottom line is exact, we conclude that  $f''$  is an epimorphism, so that the composition

$$\bar{K} \xrightarrow{f''} \tilde{K} \xrightarrow{f-f'} \pi_{p+q}X(p-r, p) \rightarrow Z$$

is an epimorphism. This map coincides with the composition

$$\bar{K} \rightarrow \pi_{p+q}X(p-r-1, p) \xrightarrow{\phi} Z,$$

so that  $\phi$  is an epimorphism as well.  $\square$

**Definition 11.8.** Let  $\mathcal{C}$  be a stable  $\infty$ -category. A *filtered object* of  $\mathcal{C}$  is a functor  $X : \mathbf{N}(\mathbf{Z}) \rightarrow \mathcal{C}$ .

Suppose that  $\mathcal{C}$  is equipped with a t-structure, and let  $X : \mathbf{N}(\mathbf{Z}) \rightarrow \mathcal{C}$  be a filtered object of  $\mathcal{C}$ . According to Lemma 11.3, we can extend  $X$  to a complex in  $\text{Gap}(\mathbf{Z} \cup \{-\infty\}, \mathcal{C})$ . Let  $\bar{X}$  be the associated object of  $\text{Gap}(\mathbf{Z}, \mathcal{C})$ , and let  $\{E_r^{p,q}, d_r\}_{r \geq 1}$  be the spectral sequence described in Construction 11.5 and Proposition 11.6. We will refer to  $\{E_r^{p,q}, d_r\}_{r \geq 1}$  as the *spectral sequence associated to the filtered object*  $X$ .

**Remark 11.9.** In the situation of Definition 11.8, Lemma 11.3 implies that  $\bar{X}$  is determined up to contractible ambiguity by  $X$ . It follows that the spectral sequence  $\{E_r^{p,q}, d_r\}_{r \geq 1}$  is independent of the choice of  $\bar{X}$ , up to canonical isomorphism.

**Example 11.10.** Let  $\mathcal{A}$  be a sufficiently nice abelian category, and let  $\mathcal{C}$  be the derived  $\infty$ -category of  $\mathcal{A}$  (see §S.13). Let  $\text{Fun}(\mathbf{N}(\mathbf{Z}), \mathcal{C})$  be the  $\infty$ -category of filtered objects of  $\mathcal{C}$ . Then the homotopy category  $\text{hFun}(\mathbf{N}(\mathbf{Z}), \mathcal{C})$  can be identified with the classical *filtered derived category* of  $\mathcal{A}$ , obtained from the category of filtered complexes of objects of  $\mathcal{A}$  by inverting all filtered quasi-isomorphisms. In this case, Definition 11.8 recovers the usual spectral sequence associated to a filtered complex.

Our next goal is to establish the convergence of the spectral sequence of Definition 11.8. We will treat only the simplest case, which will be sufficient for our applications.

**Definition 11.11.** Let  $\mathcal{C}$  be an  $\infty$ -category. We will say that  $\mathcal{C}$  *admits sequential colimits* if every diagram  $\mathbf{N}(\mathbf{Z}_{\geq 0}) \rightarrow \mathcal{C}$  has a colimit in  $\mathcal{C}$ .

If  $\mathcal{C}$  is stable and admits sequential colimits, we will say that a t-structure on  $\mathcal{C}$  is *compatible with sequential colimits* if the full subcategory  $\mathcal{C}_{\leq 0}$  is stable under the colimits of diagrams indexed by  $\mathbf{N}(\mathbf{Z}_{\geq 0})$ .

**Remark 11.12.** Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a t-structure, so that the heart of  $\mathcal{C}$  is equivalent to (the nerve of) an abelian category  $\mathcal{A}$ . Suppose that  $\mathcal{C}$  admits sequential colimits. Then  $\mathcal{C}_{\geq 0}$  admits sequential colimits, so that  $\mathbf{N}(\mathcal{A})$ , being a localization of  $\mathcal{C}_{\geq 0}$ , also admits sequential colimits. If the t-structure on  $\mathcal{C}$  is compatible with sequential colimits, then the inclusion  $\mathbf{N}(\mathcal{A}) \subseteq \mathcal{C}$  and the homological functors  $\{\pi_n : \mathcal{C} \rightarrow \mathbf{N}(\mathcal{A})\}_{n \in \mathbf{Z}}$  preserve sequential colimits. It follows that sequential colimits in the abelian category  $\mathcal{A}$  are exact: in other words, the direct limit of a sequence of monomorphisms in  $\mathcal{A}$  is again a monomorphism.

**Proposition 11.13.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a  $t$ -structure, and let  $X : \mathbf{N}(\mathbf{Z}) \rightarrow \mathcal{C}$  be a filtered object of  $\mathcal{C}$ . Assume that  $\mathcal{C}$  admits sequential colimits, and that the  $t$ -structure on  $\mathcal{C}$  is compatible with sequential colimits. Suppose furthermore that  $X(n) \simeq 0$  for  $n \ll 0$ . Then the associated spectral sequence (Definition 11.8) converges*

$$E_r^{p,q} \Rightarrow \pi_{p+q} \varinjlim(X).$$

*Proof.* Let  $\mathcal{A}$  be an abelian category such that the heart of  $\mathcal{C}$  is equivalent to (the nerve of)  $\mathcal{A}$ . The convergence assertion of the Proposition has the following meaning:

(i) For fixed  $p$  and  $q$ , the differentials  $d_r : E_r^{p,q} \rightarrow E_r^{p-r, q+r-1}$  vanish for  $r \gg 0$ .

Consequently, for sufficiently large  $r$  we obtain a sequence of epimorphisms

$$E_r^{p,q} \rightarrow E_{r+1}^{p,q} \rightarrow E_{r+2}^{p,q} \rightarrow \dots$$

Let  $E_\infty^{p,q}$  denote the colimit of this sequence (in the abelian category  $\mathcal{A}$ ).

(ii) Let  $n \in \mathbf{Z}$ , and let  $A_n = \pi_n \varinjlim(X)$ . Then there exists a filtration

$$\dots \subseteq F^{-1}A_n \subseteq F^0A_n \subseteq F^1A_n \subseteq \dots$$

of  $A_n$ , with  $F^pA_n \simeq 0$  for  $p \ll 0$ , and  $\varinjlim(F^pA_n) \simeq A_n$ .

(iii) For every  $p, q \in \mathbf{Z}$ , there exists an isomorphism  $E_\infty^{p,q} \simeq F^pA_{p+q}/F^{p-1}A_{p+q}$  in the abelian category  $\mathcal{A}$ .

To prove (i), (ii), and (iii), we first extend  $X$  to an object  $\overline{X} \in \text{Gap}(\mathbf{Z} \cup \{-\infty\}, \mathcal{C})$ , so that for each  $n \in \mathbf{Z}$  we have  $X(n) = \overline{X}(-\infty, n)$ . Without loss of generality, we may suppose that  $X(n) \simeq *$  for  $n < 0$ . This implies that  $\overline{X}(i, j) \simeq *$  for  $i, j < 0$ . It follows that  $E_r^{p-r, q+r-1}$ , as a quotient  $\pi_{p+q}\overline{X}(p-2r, p-r)$ , is zero for  $r > p$ . This proves (i).

To satisfy (ii), we set  $F^pA_n = \text{im}(\pi_n X(p) \rightarrow \pi_n \varinjlim(X))$ . It is clear that  $F^pA_n \simeq *$  for  $p < 0$ , and the isomorphism  $\varinjlim F^pA_n \simeq A_n$  follows from the compatibility of the homological functor  $\pi_n$  with sequential colimits (Remark 11.12).

To prove (iii), we note that for  $r > p$ , the object  $E_r^{p,q}$  can be identified with the image of the map  $\pi_{p+q}X(p) \simeq \pi_{p+q}\overline{X}(p-r, p) \rightarrow \pi_{p+q}\overline{X}(p-1, p+r)$ . Let  $Y = \varinjlim_r \overline{X}(p-1, p+r)$ . It follows that  $E_\infty^{p,q}$  can be identified with the image of the map  $\pi_{p+q}X(p) \xrightarrow{f} \pi_{p+q}Y$ . We have a distinguished triangle

$$X(p-1) \rightarrow \varinjlim(X) \rightarrow Y \rightarrow X(p-1)[1],$$

which induces an exact sequence

$$0 \rightarrow F^{p-1}A_{p+q} \rightarrow A_{p+q} \xrightarrow{f'} \pi_{p+q}Y.$$

We have a commutative triangle

$$\begin{array}{ccc} & A_{p+q} & \\ g \nearrow & & \searrow f' \\ \pi_{p+q}X(p) & \xrightarrow{f} & \pi_{p+q}Y. \end{array}$$

Since the image of  $g$  is  $F^pA_{p+q}$ , we obtain canonical isomorphisms

$$E_\infty^{p,q} \simeq \text{im}(f) \simeq \text{im}(f'|F^pA_{p+q}) \simeq F^pA_{p+q}/\ker(f') \simeq F^pA_{p+q}/F^{p-1}A_{p+q}.$$

This completes the proof.  $\square$

## 12 The $\infty$ -Categorical Dold-Kan Correspondence

Let  $\mathcal{A}$  be an abelian category. Then the classical Dold-Kan correspondence (see [75]) asserts that the category  $\text{Fun}(\Delta^{op}, \mathcal{A})$  of simplicial objects of  $\mathcal{A}$  is equivalent to the category  $\text{Ch}_{\geq 0}(\mathcal{A})$  of (homologically) nonnegatively graded chain complexes

$$\dots \xrightarrow{d} A_1 \xrightarrow{d} A_0 \rightarrow 0.$$

In this section, we will prove an analogue of this result when the abelian category  $\mathcal{A}$  is replaced by a stable  $\infty$ -category.

We begin by observing that if  $X_\bullet$  is a simplicial object in a stable  $\infty$ -category  $\mathcal{C}$ , then  $X_\bullet$  determines a simplicial object of the homotopy category  $\text{h}\mathcal{C}$ . The category  $\text{h}\mathcal{C}$  is not abelian, but it is additive and has the following additional property (which follows easily from the fact that  $\text{h}\mathcal{C}$  admits a triangulated structure):

- (\*) If  $i : X \rightarrow Y$  is a morphism in  $\text{h}\mathcal{C}$  which admits a left inverse, then there is an isomorphism  $Y \simeq X \oplus X'$  such that  $i$  is identified with the map  $(\text{id}, 0)$ .

These conditions are sufficient to construct a Dold-Kan correspondence in  $\text{h}\mathcal{C}$ . Consequently, every simplicial object  $X_\bullet$  of  $\mathcal{C}$  determines a chain complex

$$\dots \rightarrow C_1 \rightarrow C_0 \rightarrow 0$$

in the homotopy category  $\text{h}\mathcal{C}$ . In §11, we described another construction which gives rise to the same type of data. More precisely, Lemma 11.3 and Remark 11.2 show that every  $\mathbf{Z}_{\geq 0}$ -filtered object

$$Y(0) \xrightarrow{f_1} Y(1) \xrightarrow{f_2} \dots$$

determines a chain complex  $C_\bullet$  with values in  $\text{h}\mathcal{C}$ , where  $C_n = \text{coker}(f_i)[-n]$ . This suggests a relationship between filtered objects of  $\mathcal{C}$  and simplicial objects of  $\mathcal{C}$ . Our goal in this section is to describe this relationship in detail. Our main result, Theorem 12.8, asserts that the  $\infty$ -category of simplicial objects of  $\mathcal{C}$  is equivalent to a suitable  $\infty$ -category of (increasingly) filtered objects of  $\mathcal{C}$ . The proof will require several preliminaries.

**Lemma 12.1.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category. A square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow f' & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

*in  $\mathcal{C}$  is a pullback if and only if the induced map  $\alpha : \text{coker}(f') \rightarrow \text{coker}(f)$  is an equivalence.*

*Proof.* Form an expanded diagram

$$\begin{array}{ccccc} X' & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow f' & & \downarrow f & & \downarrow \\ Y' & \longrightarrow & Y & \longrightarrow & \text{coker}(f) \end{array}$$

where the right square is a pushout. Since  $\mathcal{C}$  is stable, the right square is also a pullback. Lemma T.4.4.2.1 implies that the left square is a pullback if and only if the outer square is a pullback, which is in turn equivalent to the assertion that  $\alpha$  is an equivalence.  $\square$

**Lemma 12.2.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category, let  $K$  be a simplicial set, and suppose that  $\mathcal{C}$  admits  $K$ -indexed colimits. Let  $\bar{\alpha} : K^\triangleright \times \Delta^1 \rightarrow \mathcal{C}$  be a natural transformation between a pair of diagrams  $\bar{p}, \bar{q} : K^\triangleright \rightarrow \mathcal{C}$ . Then  $\bar{\alpha}$  is a colimit diagram if and only if  $\text{coker}(\bar{\alpha}) : K^\triangleright \rightarrow \mathcal{C}$  is a colimit diagram.*



*Proof.* Let  $p = \bar{p}|K$ ,  $q = \bar{q}|K$ , and  $\alpha = \bar{\alpha}|K \times \Delta^1$ . Since  $\mathcal{C}$  admits  $K$ -indexed colimits, there exist colimit diagrams  $\bar{p}', \bar{q}' : K^\triangleright \rightarrow \mathcal{C}$  extending  $p$  and  $q$ , respectively. We obtain a square

$$\begin{array}{ccc} \bar{p}' & \longrightarrow & \bar{p} \\ \downarrow & & \downarrow \\ \bar{q}' & \longrightarrow & \bar{q} \end{array}$$

in the  $\infty$ -category  $\text{Fun}(K^\triangleright, \mathcal{C})$ . Let  $\infty$  denote the cone point of  $K^\triangleright$ . Using Corollary T.4.2.3.10, we deduce that  $\alpha$  is a colimit diagram if and only if the induced square

$$\begin{array}{ccc} \bar{p}'(\infty) & \longrightarrow & \bar{p}(\infty) \\ \downarrow f' & & \downarrow f \\ \bar{q}'(\infty) & \longrightarrow & \bar{q}(\infty) \end{array}$$

is a pushout. According to Lemma 12.1, this is equivalent to the assertion that the induced map  $\beta : \text{coker}(f') \rightarrow \text{coker}(f)$  is an equivalence. We conclude by observing that  $\beta$  can be identified with the natural map

$$\varinjlim(\text{coker}(\alpha)) \rightarrow \text{coker}(\bar{\alpha})(\infty),$$

which is an equivalence if and only if  $\text{coker}(\bar{\alpha})$  is a colimit diagram.  $\square$

Our next result is an analogue of Proposition S.4.4 which applies to cubical diagrams of higher dimension.

**Proposition 12.3.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category, and let  $\sigma : (\Delta^1)^n \rightarrow \mathcal{C}$  be a diagram. Then  $\sigma$  is a colimit diagram if and only if  $\sigma$  is a limit diagram.*

*Proof.* By symmetry, it will suffice to show that if  $\sigma$  is a colimit diagram, then  $\sigma$  is also a limit diagram. We work by induction on  $n$ . If  $n = 0$ , then we must show that every initial object of  $\mathcal{C}$  is also final, which follows from the assumption that  $\mathcal{C}$  has a zero object. If  $n > 0$ , then we may identify  $\sigma$  with a natural transformation  $\alpha : \sigma' \rightarrow \sigma''$  in the  $\infty$ -category  $\text{Fun}((\Delta^1)^{n-1}, \mathcal{C})$ . Assume that  $\sigma$  is a colimit diagram. Using Lemma 12.2, we deduce that  $\text{coker}(\alpha)$  is a colimit diagram. Since  $\text{coker}(\alpha) \simeq \ker(\alpha)[1]$ , we conclude that  $\ker(\alpha)$  is a colimit diagram. Applying the inductive hypothesis, we deduce that  $\ker(\alpha)$  is a limit diagram. The dual of Lemma 12.2 now implies that  $\sigma$  is a limit diagram, as desired.  $\square$

**Lemma 12.4.** *Fix  $n \geq 0$ , and let  $S$  be a subset of the open interval  $(0, 1)$  of cardinality  $\leq n$ . Let  $Y$  be the set of all sequences of real numbers  $0 \leq y_1 \leq \dots \leq y_n \leq 1$  such that  $S \subseteq \{y_1, \dots, y_n\}$ . Then  $Y$  is a contractible topological space.*

*Proof.* Let  $S$  have cardinality  $m \leq n$ , and let  $Z$  denote the set of sequences of real numbers  $0 \leq z_1 \leq \dots \leq z_{n-m} \leq 1$ . Then  $Z$  is homeomorphic to a topological  $(n-m)$ -simplex. Moreover, there is a homeomorphism  $f : Z \rightarrow Y$ , which carries a sequence  $\{z_i\}$  to a suitable reordering of the sequence  $\{z_i\} \cup S$ .  $\square$

**Lemma 12.5.** *Let  $n \leq 0$ , let  $\Delta_{\leq n}$  denote the full subcategory of  $\Delta$  spanned by the objects  $\{[m]\}_{0 \leq m \leq n}$ , and let  $\mathcal{J}$  denote the full subcategory of  $(\Delta_{\leq n})_{/[n]}$  spanned by the injective maps  $[m] \rightarrow [n]$ . Then the induced map*

$$\text{N}(\mathcal{J})^{op} \rightarrow \text{N}(\Delta_{\leq n})^{op}$$

*is cofinal.*

*Proof.* Fix  $m \leq n$ , and let  $\mathcal{J}$  denote the category of diagrams

$$[m] \leftarrow [k] \xrightarrow{i} [n]$$

where  $i$  is injective. According to Theorem T.4.1.3.1, it will suffice to show that the simplicial set  $N(\mathcal{J})$  is weakly contractible (for every  $m \leq n$ ).

Let  $X$  denote the simplicial subset of  $\Delta^m \times \Delta^n$  spanned by those nondegenerate simplices whose projection to  $\Delta^n$  is also nondegenerate. Then  $N(\mathcal{J})$  can be identified with the barycentric subdivision of  $X$ . Consequently, it will suffice to show that the topological space  $|X|$  is contractible. For this, we will show that the fibers of the map  $\phi : |X| \rightarrow |\Delta^m|$  are contractible.

We will identify the topological  $m$ -simplex  $|\Delta^m|$  with the set of all sequences of real numbers  $0 \leq x_1 \leq \dots \leq x_m \leq 1$ . Similarly, we may identify points of  $|\Delta^n|$  with sequences  $0 \leq y_1 \leq \dots \leq y_n \leq 1$ . A pair of such sequences determines a point of  $X$  if and only if each  $x_i$  belongs to the set  $\{0, y_1, \dots, y_n, 1\}$ . Consequently, the fiber of  $\phi$  over the point  $(0 \leq x_1 \leq \dots \leq x_m \leq 1)$  can be identified with the set

$$Y = \{0 \leq y_1 \leq \dots \leq y_n \leq 1 : \{x_1, \dots, x_m\} \subseteq \{0, y_1, \dots, y_n, 1\}\} \subseteq |\Delta^n|,$$

which is contractible (Lemma 12.4).  $\square$

**Corollary 12.6.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category, and let  $F : N(\Delta_{\leq n})^{op} \rightarrow \mathcal{C}$  be a functor such that  $F([m]) \simeq 0$  for all  $m < n$ . Then there is a canonical isomorphism  $\varinjlim(F) \simeq X[n]$  in the homotopy category  $h\mathcal{C}$ , where  $X = F([n])$ .*

*Proof.* Let  $\mathcal{J}$  be as in Lemma 12.5, let  $G''$  denote the composition  $N(\mathcal{J})^{op} \rightarrow N(\Delta_{\leq n})^{op} \xrightarrow{F} \mathcal{C}$ , and let  $G$  denote the constant map  $N(\mathcal{J})^{op} \rightarrow \mathcal{C}$  taking the value  $X$ . Let  $\mathcal{J}_0$  denote the full subcategory of  $\mathcal{J}$  obtained by deleting the initial object. There is a canonical map  $\alpha : G \rightarrow G''$ , and  $G' = \ker(\alpha)$  is a left Kan extension of  $G'|N(\mathcal{J}_0)^{op}$ . We obtain a distinguished triangle

$$\varinjlim(G') \rightarrow \varinjlim(G) \rightarrow \varinjlim(G'') \rightarrow \varinjlim(G')[1]$$

in the homotopy category  $h\mathcal{C}$ . Lemma 12.5 yields an equivalence  $\varinjlim(F) \simeq \varinjlim(G'')$ , and Lemma T.4.3.2.7 implies the existence of an equivalence  $\varinjlim(G') \simeq \varinjlim(G'|N(\mathcal{J}_0)^{op})$ .

We now observe that the simplicial set  $N(\mathcal{J})^{op}$  can be identified with the barycentric subdivision of the standard  $n$ -simplex  $\Delta^n$ , and that  $N(\mathcal{J}_0)^{op}$  can be identified with the barycentric subdivision of its boundary  $\partial\Delta^n$ . It follows (see §T.4.4.4) that we may identify the map  $\varinjlim(G') \rightarrow \varinjlim(G)$  with the map  $\beta : X \otimes (\partial\Delta^n) \rightarrow X \otimes \Delta^n$ . The cokernel of  $\beta$  is canonically isomorphic (in  $h\mathcal{C}$ ) to the  $n$ -fold suspension  $X[n]$  of  $X$ .  $\square$

**Lemma 12.7.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category, let  $n \geq 0$ , and let  $F : N(\Delta_{+, \leq n})^{op} \rightarrow \mathcal{C}$  be a functor (here  $\Delta_{+, \leq n}$  denotes the full subcategory of  $\Delta_+$  spanned by the objects  $\{[k]\}_{-1 \leq k \leq n}$ ). The following conditions are equivalent:*

- (i) *The functor  $F$  is a left Kan extension of  $F|N(\Delta_{\leq n})^{op}$ .*
- (ii) *The functor  $F$  is a right Kan extension of  $F|N(\Delta_{+, \leq n-1})^{op}$ .*

*Proof.* Condition (ii) is equivalent to the assertion that the composition

$$F' : N(\Delta_{+, \leq n-1})_{[n]}^{op} \rightarrow N(\Delta_{+, \leq n})^{op} \xrightarrow{F} \mathcal{C}$$

is a limit diagram. Since the source of  $F'$  is isomorphic to  $(\Delta^1)^{n+1}$ , Proposition 12.3 asserts that  $F'$  is a limit diagram if and only if  $F'$  is a colimit diagram. In view of Lemma 12.5,  $F'$  is a colimit diagram if and only if  $F$  is a colimit diagram, which is equivalent to (i).  $\square$

**Theorem 12.8** ( $\infty$ -Categorical Dold-Kan Correspondence). *Let  $\mathcal{C}$  be a stable  $\infty$ -category. Then the  $\infty$ -categories  $\text{Fun}(N(\mathbf{Z}_{\geq 0}), \mathcal{C})$  and  $\text{Fun}(N(\Delta)^{op}, \mathcal{C})$  are (canonically) equivalent to one another.*

*Proof.* Our first step is to describe the desired equivalence in more precise terms. Let  $\mathcal{J}_+$  denote the full subcategory of  $\mathbf{N}(\mathbf{Z}_{\geq 0}) \times \mathbf{N}(\mathbf{\Delta}_+)^{op}$  spanned by those pairs  $(n, [m])$ , where  $m \leq n$ , and let  $\mathcal{J}$  be the full subcategory of  $\mathcal{J}_+$  spanned by those pairs  $(n, [m])$  where  $0 \leq m \leq n$ . We observe that there is a natural projection  $p : \mathcal{J} \rightarrow \mathbf{N}(\mathbf{\Delta})^{op}$ , and a natural embedding  $i : \mathbf{N}(\mathbf{Z}_{\geq 0}) \rightarrow \mathcal{J}_+$ , which carries  $n \geq 0$  to the object  $(n, [-1])$ .

Let  $\text{Fun}^0(\mathcal{J}, \mathcal{C})$  denote the full subcategory of  $\text{Fun}(\mathcal{J}, \mathcal{C})$  spanned by those functors  $F : \mathcal{J} \rightarrow \mathcal{C}$  such that, for every  $s \leq m \leq n$ , the image under  $F$  of the natural map  $(m, [s]) \rightarrow (n, [s])$  is an equivalence in  $\mathcal{C}$ . Let  $\text{Fun}^0(\mathcal{J}_+, \mathcal{C})$  denote the full subcategory of  $\text{Fun}(\mathcal{J}_+, \mathcal{C})$  spanned by functors  $F_+ : \mathcal{J}_+ \rightarrow \mathcal{C}$  such that  $F = F_+|_{\mathcal{J}}$  belongs to  $\text{Fun}^0(\mathcal{J}, \mathcal{C})$ , and  $F_+$  is a left Kan extension of  $F$ . Composition with  $p$ , composition with  $i$ , and restriction from  $\mathcal{J}_+$  to  $\mathcal{J}$  yields a diagram of  $\infty$ -categories

$$\text{Fun}(\mathbf{N}(\mathbf{\Delta})^{op}, \mathcal{C}) \xrightarrow{G} \text{Fun}^0(\mathcal{J}, \mathcal{C}) \xleftarrow{G'} \text{Fun}^0(\mathcal{J}_+, \mathcal{C}) \xrightarrow{G''} \text{Fun}(\mathbf{N}(\mathbf{Z}_{\geq 0}), \mathcal{C}).$$

We will prove that  $G$ ,  $G'$ , and  $G''$  are equivalences of  $\infty$ -categories.

To show that  $G$  is an equivalence of  $\infty$ -categories, we let  $\mathcal{J}^{\leq k}$  denote the full subcategory of  $\mathcal{J}$  spanned by pairs  $(n, [m])$  where  $m \leq n \leq k$ , and let  $\mathcal{J}^k$  denote the full subcategory of  $\mathcal{J}$  spanned by those pairs  $(n, [m])$  where  $m \leq n = k$ . Then the projection  $p$  restricts to an equivalence  $\mathcal{J}^k \rightarrow \mathbf{N}(\mathbf{\Delta}_{\leq n})^{op}$ . Let  $\text{Fun}^0(\mathcal{J}^{\leq k}, \mathcal{C})$  denote the full subcategory of  $\text{Fun}(\mathcal{J}^{\leq k}, \mathcal{C})$  spanned by those functors  $F : \mathcal{J}^{\leq k} \rightarrow \mathcal{C}$  such that, for every  $s \leq m \leq n \leq k$ , the image under  $F$  of the natural map  $(m, [s]) \rightarrow (n, [s])$  is an equivalence in  $\mathcal{C}$ . We observe that this is equivalent to the condition that  $F$  be a right Kan extension of  $F|_{\mathcal{J}^k}$ . Using Proposition T.4.3.2.15, we deduce that the restriction map  $r : \text{Fun}^0(\mathcal{J}^{\leq k}, \mathcal{C}) \rightarrow \text{Fun}^0(\mathcal{J}^k, \mathcal{C})$  is an equivalence of  $\infty$ -categories. Composition with  $p$  induces a functor  $G_k : \text{Fun}(\mathbf{N}(\mathbf{\Delta}_{\leq k})^{op}, \mathcal{C}) \rightarrow \text{Fun}^0(\mathcal{J}^{\leq k}, \mathcal{C})$  which is a section of  $r$ . It follows that  $G_k$  is an equivalence of  $\infty$ -categories. We can identify  $G$  with the homotopy inverse limit of the functors  $\varprojlim(G_k)$ , so that  $G$  is also an equivalence of  $\infty$ -categories.

The fact that  $G'$  is an equivalence of  $\infty$ -categories follows immediately from Proposition T.4.3.2.15, since for each  $n \geq 0$  the simplicial set  $\mathcal{J}_{/(n, [-1])}$  is finite and  $\mathcal{C}$  admits finite colimits.

We now show that  $G''$  is an equivalence of  $\infty$ -categories. Let  $\mathcal{J}_+^{\leq k}$  denote the full subcategory of  $\mathcal{J}_+$  spanned by pairs  $(n, [m])$  where either  $m \leq n \leq k$  or  $m = -1$ . We let  $\mathcal{D}(k)$  denote the full subcategory of  $\text{Fun}(\mathcal{J}_+^{\leq k}, \mathcal{C})$  spanned by those functors  $F : \mathcal{J}_+^{\leq k} \rightarrow \mathcal{C}$  with the following pair of properties:

- (i) For every  $s \leq m \leq n \leq k$ , the image under  $F$  of the natural map  $(m, [s]) \rightarrow (n, [s])$  is an equivalence in  $\mathcal{C}$ .
- (ii) For every  $n \leq k$ ,  $F$  is a left Kan extension of  $F|_{\mathcal{J}^{\leq k}}$  at  $(n, [-1])$ .

Then  $\text{Fun}^0(\mathcal{J}_+, \mathcal{C})$  is the inverse limit of the tower of restriction maps

$$\dots \rightarrow \mathcal{D}(1) \rightarrow \mathcal{D}(0) \rightarrow \mathcal{D}(-1) = \text{Fun}(\mathbf{N}(\mathbf{Z}_{\geq 0}), \mathcal{C}).$$

To complete the proof, we will show that for each  $k \geq 0$ , the restriction map  $\mathcal{D}(k) \rightarrow \mathcal{D}(k-1)$  is a trivial Kan fibration.

Let  $\mathcal{J}_0^{\leq k}$  be the full subcategory of  $\mathcal{J}_+^{\leq k}$  obtained by removing the object  $(k, [k])$ , and let  $\mathcal{D}'(k)$  be the full subcategory of  $\text{Fun}(\mathcal{J}_0^{\leq k}, \mathcal{C})$  spanned by those functors  $F$  which satisfy condition (i) and satisfy (ii) for  $n < k$ . We have restriction maps

$$\mathcal{D}(k) \xrightarrow{\theta} \mathcal{D}'(k) \xrightarrow{\theta'} \mathcal{D}(k-1).$$

We observe that a functor  $F : \mathcal{J}_0^{\leq k}$  belongs to  $\mathcal{D}'(k)$  if and only if  $F|_{\mathcal{J}_+^{\leq k-1}}$  belongs to  $\mathcal{D}(k-1)$  and  $F$  is a left Kan extension of  $F|_{\mathcal{J}_+^{\leq k-1}}$ . Using Proposition T.4.3.2.15, we conclude that  $\theta'$  is a trivial Kan fibration.

We will prove that  $\theta$  is a trivial Kan fibration by a similar argument. According to Proposition T.4.3.2.15, it will suffice to show that a functor  $F : \mathcal{J}_+^{\leq k} \rightarrow \mathcal{C}$  belongs to  $\mathcal{D}(k)$  if and only if  $F|_{\mathcal{J}_0^{\leq k}}$  belongs to  $\mathcal{D}'(k)$  and  $F$  is a right Kan extension of  $F|_{\mathcal{J}_0^{\leq k}}$ . This follows immediately from Lemma 12.7 and the observation that the inclusion  $\mathcal{J}^k \subseteq \mathcal{J}^{\leq k}$  is cofinal.  $\square$

**Remark 12.9.** Let  $\mathcal{C}$  be a stable  $\infty$ -category. We may informally describe the equivalence of Theorem 12.8 as follows. To a simplicial object  $C_\bullet$  of  $\mathcal{C}$ , we assign the filtered object

$$D(0) \rightarrow D(1) \rightarrow D(2) \rightarrow \dots$$

where  $D(k)$  is the colimit of the  $k$ -skeleton of  $C_\bullet$ . In particular, we observe that colimits  $\varinjlim D(j)$  can be identified with geometric realizations of the simplicial object  $C_\bullet$ .

**Remark 12.10.** Let  $\mathcal{C}$  be a stable  $\infty$ -category, and let  $X$  be a simplicial object of  $\mathcal{C}$ . Using the Dold-Kan correspondence, we can associate to  $X$  a chain complex

$$\dots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow 0$$

in the triangulated category  $\mathrm{h}\mathcal{C}$ . More precisely, for each  $n \geq 0$ , let  $L_n \in \mathcal{C}$  denote the  $n$ th *latching object* of  $X$  (see §T.A.2.9), so that  $X$  determines a canonical map  $\alpha : L_n \rightarrow X_n$ . Then  $C_n \simeq \mathrm{coker}(\alpha)$ , where the cokernel can be formed either in the  $\infty$ -category  $\mathcal{C}$  or in its homotopy category  $\mathrm{h}\mathcal{C}$  (since  $L_n$  is actually a direct summand of  $X_n$ ).

Using Theorem 12.8, we can also associate to  $X$  a filtered object

$$D(0) \rightarrow D(1) \rightarrow D(2) \rightarrow \dots$$

of  $\mathcal{C}$ . Using Lemma 11.3 and Remark 11.2, we can associate to this filtered object another chain complex

$$\dots \rightarrow C'_1 \rightarrow C'_0 \rightarrow 0$$

with values in  $\mathrm{h}\mathcal{C}$ . For each  $n \geq 0$ , let  $X(n)$  denote the restriction of  $X$  to  $\mathbf{N}(\Delta_{\leq n}^{op})$ , and let  $X'(n)$  be a left Kan extension of  $X(n-1)$  to  $\mathbf{N}(\Delta_{\leq n}^{op})$ . Then we have a canonical map  $\beta = X'(n) \rightarrow X(n)$ , which induces an equivalence  $X'(n)_m \rightarrow X(n)_m$  for  $m < n$ , while  $X'(n)_n$  can be identified with the latching object  $L_n$ . Let  $X''(n) = \mathrm{coker}(\beta)$ . Then  $X''(n)_m = 0$  for  $m < n$ , while  $X''(n)_n \simeq C_n$ . Corollary 12.6 determines a canonical isomorphism  $\varinjlim X''(n) \simeq C_n[n]$  in the homotopy category  $\mathrm{h}\mathcal{C}$ . The map  $D(n-1) \rightarrow D(n)$  can be identified with the composition

$$D(n-1) \simeq \varinjlim X(n-1) \simeq \varinjlim X'(n) \rightarrow \varinjlim X(n) \simeq D(n).$$

It follows that  $C'_n \simeq \mathrm{coker}(D(n-1) \rightarrow D(n))[-n] \simeq X''(n)_n[-n]$  is canonically isomorphic to  $C_n$ . It is not difficult to show that these isomorphisms are compatible with the differentials, so that we obtain an isomorphism of chain complexes  $C_\bullet \simeq C'_\bullet$  with values in the triangulated category  $\mathrm{h}\mathcal{C}$ .

**Remark 12.11.** Let  $\mathcal{C}$  be a stable  $\infty$ -category, let  $X_\bullet$  be a simplicial object of  $\mathcal{C}$ , let

$$D(0) \rightarrow D(1) \rightarrow \dots$$

be the associated filtered object. Using the classical Dold-Kan correspondence and Remark 12.10, we conclude that each  $X_n$  is equivalent to a finite coproduct of objects of the form  $\mathrm{coker}(D(m-1) \rightarrow D(m))[-m]$ , where  $0 \leq m \leq n$  (here  $D(-1) \simeq 0$  by convention).

**Remark 12.12.** Let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a t-structure, whose heart is equivalent to (the nerve of) an abelian category  $\mathcal{A}$ . Let  $X_\bullet$  be a simplicial object of  $\mathcal{C}$ , and let

$$D(0) \rightarrow D(1) \rightarrow D(2) \rightarrow \dots$$

be the associated filtered object (Theorem 12.8). Using Definition 11.8 (and Lemma 11.3), we can associate to this filtered object a spectral sequence  $\{E_r^{p,q}, d_r\}_{r \geq 1}$  in the abelian category  $\mathcal{A}$ . In view of Remarks 11.7 and 12.10, for each  $q \in \mathbf{Z}$  we can identify the complex  $(E_1^{p,q}, d_1)$  with the normalized chain complex associated to the simplicial object  $\pi_q X_\bullet$  of  $\mathcal{A}$ . Under the hypotheses of Proposition 11.13, this spectral sequence converges to a filtration on the homotopy groups  $\pi_{p+q} \varinjlim (D(n)) \simeq \pi_{p+q} |X_\bullet|$ .

It is possible to consider a slight variation on the spectral sequence described above. Namely, one can construct a new spectral sequence  $\{\bar{E}_r^{p,q}, d_r\}_{r \geq 1}$  which is isomorphic to  $\{E_r^{p,q}, d_r\}_{r \geq 1}$  from the  $E_2$ -page onward, but with  $\bar{E}_1^{p,q}$  given by the *unnormalized* chain complex of  $\pi_q X_\bullet$ . We can then write simply  $\bar{E}_1^{p,q} \simeq \pi_q X_p$ .

## 13 Homological Algebra

Let  $\mathcal{A}$  be an abelian category. In classical homological algebra, it is customary to associate to  $\mathcal{A}$  a certain triangulated category, called the *derived category* of  $\mathcal{A}$ , the objects of which are chain complexes with values in  $\mathcal{A}$ . In this section, we will review the theory of derived categories from the perspective of higher category theory. To simplify the discussion, we primarily consider only abelian categories  $\mathcal{A}$  which have enough projective objects (the dual case of abelian categories with enough injective objects can be understood by passing to the opposite category).

We begin by considering an arbitrary additive category  $\mathcal{A}$ . Let  $\text{Ch}(\mathcal{A})$  denote the category whose objects are chain complexes

$$\dots \rightarrow A_1 \rightarrow A_0 \rightarrow A_{-1} \rightarrow \dots$$

with values in  $\mathcal{A}$ . The category  $\text{Ch}(\mathcal{A})$  is naturally enriched over simplicial sets. For  $A_\bullet, B_\bullet \in \text{Ch}(\mathcal{A})$ , the simplicial set  $\text{Map}_{\text{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)$  is characterized by the property that for every finite simplicial set  $K$  there is a natural bijection

$$\text{Hom}_{\text{Set}_\Delta}(K, \text{Map}_{\text{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)) \simeq \text{Hom}_{\text{Ch}(\mathcal{A})}(A_\bullet \otimes C_\bullet(K), B_\bullet).$$

Here  $C_\bullet(K)$  denotes the normalized chain complex for computing the homology of  $K$ , so that  $C_n(K)$  is a free abelian group whose generators are in bijection with the nondegenerate  $n$ -simplices of  $K$ . Unwinding the definitions, we see that the vertices of  $\text{Map}_{\text{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)$  are just the maps of chain complexes from  $A_\bullet$  to  $B_\bullet$ . An edge  $e$  of  $\text{Map}_{\text{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)$  is determined by three pieces of data:

- (i) A vertex  $d_0(e)$ , corresponding to a chain map  $f : A_\bullet \rightarrow B_\bullet$ .
- (ii) A vertex  $d_1(e)$ , corresponding to a chain map  $g : A_\bullet \rightarrow B_\bullet$ .
- (iii) A map  $h : A_\bullet \rightarrow B_{\bullet+1}$ , which determines a chain homotopy from  $f$  to  $g$ .

**Remark 13.1.** Let  $Ab$  be the category of abelian groups, and let  $\text{Ch}_{\geq 0}(Ab)$  denote the full subcategory of  $\text{Ch}(Ab)$  spanned by those complexes  $A_\bullet$  such that  $A_n \simeq 0$  for all  $n < 0$ . The classical Dold-Kan correspondence (see [75]) asserts that  $\text{Ch}_{\geq 0}(Ab)$  is equivalent to the category of *simplicial* abelian groups. In particular, there is a forgetful functor  $\theta : \text{Ch}_{\geq 0}(Ab) \rightarrow \text{Set}_\Delta$ .

Given a pair of complexes  $A_\bullet, B_\bullet \in \text{Ch}(\mathcal{A})$ , the mapping space  $\text{Map}_{\text{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)$  can be defined as follows:

- (1) First, we extract the mapping complex

$$[A_\bullet, B_\bullet] \in \text{Ch}(Ab),$$

where  $[A_\bullet, B_\bullet]_n = \prod \text{Hom}_{\mathcal{A}}(A_m, B_{n+m})$ .

- (2) The inclusion  $\text{Ch}_{\geq 0}(Ab) \subseteq \text{Ch}(Ab)$  has a right adjoint, which associates to an arbitrary chain complex  $M_\bullet$  the truncated complex

$$\dots \rightarrow M_1 \rightarrow \ker(M_0 \rightarrow M_{-1}) \rightarrow 0 \rightarrow \dots$$

Applying this functor to  $[A_\bullet, B_\bullet]$ , we obtain a new complex  $[A_\bullet, B_\bullet]_{\geq 0}$ , whose degree zero term coincides with the set of chain maps from  $A_\bullet$  to  $B_\bullet$ .

- (3) Applying the Dold-Kan correspondence  $\theta$ , we can convert the chain complex  $[A_\bullet, B_\bullet]_{\geq 0}$  into a simplicial set  $\text{Map}_{\text{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)$ .

Because every simplicial abelian group is a Kan complex, the simplicial category  $\text{Ch}(\mathcal{A})$  is automatically *fibrant*.

**Remark 13.2.** Let  $\mathcal{A}$  be an additive category, and let  $A_\bullet, B_\bullet \in \text{Ch}(\mathcal{A})$ . The homotopy group

$$\pi_n \text{Map}_{\text{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)$$

can be identified with the group of chain-homotopy classes of maps from  $A_\bullet$  to  $B_{\bullet+n}$ .

**Example 13.3.** Let  $\mathcal{A}$  be an abelian category, and let  $A_\bullet, B_\bullet \in \text{Ch}(\mathcal{A})$ . Suppose that  $A_n \simeq 0$  for  $n < 0$ , and that  $B_n \simeq 0$  for  $n > 0$ . Then the simplicial set  $\text{Map}_{\text{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)$  is constant, with value  $\text{Hom}_{\mathcal{A}}(H_0(A_\bullet), H_0(B_\bullet))$ .

**Lemma 13.4.** *Let  $\mathcal{A}$  be an additive category. Then:*

(1) *Let*

$$\begin{array}{ccc} A_\bullet & \xrightarrow{f} & B_\bullet \\ \downarrow & & \downarrow \\ A'_\bullet & \longrightarrow & B'_\bullet \end{array}$$

*be a pushout diagram in the (ordinary) category  $\text{Ch}(\mathcal{A})$ , and suppose that  $f$  is degreewise split (so that each  $B_n \simeq A_n \oplus C_n$ , for some  $C_n \in \mathcal{A}$ ). Then the above diagram determines a homotopy pushout square in the  $\infty$ -category  $\text{N}(\text{Ch}(\mathcal{A}))$ .*

(2) *The  $\infty$ -category  $\text{N}(\text{Ch}(\mathcal{A}))$  is stable.*

*Proof.* To prove (1), it will suffice (Theorem T.4.2.4.1) to show that for every  $D_\bullet \in \text{Ch}(\mathcal{A})$ , the associated diagram of simplicial sets

$$\begin{array}{ccc} \text{Map}_{\text{Ch}(\mathcal{A})}(B'_\bullet, D_\bullet) & \longrightarrow & \text{Map}_{\text{Ch}(\mathcal{A})}(A'_\bullet, D_\bullet) \\ \downarrow & & \downarrow \\ \text{Map}_{\text{Ch}(\mathcal{A})}(B_\bullet, D_\bullet) & \xrightarrow{f'} & \text{Map}_{\text{Ch}(\mathcal{A})}(A_\bullet, D_\bullet) \end{array}$$

is homotopy Cartesian. The above diagram is obviously a pullback, it will suffice to prove that  $f'$  is a Kan fibration. This follows from the fact that  $f'$  is the map of simplicial sets associated (under the Dold-Kan correspondence) to a map between complexes of abelian groups which is surjective in positive (homological) degrees.

It follows from (1) that the  $\infty$ -category  $\text{N}(\text{Ch}(\mathcal{A}))$  admits pushouts: it suffices to observe that any morphism  $f : A_\bullet \rightarrow B_\bullet$  is chain homotopy-equivalent to a morphism which is degreewise split (replace  $B_\bullet$  by the mapping cylinder of  $f$ ). It is obvious that  $\text{N}(\text{Ch}(\mathcal{A}))$  has a zero object (since  $\text{Ch}(\mathcal{A})$  has a zero object). Moreover, we can use (1) to describe the suspension functor on  $\text{Ch}(\mathcal{A})$ : for each  $A_\bullet \in \text{Ch}(\mathcal{A})$ , let  $C(A_\bullet)$  denote the cone of  $A_\bullet$ , so that  $C(A_\bullet) \simeq 0$  and there is a pushout diagram

$$\begin{array}{ccc} A_\bullet & \longrightarrow & C(A_\bullet) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A_{\bullet-1} \end{array}$$

It follows that the suspension functor  $\Sigma$  can be identified with the shift functor

$$A_\bullet \mapsto A_{\bullet-1}.$$

In particular, we conclude that  $\Sigma$  is an equivalence of  $\infty$ -categories, so that  $\text{Ch}(\mathcal{A})$  is stable (Proposition 8.28).  $\square$

**Remark 13.5.** Let  $\mathcal{A}$  be an additive category, and let  $\text{Ch}'(\mathcal{A})$  be a full subcategory of  $\text{Ch}(\mathcal{A})$ . Suppose that  $\text{Ch}'(\mathcal{A})$  is stable under translations and the formation of mapping cones. Then the proof of Lemma 13.4 shows that  $\text{N}(\text{Ch}'(\mathcal{A}))$  is a stable subcategory of  $\text{N}(\text{Ch}(\mathcal{A}))$ . In particular, if  $\text{Ch}^-(\mathcal{A})$  denotes the full subcategory of  $\text{Ch}(\mathcal{A})$  spanned by those complexes  $A_\bullet$  such that  $A_n \simeq 0$  for  $n \ll 0$ , then  $\text{N}(\text{Ch}^-(\mathcal{A}))$  is a stable subcategory of  $\text{N}(\text{Ch}(\mathcal{A}))$ .

**Definition 13.6.** Let  $\mathcal{A}$  be an abelian category with enough projective objects. We let  $\mathcal{D}^-(\mathcal{A})$  denote the nerve of the simplicial category  $\text{Ch}^-(\mathcal{A}_0)$ , where  $\mathcal{A}_0 \subseteq \mathcal{A}$  is the full subcategory spanned by the projective objects of  $\mathcal{A}$ . We will refer to  $\mathcal{D}^-(\mathcal{A})$  as the *derived  $\infty$ -category of  $\mathcal{A}$* .

**Remark 13.7.** The homotopy category  $\text{h}\mathcal{D}^-(\mathcal{A})$  can be described as follows: objects are given by (bounded above) chain complexes of projective objects of  $\mathcal{A}$ , and morphisms are given by homotopy classes of chain maps. Consequently,  $\text{h}\mathcal{D}^-(\mathcal{A})$  can be identified with the derived category of  $\mathcal{A}$  studied in classical homological algebra (with appropriate boundedness conditions imposed).

**Lemma 13.8.** *Let  $\mathcal{A}$  be an abelian category, and let  $P_\bullet \in \text{Ch}(\mathcal{A})$  be a complex of projective objects of  $\mathcal{A}$  such that  $P_n \simeq 0$  for  $n \ll 0$ . Let  $Q_\bullet \rightarrow Q'_\bullet$  be a quasi-isomorphism in  $\text{Ch}(\mathcal{A})$ . Then the induced map*

$$\text{Map}_{\text{Ch}(\mathcal{A})}(P_\bullet, Q_\bullet) \rightarrow \text{Map}_{\text{Ch}(\mathcal{A})}(P_\bullet, Q'_\bullet)$$

*is a homotopy equivalence.*

*Proof.* We observe that  $P_\bullet$  is a homotopy colimit of its naive truncations

$$\dots \rightarrow 0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots$$

It therefore suffices to prove the result for each of these truncations, so we may assume that  $P_\bullet$  is concentrated in finitely many degrees. Working by induction, we can reduce to the case where  $P_\bullet$  is concentrated in a single degree. Shifting, we can reduce to the case where  $P_\bullet$  consists of a single projective object  $P$  concentrated in degree zero. Since  $P$  is projective, we have isomorphisms

$$\text{Ext}_{\text{N}(\text{Ch}(\mathcal{A}))}^i(P_\bullet, Q_\bullet) \simeq \text{Hom}_{\mathcal{A}}(P, H_{-i}(Q_\bullet)) \simeq \text{Hom}_{\mathcal{A}}(P, H_{-i}(Q'_\bullet)) \simeq \text{Ext}_{\text{N}(\text{Ch}(\mathcal{A}))}^i(P_\bullet, Q'_\bullet).$$

□

**Lemma 13.9.** *Let  $\mathcal{A}$  be an abelian category. Suppose that  $P_\bullet, Q_\bullet \in \text{Ch}(\mathcal{A})$  have the following properties:*

- (1) *Each  $P_n$  is projective, and  $P_n \simeq 0$  for  $n < 0$ .*
- (2) *The homologies  $H_n(Q_\bullet)$  vanish for  $n > 0$ .*

*Then the space  $\text{Map}_{\text{Ch}(\mathcal{A})}(P_\bullet, Q_\bullet)$  is discrete, and we have a canonical isomorphism of abelian groups*

$$\text{Ext}^0(P_\bullet, Q_\bullet) \simeq \text{Hom}_{\mathcal{A}}(H_0(P_\bullet), H_0(Q_\bullet)).$$

*Proof.* Let  $Q'_\bullet$  be the complex

$$\dots \rightarrow 0 \rightarrow \text{coker}(Q_1 \rightarrow Q_0) \rightarrow Q_{-1} \rightarrow \dots$$

Condition (2) implies that the canonical map  $Q_\bullet \rightarrow Q'_\bullet$  is a quasi-isomorphism. In view of (1) and Lemma 13.8, it will suffice to prove the result after replacing  $Q_\bullet$  by  $Q'_\bullet$ . The result now follows from Example 13.3. □

**Proposition 13.10.** *Let  $\mathcal{A}$  be an abelian category with enough projective objects. Then:*

- (1) *The  $\infty$ -category  $\mathcal{D}^-(\mathcal{A})$  is stable.*

- (2) Let  $\mathcal{D}_{\geq 0}^-(\mathcal{A})$  be the full subcategory of  $\mathcal{D}^-(\mathcal{A})$  spanned by those complexes  $A_\bullet$  such that the homology objects  $H_n(A_\bullet) \in \mathcal{A}$  vanish for  $n < 0$ , and let  $\mathcal{D}_{\leq 0}^-(\mathcal{A})$  be defined similarly. Then  $(\mathcal{D}_{\geq 0}^-(\mathcal{A}), \mathcal{D}_{\leq 0}^-(\mathcal{A}))$  determines a  $t$ -structure on  $\mathcal{D}^-(\mathcal{A})$ .
- (3) The heart of  $\mathcal{D}^-(\mathcal{A})$  is equivalent to (the nerve of) the abelian category  $\mathcal{A}$ .

*Proof.* Assertion (1) follows from Remark 13.5.

To prove (2), we first make the following observation:

- (\*) For any object  $A_\bullet \in \text{Ch}(\mathcal{A})$ , there exists a map  $f : P_\bullet \rightarrow A_\bullet$  where each  $P_n$  is projective,  $P_n \simeq 0$  for  $n < 0$ , and the induced map  $H_k(P_\bullet) \rightarrow H_k(A_\bullet)$  is an isomorphism for  $k \geq 0$ .

This is proven by a standard argument in homological algebra, using the assumption that  $\mathcal{A}$  has enough projectives. We also note that if  $A_\bullet \in \mathcal{D}^-(\mathcal{A})$  and the homologies  $H_n(A_\bullet)$  vanish for  $n < 0$ , then  $f$  is a quasi-isomorphism between projective complexes and therefore a chain homotopy equivalence.

It is obvious that  $\mathcal{D}_{\leq 0}^-(\mathcal{A})[-1] \subseteq \mathcal{D}_{\leq 0}^-(\mathcal{A})$  and  $\mathcal{D}_{\geq 0}^-(\mathcal{A})[1] \subseteq \mathcal{D}_{\geq 0}^-(\mathcal{A})$ . Suppose now that  $A_\bullet \in \mathcal{D}_{\geq 0}^-(\mathcal{A})$  and  $B_\bullet \in \mathcal{D}_{\leq -1}^-(\mathcal{A})$ ; we wish to show that  $\text{Ext}_{\mathcal{D}^-(\mathcal{A})}^0(A_\bullet, B_\bullet) \simeq 0$ . Using (\*), we may reduce to the case where  $A_n \simeq 0$  for  $n < 0$ . The desired result now follows immediately from Lemma 13.9. Finally, choose an arbitrary object  $A_\bullet \in \mathcal{D}^-(\mathcal{A})$ , and let  $f : P_\bullet \rightarrow A_\bullet$  be as in (\*). It is easy to see that  $\text{coker}(f) \in \mathcal{D}_{\leq -1}^-(\mathcal{A})$ . This completes the proof of (2).

To prove (3), we begin by observing that the functor  $A_\bullet \mapsto H_0(A_\bullet)$  determines a functor  $\theta : \text{N}(\text{Ch}(\mathcal{A})) \rightarrow \text{N}(\mathcal{A})$ . Let  $\mathcal{C} \subseteq \text{N}(\text{Ch}(\mathcal{A}))$  be the full subcategory spanned by complexes  $P_\bullet$  such that each  $P_n$  is projective,  $P_n \simeq 0$  for  $n < 0$ , and  $H_n(P_\bullet) \simeq 0$  for  $n \neq 0$ . Assertion (\*) implies that the inclusion  $\mathcal{C} \subseteq \mathcal{D}^-(\mathcal{A})^\heartsuit$  is an equivalence of  $\infty$ -categories. Lemma 13.9 implies that  $\theta|_{\mathcal{C}}$  is fully faithful. Finally, we can apply (\*) in the case where  $A_\bullet$  is concentrated in degree zero to deduce that  $\theta|_{\mathcal{C}}$  is essentially surjective. This proves (3).  $\square$

**Remark 13.11.** Let  $\mathcal{A}$  be an abelian category with enough projective objects. Then  $\mathcal{D}^-(\mathcal{A})$  is a colocalization of  $\text{N}(\text{Ch}^-(\mathcal{A}))$ . To prove this, it will suffice to show that for every  $A_\bullet \in \text{Ch}^-(\mathcal{A})$ , there exists a map of chain complexes  $f : P_\bullet \rightarrow A_\bullet$  where  $P_\bullet \in \mathcal{D}^-(\mathcal{A})$ , and such that  $f$  induces a homotopy equivalence

$$\text{Map}_{\text{Ch}(\mathcal{A})}(Q_\bullet, P_\bullet) \rightarrow \text{Map}_{\text{Ch}(\mathcal{A})}(Q_\bullet, A_\bullet)$$

for every  $Q_\bullet \in \mathcal{D}^-(\mathcal{A})$  (Proposition T.5.2.7.8). According Lemma 13.8, it will suffice to choose  $f$  to be a quasi-isomorphism; the existence now follows from (\*) in the proof of Proposition 13.10.

Let  $L : \text{N}(\text{Ch}^-(\mathcal{A})) \rightarrow \mathcal{D}^-(\mathcal{A})$  be a right adjoint to the inclusion. Roughly speaking, the functor  $L$  associates to each complex  $A_\bullet$  a projective resolution  $P_\bullet$  as above. We observe that, if  $f : A_\bullet \rightarrow B_\bullet$  is a map of complexes, then  $Lf$  is a chain homotopy equivalence if and only if  $f$  is a quasi-isomorphism. Consequently, we may regard  $\mathcal{D}^-(\mathcal{A})$  as the  $\infty$ -category obtained from  $\text{N}(\text{Ch}^-(\mathcal{A}))$  by inverting all quasi-isomorphism.

## 14 The Universal Property of $\mathcal{D}^-(\mathcal{A})$

In this section, we will apply the results of §T.5.5.8 and §T.5.5.9 to characterize the derived  $\infty$ -category  $\mathcal{D}^-(\mathcal{A})$  by a universal mapping property. Here  $\mathcal{A}$  denotes an abelian category with enough projective objects; to simplify the discussion, we will assume that  $\mathcal{A}$  is small.

Let  $\mathcal{A}_0 \subseteq \mathcal{A}$  be the full subcategory of  $\mathcal{A}$  spanned by the projective objects, and let  $\mathbf{A}$  denote the category of product-preserving functors from  $\mathcal{A}_0^{op}$  to the category of simplicial sets, as in §T.5.5.9. Let  $\mathcal{A}^\vee$  denote the category of product-preserving functors from  $\mathcal{A}_0^{op}$  to sets, so that we can identify  $\mathbf{A}$  with the category of simplicial objects of  $\mathcal{A}^\vee$ . Our first goal is to understand the category  $\mathcal{A}^\vee$ .

**Lemma 14.1.** *Let  $\mathcal{A}$  be an abelian category with enough projective objects, and let  $\mathcal{B}$  be an arbitrary category which admits finite colimits. Let  $\mathcal{C}$  be the category of right exact functors from  $\mathcal{A}$  to  $\mathcal{B}$ , and let  $\mathcal{C}'$  be the category of coproduct-preserving functors from  $\mathcal{A}_0$  to  $\mathcal{B}$ . Then the restriction functor  $\theta : \mathcal{C} \rightarrow \mathcal{C}'$  is an equivalence of categories.*



*Proof.* We will describe an explicit construction of an inverse functor. Let  $f : \mathcal{A}_0 \rightarrow \mathcal{B}$  be a functor which preserves finite coproducts. Let  $A \in \mathcal{A}$  be an arbitrary object. Since  $\mathcal{A}$  has enough projectives, there exists a projective resolution

$$\dots \rightarrow P_1 \xrightarrow{u} P_0 \rightarrow A \rightarrow 0.$$

We now define  $F(A)$  to be the coequalizer of the map

$$f(P_1) \begin{array}{c} \xrightarrow{f(0)} \\ \xrightarrow{f(u)} \end{array} f(P_0).$$

Of course, this definition appears to depend not only on  $A$  but on a choice of projective resolution. However, because any two projective resolutions of  $A$  are chain homotopy equivalent to one another,  $F(A)$  is well-defined up to canonical isomorphism. It is easy to see that  $F : \mathcal{A} \rightarrow \mathcal{B}$  is a right exact functor which extends  $f$ , and that  $F$  is uniquely determined (up to unique isomorphism) by these properties.  $\square$

**Proposition 14.2.** *Let  $\mathcal{A}$  be an abelian category with enough projective objects. Then:*

- (1) *The category  $\mathcal{A}^\vee$  can be identified with the category of Ind-objects of  $\mathcal{A}$ .*
- (2) *The category  $\mathcal{A}^\vee$  is abelian.*
- (3) *The abelian category  $\mathcal{A}^\vee$  has enough projective objects.*

*Proof.* Assertion (1) follows immediately from Lemma 14.1 (taking  $\mathcal{B}$  to be the opposite of the category of sets). Part (2) follows formally from (1) and the assumption that  $\mathcal{A}$  is an abelian category (see, for example, [3]). We may identify  $\mathcal{A}$  with a full subcategory of  $\mathcal{A}^\vee$  via the Yoneda embedding. Moreover, if  $P$  is a projective object of  $\mathcal{A}$ , then  $P$  is also projective when viewed as an object of  $\mathcal{A}^\vee$ . An arbitrary object of  $\mathcal{A}^\vee$  can be written as a filtered colimit  $A = \varinjlim \{A_\alpha\}$ , where each  $A_\alpha \in \mathcal{A}$ . Using the assumption that  $\mathcal{A}$  has enough projective objects, we can choose epimorphisms  $P_\alpha \rightarrow A_\alpha$ , where each  $P_\alpha$  is projective. We then have an epimorphism  $\oplus P_\alpha \rightarrow A$ . Since  $\oplus P_\alpha$  is projective, we conclude that  $\mathcal{A}^\vee$  has enough projectives.  $\square$

**Warning 14.3.** Let  $\mathcal{A}$  be an abelian category with enough projective objects, and let  $\mathbf{A}$  be the category of product-preserving functors  $\mathcal{A}_0^{op} \rightarrow \text{Set}_\Delta$ . The Dold-Kan correspondence determines an equivalence of categories  $\theta : \mathbf{A} \simeq \text{Ch}_{\geq 0}(\mathcal{A}^\vee)$ . However, this is *not* an equivalence of simplicial categories. Let  $K$  be a simplicial set, and let  $\mathbf{Z}K$  denote the free simplicial abelian group generated by  $K$  (so that the group of  $n$ -simplices of  $\mathbf{Z}K$  is the free abelian group generated by the set of  $n$ -simplices of  $K$ , for each  $n \geq 0$ ). Then  $\mathbf{A}$  is tensored over the category of simplicial sets in two different ways:

- (i) Given a simplicial set  $K$  and an object  $A_\bullet \in \mathbf{A}$  viewed as a simplicial object of  $\mathcal{A}^\vee$ , we can form the tensor product  $A_\bullet \otimes K$  given by the formula  $(A_\bullet \otimes K)_n = A_n \otimes (\mathbf{Z}K)_n$ .
- (ii) Given a simplicial set  $K$  and an object  $A_\bullet \in \mathbf{A}$ , we can construct a new object  $A_\bullet \odot K$ , which is characterized by the existence of an isomorphism

$$\theta(A_\bullet \odot K) \simeq \theta(A_\bullet) \otimes \theta'(\mathbf{Z}K)$$

in the category  $\text{Ch}(\mathcal{A}^\vee)$ . Here  $\theta'(\mathbf{Z}K)$  denotes the object of  $\text{Ch}(\mathcal{A}^\vee)$  determined by  $\mathbf{Z}K$ .

However, it is easy to see that both of these simplicial structures on  $\mathbf{A}$  are compatible with the model structure of Proposition T.5.5.9.1. Moreover, the classical *Alexander-Whitney map* determines a natural transformation  $A_\bullet \otimes K \rightarrow A_\bullet \odot K$ , which endows  $\theta^{-1} : \text{Ch}_{\geq 0}(\mathcal{A}^\vee) \rightarrow \mathbf{A}$  with the structure of a simplicial functor.

We observe that every object of  $\mathbf{A}$  is fibrant, and that an object of  $\mathbf{A}$  is cofibrant if and only if it corresponds (under  $\theta$ ) to a complex of projective objects of  $\mathcal{A}^\vee$ . Applying Corollary T.A.3.1.12, we obtain an equivalence of  $\infty$ -categories  $\mathcal{D}_{\geq 0}^-(\mathcal{A}^\vee) \rightarrow \mathbf{N}(\mathbf{A}^\circ)$ . Here  $\mathcal{D}_{\geq 0}^-(\mathcal{A}^\vee)$  denotes the full subcategory of  $\mathcal{D}^-(\mathcal{A}^\vee)$  spanned by those complexes  $P_\bullet$  such that  $P_n \simeq 0$  for  $n < 0$ . Composing with the equivalence of Corollary T.5.5.9.3, we obtain the following result:

**Proposition 14.4.** *Let  $\mathcal{A}$  be an abelian category with enough projective objects. Then there exists an equivalence of  $\infty$ -categories*

$$\psi : \mathcal{D}_{\geq 0}^-(\mathcal{A}^\vee) \rightarrow \mathcal{P}_\Sigma(\mathcal{N}(\mathcal{A}_0))$$

whose composition with the inclusion  $\mathcal{N}(\mathcal{A}_0) \subseteq \mathcal{D}_{\geq 0}^-(\mathcal{A}^\vee)$  is equivalent to the Yoneda embedding  $\mathcal{N}(\mathcal{A}_0) \rightarrow \mathcal{P}_\Sigma(\mathcal{N}(\mathcal{A}_0))$ .

**Remark 14.5.** We can identify  $\mathcal{D}^-(\mathcal{A})$  with a full subcategory of  $\mathcal{D}^-(\mathcal{A}^\vee)$ . Moreover, an object  $P_\bullet \in \mathcal{D}^-(\mathcal{A}^\vee)$  belongs to the essential image of  $\mathcal{D}^-(\mathcal{A})$  if and only if the homologies  $H_n(P_\bullet)$  belong to  $\mathcal{A}$ , for all  $n \in \mathbf{Z}$ .

**Proposition 14.6.** *Let  $\mathcal{A}$  be an abelian category with enough projective objects. Then the  $t$ -structure on  $\mathcal{D}^-(\mathcal{A})$  is right bounded and left complete.*

*Proof.* The right boundedness of  $\mathcal{D}^-(\mathcal{A})$  is obvious. To prove the left completeness, we must show that  $\mathcal{D}^-(\mathcal{A})$  is a homotopy inverse limit of the tower of  $\infty$ -categories

$$\dots \rightarrow \mathcal{D}^-(\mathcal{A})_{\leq 1} \rightarrow \mathcal{D}^-(\mathcal{A})_{\leq 0} \rightarrow \dots$$

Invoking the right boundedness of  $\mathcal{D}^-(\mathcal{A})$ , we may reduce to proving that for each  $n \in \mathbf{Z}$ ,  $\mathcal{D}^-(\mathcal{A})_{\geq n}$  is a homotopy inverse limit of the tower

$$\dots \rightarrow \mathcal{D}^-(\mathcal{A})_{\leq 1, \geq n} \rightarrow \mathcal{D}^-(\mathcal{A})_{\leq 0, \geq n} \rightarrow \dots$$

Shifting if necessary, we may suppose that  $n = 0$ . Using Remark 14.5, we can replace  $\mathcal{A}$  by  $\mathcal{A}^\vee$ . For each  $k \geq 0$ , we let  $\mathcal{P}_{\Sigma}^{\leq k}(\mathcal{N}(\mathcal{A}_0))$  denote the  $\infty$ -category of product-preserving functors from  $\mathcal{N}(\mathcal{A}_0)^{op}$  to  $\tau_{\leq k} \mathcal{S}$ ; equivalently, we can define  $\mathcal{P}_{\Sigma}^{\leq k}(\mathcal{N}(\mathcal{A}_0))$  to be the  $\infty$ -category of  $k$ -truncated objects of  $\mathcal{P}_\Sigma(\mathcal{N}(\mathcal{A}_0))$ . We observe that the equivalence  $\psi$  of Proposition 14.4 restricts to an equivalence

$$\psi(k) : \mathcal{D}_{\geq 0}^-(\mathcal{A}^\vee)_{\leq k} \rightarrow \mathcal{P}_{\Sigma}^{\leq k}(\mathcal{N}(\mathcal{A}_0)).$$

Consequently, it will suffice to show that  $\mathcal{P}_\Sigma(\mathcal{N}(\mathcal{A}_0))$  is a homotopy inverse limit for the tower

$$\dots \rightarrow \mathcal{P}_{\Sigma}^{\leq 1}(\mathcal{N}(\mathcal{A}_0)) \rightarrow \mathcal{P}_{\Sigma}^{\leq 0}(\mathcal{N}(\mathcal{A}_0)).$$

Since the truncation functors on  $\mathcal{S}$  commute with finite products (Lemma T.6.5.1.2), we may reduce to the problem of showing that  $\mathcal{S}$  is a homotopy inverse limit of the tower

$$\dots \rightarrow \tau_{\leq 1} \mathcal{S} \rightarrow \tau_{\leq 0} \mathcal{S}.$$

This amounts to the classical fact that every space  $X$  can be recovered as the limit of its Postnikov tower (see for example §T.7.2.1). □

Our goal is to characterize the derived  $\infty$ -category  $\mathcal{D}^-(\mathcal{A})$  by a universal mapping property. Propositions 14.4 and T.5.5.8.15 give a characterization of  $\mathcal{D}_{\geq 0}^-(\mathcal{A}^\vee)$  of the right flavor. The next step is to understand the embedding of  $\mathcal{D}_{\geq 0}^-(\mathcal{A})$  into  $\mathcal{D}_{\geq 0}^-(\mathcal{A}^\vee)$ .

**Definition 14.7.** Let  $\mathcal{C}$  and  $\mathcal{C}'$  be stable  $\infty$ -categories equipped with  $t$ -structures. We will say that a functor  $f : \mathcal{C} \rightarrow \mathcal{C}'$  is *right  $t$ -exact* if it is exact, and carries  $\mathcal{C}_{\geq 0}$  into  $\mathcal{C}'_{\geq 0}$ .

**Lemma 14.8.** (1) *Let  $\mathcal{C}$  be an  $\infty$ -category which admits finite coproducts and geometric realizations. Then  $\mathcal{C}$  admits all finite colimits. Conversely, if  $\mathcal{C}$  is an  $n$ -category which admits finite colimits, then  $\mathcal{C}$  admits geometric realizations.*

- (2) Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between  $\infty$ -categories which admit finite coproducts and geometric realizations. If  $F$  preserves finite coproducts and geometric realizations, then  $F$  is right exact. The converse holds if  $\mathcal{C}$  and  $\mathcal{D}$  are  $n$ -categories.

*Proof.* We will prove (1); the proof of (2) follows by the same argument. Now suppose that  $\mathcal{C}$  admits finite coproducts and geometric realizations of simplicial objects. We wish to show that  $\mathcal{C}$  admits all finite colimits. According to Proposition T.4.4.3.2, it will suffice to show that  $\mathcal{C}$  admits coequalizers. Let  $\Delta_s^{\leq 1}$  be the full subcategory of  $\Delta$  spanned by the objects  $[0]$  and  $[1]$ , and *injective* maps between them, so that a coequalizer diagram in  $\mathcal{C}$  can be identified with a functor  $N(\Delta_s^{\leq 1})^{op} \rightarrow \mathcal{C}$ . Let  $j : N(\Delta_s^{\leq 1})^{op} \rightarrow N(\Delta)^{op}$  be the inclusion functor. We claim that every diagram  $f : N(\Delta_s^{\leq 1})^{op} \rightarrow \mathcal{C}$  has a left Kan extension along  $j$ . To prove this, it suffices to show that for each  $n \geq 0$ , the associated diagram

$$N(\Delta_s^{\leq 1})^{op} \times_{N(\Delta)^{op}} (N(\Delta)^{op})_{[n]} \rightarrow \mathcal{C}$$

has a colimit. We now observe that this last colimit is equivalent to a coproduct: more precisely, we have  $(j_! f)([n]) \simeq f([0]) \amalg f([1]) \amalg \dots \amalg f([1])$ , where there are precisely  $n$  summands equivalent to  $f([1])$ . Since  $\mathcal{C}$  admits finite coproducts, the desired Kan extension  $j_! f$  exists. We now observe that  $\varinjlim(f)$  can be identified with  $\varinjlim(j_! f)$ , and the latter exists in virtue of our assumption that  $\mathcal{C}$  admits geometric realizations for simplicial objects.

Now suppose that  $\mathcal{C}$  is an  $n$ -category which admits finite colimits; we wish to show that  $\mathcal{C}$  admits geometric realizations. Passing to a larger universe if necessary, we may suppose that  $\mathcal{C}$  is small. Let  $\mathcal{D} = \text{Ind}(\mathcal{C})$ , and let  $\mathcal{C}' \subseteq \mathcal{D}$  denote the essential image of the Yoneda embedding  $j : \mathcal{C} \rightarrow \mathcal{D}$ . Then  $\mathcal{D}$  admits small colimits (Theorem T.5.5.1.1) and  $j$  is fully faithful (Proposition T.5.1.3.1); it will therefore suffice to show that  $\mathcal{C}'$  is stable under geometric realization of simplicial objects in  $\mathcal{D}$ .

Fix a simplicial object  $U_\bullet : N(\Delta)^{op} \rightarrow \mathcal{C}' \subseteq \mathcal{D}$ . Let  $V_\bullet : N(\Delta)^{op} \rightarrow \mathcal{D}$  be a left Kan extension of  $U_\bullet | N(\Delta^{\leq n})^{op}$ , and  $\alpha_\bullet : V_\bullet \rightarrow U_\bullet$  the induced map. The geometric realization of  $V_\bullet$  can be identified with the colimit of  $U_\bullet | N(\Delta^{\leq n})^{op}$ , and therefore belongs to  $\mathcal{C}'$  since  $\mathcal{C}'$  is stable under finite colimits in  $\mathcal{D}$  (Proposition T.5.3.5.14). Consequently, it will suffice to prove that  $\alpha_\bullet$  induces an equivalence from the geometric realization of  $V_\bullet$  to the geometric realization of  $U_\bullet$ .

Let  $L : \mathcal{P}(\mathcal{C}) \rightarrow \mathcal{D}$  be a left adjoint to the inclusion. Let  $|U_\bullet|$  and  $|V_\bullet|$  be colimits of  $U_\bullet$  and  $V_\bullet$  in the  $\infty$ -category  $\mathcal{P}(\mathcal{C})$ , and let  $|\alpha_\bullet| : |V_\bullet| \rightarrow |U_\bullet|$  be the induced map. We wish to show that  $L|\alpha_\bullet|$  is an equivalence in  $\mathcal{D}$ . Since  $\mathcal{C}$  is an  $n$ -category, we have inclusions  $\text{Ind}(\mathcal{C}) \subseteq \text{Fun}(\mathcal{C}^{op}, \tau_{\leq n-1} \mathcal{S}) \subseteq \mathcal{P}(\mathcal{C})$ . It follows that  $L$  factors through the truncation functor  $\tau_{\leq n-1} : \mathcal{P}(\mathcal{C}) \rightarrow \mathcal{P}(\mathcal{C})$ . Consequently, it will suffice to prove that  $\tau_{\leq n-1}|\alpha_\bullet|$  is an equivalence in  $\mathcal{P}(\mathcal{C})$ . For this, it will suffice to show that the morphism  $|\alpha_\bullet|$  is  $n$ -connective (in the sense of Definition T.6.5.1.10). This follows from Lemma T.6.5.3.10, since  $\alpha_k : V_k \rightarrow U_k$  is an equivalence for  $k \leq n$ .  $\square$

**Lemma 14.9.** *Let  $\mathcal{C}$  and  $\mathcal{C}'$  be stable  $\infty$ -categories equipped with  $t$ -structures. Let  $\theta : \text{Fun}(\mathcal{C}, \mathcal{C}') \rightarrow \text{Fun}(\mathcal{C}_{\geq 0}, \mathcal{C}')$  be the restriction map. Then:*

- (1) *If  $\mathcal{C}$  is right-bounded, then  $\theta$  induces an equivalence from the full subcategory of  $\text{Fun}(\mathcal{C}, \mathcal{C}')$  spanned by the right  $t$ -exact functors to the full subcategory of  $\text{Fun}(\mathcal{C}_{\geq 0}, \mathcal{C}'_{\geq 0})$  spanned by the right exact functors.*
- (2) *Let  $\mathcal{C}$  and  $\mathcal{C}'$  be left complete. Then the  $\infty$ -categories  $\mathcal{C}_{\geq 0}$  and  $\mathcal{C}'_{\geq 0}$  admit geometric realizations of simplicial objects. Furthermore, a functor  $F : \mathcal{C}_{\geq 0} \rightarrow \mathcal{C}'_{\geq 0}$  is right exact if and only if it preserves finite coproducts and geometric realizations of simplicial objects.*

*Proof.* We first prove (1). If  $\mathcal{C}$  is right bounded, then  $\text{Fun}(\mathcal{C}, \mathcal{C}')$  is the (homotopy) inverse limit of the tower

$$\dots \rightarrow \text{Fun}(\mathcal{C}_{\geq -1}, \mathcal{C}') \rightarrow \text{Fun}(\mathcal{C}_{\geq 0}, \mathcal{C}'),$$

where the functors are given by restriction. The full subcategory of right  $t$ -exact functors is then given by the homotopy inverse limit

$$\dots \rightarrow \text{Fun}'(\mathcal{C}_{\geq -1}, \mathcal{C}'_{\geq -1}) \xrightarrow{\theta(0)} \text{Fun}(\mathcal{C}_{\geq 0}, \mathcal{C}'_{\geq 0})$$

where  $\text{Fun}'(\mathcal{C}, \mathcal{D})$  denotes the full subcategory of  $\text{Fun}(\mathcal{C}, \mathcal{D})$  spanned by the right exact functors. To complete the proof, it will suffice to show that this tower is essentially constant; in other words, that each  $\theta(n)$  is an equivalence of  $\infty$ -categories. Without loss of generality, we may suppose that  $n = 0$ . Choose shift functors on the  $\infty$ -categories  $\mathcal{C}$  and  $\mathcal{C}'$ , and define

$$\psi : \text{Fun}'(\mathcal{C}_{\geq 0}, \mathcal{C}'_{\geq 0}) \rightarrow \text{Fun}'(\mathcal{C}_{\geq -1}, \mathcal{C}'_{\geq -1})$$

by the formula  $\psi(F) = \Sigma^{-1} \circ F \circ \Sigma$ . We claim that  $\psi$  is a homotopy inverse to  $\theta(0)$ . To prove this, we observe that the right exactness of  $F \in \text{Fun}'(\mathcal{C}_{\geq 0}, \mathcal{C}'_{\geq 0})$ ,  $G \in \text{Fun}'(\mathcal{C}_{\geq -1}, \mathcal{C}'_{\geq -1})$  determines canonical equivalences

$$(\theta(0) \circ \psi)(F) \simeq F$$

$$(\psi \circ \theta(0))(G) \simeq G.$$

We now prove (2). Since  $\mathcal{C}$  is left complete,  $\mathcal{C}_{\geq 0}$  is the (homotopy) inverse limit of the tower of  $\infty$ -categories  $\{(\mathcal{C}_{\geq 0})_{\leq n}\}$  with transition maps given by right exact truncation functors. Lemma 14.8 implies that each  $(\mathcal{C}_{\geq 0})_{\leq n}$  admits geometric realizations of simplicial objects, and that each of the truncation functors preserves geometric realizations of simplicial objects. It follows that  $\mathcal{C}_{\geq 0}$  admits geometric realizations for simplicial objects. Similarly,  $\mathcal{C}'_{\geq 0}$  admits geometric realizations for simplicial objects.

If  $F$  preserves finite coproducts and geometric realizations of simplicial objects, then  $F$  is right exact (Lemma 14.8). Conversely, suppose that  $F$  is right exact; we wish to prove that  $F$  preserves geometric realizations of simplicial objects. It will suffice to show that each composition

$$\mathcal{C}_{\geq 0} \xrightarrow{F} \mathcal{C}'_{\geq 0} \xrightarrow{\tau_{\leq n}} (\mathcal{C}'_{\geq 0})_{\leq n}$$

preserves geometric realizations of simplicial objects. We observe that, in virtue of the right exactness of  $F$ , this functor is equivalent to the composition

$$\mathcal{C}_{\leq 0} \xrightarrow{\tau_{\leq n}} (\mathcal{C}_{\geq 0})_{\leq n} \xrightarrow{\tau_{\leq n} \circ F} (\mathcal{C}'_{\geq 0})_{\leq n}.$$

It will therefore suffice to prove that  $\tau_{\leq n} \circ F$  preserves geometric realizations of simplicial objects, which follows from Lemma 14.8 since both the source and target are equivalent to  $n$ -categories.  $\square$

**Lemma 14.10.** *Let  $\mathcal{A}$  be a small abelian category with enough projective objects, and let  $\mathcal{C} \subseteq \mathcal{P}_{\Sigma}(\mathcal{N}(\mathcal{A}_0))$  be the essential image of  $\mathcal{D}_{\geq 0}^-(\mathcal{A}) \subseteq \mathcal{D}_{\geq 0}^-(\mathcal{A}^{\vee})$  under the equivalence  $\psi : \mathcal{D}_{\geq 0}^-(\mathcal{A}^{\vee}) \rightarrow \mathcal{P}_{\Sigma}(\mathcal{N}(\mathcal{A}_0))$  of Proposition 14.4. Then  $\mathcal{C}$  is the smallest full subcategory of  $\mathcal{P}(\mathcal{N}(\mathcal{A}_0))$  which is closed under geometric realization and contains the essential image of the Yoneda embedding.*

*Proof.* It is clear that  $\mathcal{C}$  contains the essential image of the Yoneda embedding. Lemma 14.9 implies that  $\mathcal{D}_{\geq 0}^-(\mathcal{A})$  admits geometric realizations and that the inclusion  $\mathcal{D}_{\geq 0}^-(\mathcal{A}) \subseteq \mathcal{D}_{\geq 0}^-(\mathcal{A}^{\vee})$  preserves geometric realizations. It follows that  $\mathcal{C}$  is closed under geometric realizations in  $\mathcal{P}(\mathcal{N}(\mathcal{A}_0))$ .

To complete the proof, we will show that every object of  $X \in \mathcal{D}_{\geq 0}^-(\mathcal{A})$  can be obtained as the geometric realization, in  $\mathcal{D}_{\geq 0}^-(\mathcal{A}^{\vee})$ , of a simplicial object  $P_{\bullet}$  such that each  $P_n \in \mathcal{D}_{\geq 0}^-(\mathcal{A}^{\vee})$  consists of a projective object of  $\mathcal{A}$ , concentrated in degree zero. In fact, we can take  $P_{\bullet}$  to be the simplicial object of  $\mathcal{A}_0$  which corresponds to  $X \in \text{Ch}_{\geq 0}(\mathcal{A}_0)$  under the Dold-Kan correspondence. It follows from Theorem T.4.2.4.1 and Proposition T.5.5.9.14 that  $X$  can be identified with the geometric realization of  $P_{\bullet}$ .  $\square$

We are now ready to establish our characterization of  $\mathcal{D}_{\geq 0}^-(\mathcal{A})$ .

**Theorem 14.11.** *Let  $\mathcal{A}$  be an abelian category with enough projective objects,  $\mathcal{A}_0 \subseteq \mathcal{A}$  the full subcategory spanned by the projective objects, and  $\mathcal{C}$  an arbitrary  $\infty$ -category which admits geometric realizations. Let  $\text{Fun}'(\mathcal{D}_{\geq 0}^-(\mathcal{A}), \mathcal{C})$  denote the full subcategory of  $\text{Fun}(\mathcal{D}_{\geq 0}^-(\mathcal{A}), \mathcal{C})$  spanned by those functors which preserve geometric realizations. Then:*

(1) *The restriction map*

$$\mathrm{Fun}'(\mathcal{D}_{\geq 0}^-(\mathcal{A}), \mathcal{C}) \rightarrow \mathrm{Fun}(\mathcal{N}(\mathcal{A}_0), \mathcal{C})$$

*is an equivalence of  $\infty$ -categories.*

(2) *A functor  $F \in \mathrm{Fun}'(\mathcal{D}_{\geq 0}^-(\mathcal{A}), \mathcal{C})$  preserves finite coproducts if and only if the restriction  $F|_{\mathcal{N}(\mathcal{A}_0)}$  preserves finite coproducts.*

*Proof.* Part (1) follows from Lemma 14.10, Remark T.5.3.5.9, and Proposition T.4.3.2.15. The “only if” direction of (2) is obvious. To prove the “if” direction, let us suppose that  $F|_{\mathcal{N}(\mathcal{A}_0)}$  preserves finite coproducts. We may assume without loss of generality that  $\mathcal{C}$  admits filtered colimits (Lemma T.5.3.5.7), so that  $F$  extends to a functor  $F' : \mathcal{D}_{\geq 0}^-(\mathcal{A}^\vee)$  which preserves filtered colimits and geometric realizations (Propositions 14.4 and T.5.5.8.15). It follows from Proposition T.5.5.8.15 that  $F'$  preserves finite coproducts, so that  $F = F'|_{\mathcal{D}_{\geq 0}^-(\mathcal{A})}$  also preserves finite coproducts.  $\square$

**Corollary 14.12.** *Let  $\mathcal{A}$  be an abelian category with enough projective objects, and let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a left complete t-structure. Then the restriction functor*

$$\mathrm{Fun}(\mathcal{D}^-(\mathcal{A}), \mathcal{C}) \rightarrow \mathrm{Fun}(\mathcal{N}(\mathcal{A}_0), \mathcal{C})$$

*induces an equivalence from the full subcategory of  $\mathrm{Fun}(\mathcal{D}^-(\mathcal{A}), \mathcal{C})$  spanned by the right t-exact functors to the full subcategory of  $\mathrm{Fun}(\mathcal{N}(\mathcal{A}_0), \mathcal{C}_{\geq 0})$  spanned by functors which preserve finite coproducts (here  $\mathcal{A}_0$  denotes the full subcategory of  $\mathcal{A}$  spanned by the projective objects).*

*Proof.* Let  $\mathrm{Fun}'(\mathcal{D}^-(\mathcal{A}), \mathcal{C})$  be the full subcategory of  $\mathrm{Fun}(\mathcal{D}^-(\mathcal{A}), \mathcal{C})$  spanned by the right t-exact functors. Lemma 14.9 implies that  $\mathrm{Fun}'(\mathcal{D}^-(\mathcal{A}), \mathcal{C})$  is equivalent (via restriction) to the full subcategory

$$\mathrm{Fun}'(\mathcal{D}_{\geq 0}^-(\mathcal{A}), \mathcal{C}_{\geq 0}) \subseteq \mathrm{Fun}(\mathcal{D}_{\geq 0}^-(\mathcal{A}), \mathcal{C}_{\geq 0})$$

spanned by those functors which preserve finite coproducts and geometric realizations of simplicial objects. Theorem 14.11 and Proposition T.5.5.8.15 allow us to identify  $\mathrm{Fun}'(\mathcal{D}_{\geq 0}^-(\mathcal{A}), \mathcal{C}_{\geq 0})$  with the  $\infty$ -category of finite-coproduct preserving functors from  $\mathcal{N}(\mathcal{A}_0)$  into  $\mathcal{C}_{\geq 0}$ .  $\square$

**Corollary 14.13.** *Let  $\mathcal{A}$  be an abelian category with enough projective objects, let  $\mathcal{C}$  be a stable  $\infty$ -category equipped with a left complete t-structure, and let  $\mathcal{E} \subseteq \mathrm{Fun}(\mathcal{D}^-(\mathcal{A}), \mathcal{C})$  be the full subcategory spanned by those right t-exact functors which carry projective objects of  $\mathcal{A}$  into the heart of  $\mathcal{C}$ . Then  $\mathcal{E}$  is equivalent to (the nerve of) the ordinary category of right exact functors from  $\mathcal{A}$  to the heart of  $\mathcal{C}$ .*

*Proof.* Corollary 14.12 implies that the restriction map

$$\mathcal{E} \rightarrow \mathrm{Fun}(\mathcal{N}(\mathcal{A}_0), \mathcal{C}^\heartsuit)$$

is fully faithful, and that the essential image of  $\theta$  consists of the collection of coproduct-preserving functors from  $\mathcal{N}(\mathcal{A}_0)$  to  $\mathcal{C}^\heartsuit$ . Lemma 14.1 allows us to identify the latter  $\infty$ -category with the nerve of the category of right exact functors from  $\mathcal{A}$  to the heart of  $\mathcal{C}$ .  $\square$

If  $\mathcal{A}$  and  $\mathcal{C}$  are as in Proposition 14.12, then any right exact functor from  $\mathcal{A}$  to  $\mathcal{C}^\heartsuit$  can be extended (in an essentially unique way) to a functor  $\mathcal{D}^-(\mathcal{A}) \rightarrow \mathcal{C}$ . In particular, if the abelian category  $\mathcal{C}^\heartsuit$  has enough projective objects, then we obtain an induced map  $\mathcal{D}^-(\mathcal{C}^\heartsuit) \rightarrow \mathcal{C}$ .

**Example 14.14.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be abelian categories equipped with enough projective objects. Then any right-exact functor  $f : \mathcal{A} \rightarrow \mathcal{B}$  extends to a right t-exact functor  $F : \mathcal{D}^-(\mathcal{A}) \rightarrow \mathcal{D}^-(\mathcal{B})$ . One typically refers to  $F$  as the *left derived functor* of  $f$ .

**Example 14.15.** Let  $\mathrm{Sp}$  be the stable  $\infty$ -category of spectra (see §9), with its natural t-structure. Then the heart of  $\mathrm{Sp}$  is equivalent to the category  $\mathcal{A}$  of abelian groups. We therefore obtain a functor  $\mathcal{D}^-(\mathcal{A}) \rightarrow \mathrm{Sp}$ , which carries a complex of abelian groups to the corresponding *generalized Eilenberg-MacLane spectrum*.

## 15 Presentable Stable $\infty$ -Categories

In this section, we will study the class of *presentable* stable  $\infty$ -categories: that is, stable  $\infty$ -categories which admit small colimits and are generated (under colimits) by a set of small objects. In the stable setting, the condition of presentability can be formulated in reasonably simple terms.

**Proposition 15.1.** (1) *A stable  $\infty$ -category  $\mathcal{C}$  admits small colimits if and only if  $\mathcal{C}$  admits small coproducts.*

(2) *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be an exact functor between stable  $\infty$ -categories satisfying (1). Then  $F$  preserves small colimits if and only if  $F$  preserves small coproducts.*

(3) *Let  $\mathcal{C}$  be a stable  $\infty$ -category satisfying (1), and let  $X$  be an object of  $\mathcal{C}$ . Then  $X$  is compact if and only if the following condition is satisfied:*

(\*) *For every map  $f : X \rightarrow \coprod_{\alpha \in A} Y_\alpha$  in  $\mathcal{C}$ , there exists a finite subset  $A_0 \subseteq A$  such that  $f$  factors (up to homotopy) through  $\coprod_{\alpha \in A_0} Y_\alpha$ .*

*Proof.* The “only if” direction of (1) is obvious, and the converse follows from Proposition T.4.4.3.2. Assertion (2) can be proven in the same way.

The “only if” direction of (3) follows from the fact that an arbitrary coproduct  $\coprod_{\alpha \in A} Y_\alpha$  can be obtained as a filtered colimit of finite coproducts  $\coprod_{\alpha \in A_0} Y_\alpha$  (see §T.4.2.3). Conversely, suppose that an object  $X \in \mathcal{C}$  satisfies (\*); we wish to show that  $X$  is compact. Let  $f : \mathcal{C} \rightarrow \widehat{\mathcal{S}}$  be the functor corepresented by  $X$  (recall that  $\widehat{\mathcal{S}}$  denotes the  $\infty$ -category of spaces which are not necessarily small). Proposition T.5.1.3.2 implies that  $f$  is left exact. According to Proposition 10.12, we can assume that  $f = \Omega^\infty \circ F$ , where  $F : \mathcal{C} \rightarrow \widehat{\mathcal{S}p}$  is an exact functor; here  $\widehat{\mathcal{S}p}$  denotes the  $\infty$ -category of spectra which are not necessarily small. We wish to prove that  $f$  preserves filtered colimits. Since  $\Omega^\infty$  preserves filtered colimits, it will suffice to show that  $F$  preserves *all* colimits. In view of (2), it will suffice to show that  $F$  preserves coproducts. In virtue of Remark 9.10, we are reduced to showing that each of the induced functors

$$\mathcal{C} \xrightarrow{F} \widehat{\mathcal{S}p} \xrightarrow{\pi_n} \mathbf{N}(\mathcal{A}b)$$

preserves coproducts, where  $\mathcal{A}b$  denotes the category of (not necessarily small) abelian groups. Shifting if necessary, we may suppose  $n = 0$ . In other words, we must show that for any collection of objects  $\{Y_\alpha\}_{\alpha \in A}$ , the natural map

$$\theta : \bigoplus \mathrm{Ext}_{\mathcal{C}}^0(X, Y_\alpha) \rightarrow \mathrm{Ext}_{\mathcal{C}}^0(X, \coprod Y_\alpha)$$

is an isomorphism of abelian groups. The surjectivity of  $\theta$  amounts to the assumption (\*), while the injectivity follows from the observations that each  $Y_\alpha$  is a retract of the coproduct  $\coprod Y_\alpha$  and that the natural map  $\bigoplus \mathrm{Ext}_{\mathcal{C}}^0(X, Y_\alpha) \rightarrow \prod \mathrm{Ext}_{\mathcal{C}}^0(X, Y_\alpha)$  is injective.  $\square$

If  $\mathcal{C}$  is a stable  $\infty$ -category, then we will say that an object  $X \in \mathcal{C}$  *generates*  $\mathcal{C}$  if the condition  $\pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y) \simeq *$  implies that  $Y$  is a zero object of  $\mathcal{C}$ .

**Corollary 15.2.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category. Then  $\mathcal{C}$  is presentable if and only if the following conditions are satisfied:*

- (1) *The  $\infty$ -category  $\mathcal{C}$  admits small coproducts.*
- (2) *The homotopy category  $\mathrm{h}\mathcal{C}$  is locally small.*
- (3) *There exists regular cardinal  $\kappa$  and a  $\kappa$ -compact generator  $X \in \mathcal{C}$ .*

*Proof.* Suppose first that  $\mathcal{C}$  is presentable. Conditions (1) and (2) are obvious. To establish (3), we may assume without loss of generality that  $\mathcal{C}$  is an accessible localization of  $\mathcal{P}(\mathcal{D})$ , for some small  $\infty$ -category  $\mathcal{D}$ . Let  $F : \mathcal{P}(\mathcal{D}) \rightarrow \mathcal{C}$  be the localization functor and  $G$  its right adjoint. Let  $j : \mathcal{D} \rightarrow \mathcal{P}(\mathcal{D})$  be the Yoneda embedding, and let  $X$  be a coproduct of all suspensions (see §3) of objects of the form  $F(j(D))$ , where  $D \in \mathcal{D}$ . Since  $\mathcal{C}$  is presentable,  $X$  is  $\kappa$ -compact provided that  $\kappa$  is sufficiently large. We claim that  $X$  generates  $\mathcal{C}$ . To prove this, we consider an arbitrary  $Y \in \mathcal{C}$  such that  $\pi_0 \text{Map}_{\mathcal{C}}(X, Y) \simeq *$ . It follows that the space

$$\text{Map}_{\mathcal{C}}(F(j(D)), Y) \simeq \text{Map}_{\mathcal{P}(\mathcal{D})}(j(D), G(Y)) \simeq G(Y)(D)$$

is contractible for all  $D \in \mathcal{D}$ , so that  $G(Y)$  is a final object of  $\mathcal{P}(\mathcal{D})$ . Since  $G$  is fully faithful, we conclude that  $Y$  is a final object of  $\mathcal{C}$ , as desired.

Conversely, suppose that (1), (2), and (3) are satisfied. We first claim that  $\mathcal{C}$  is itself locally small. It will suffice to show that for every morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$  and every  $n \geq 0$ , the homotopy group  $\pi_n(\text{Hom}_{\mathcal{C}}^{\mathbb{R}}(X, Y), f)$  is small. We note that  $\text{Hom}_{\mathcal{C}}^{\mathbb{R}}(X, Y)$  is equivalent to the loop space of  $\text{Hom}_{\mathcal{C}}^{\mathbb{R}}(X, Y[1])$ ; the question is therefore independent of base point, so we may assume that  $f$  is the zero map. We conclude that the relevant homotopy group can be identified with  $\text{Hom}_{\mathfrak{h}\mathcal{C}}(X[n], Y)$ , which is small in virtue of assumption (2).

Fix a regular cardinal  $\kappa$  and a  $\kappa$ -compact object  $X$  which generates  $\mathcal{C}$ . We now define a transfinite sequence of full subcategories

$$\mathcal{C}(0) \subseteq \mathcal{C}(1) \subseteq \dots$$

as follows. Let  $\mathcal{C}(0)$  be the full subcategory of  $\mathcal{C}$  spanned by the objects  $\{X[n]\}_{n \in \mathbb{Z}}$ . If  $\lambda$  is a limit ordinal, let  $\mathcal{C}(\lambda) = \bigcup_{\beta < \lambda} \mathcal{C}(\beta)$ . Finally, let  $\mathcal{C}(\alpha + 1)$  be the full subcategory of  $\mathcal{C}$  spanned by all objects which can be obtained as the colimit of  $\kappa$ -small diagrams in  $\mathcal{C}(\alpha)$ . Since  $\mathcal{C}$  is locally small, it follows that each  $\mathcal{C}(\alpha)$  is essentially small. It follows by induction that each  $\mathcal{C}(\alpha)$  consists of  $\kappa$ -compact objects of  $\mathcal{C}$  and is stable under translation. Finally, we observe that  $\mathcal{C}(\kappa)$  is stable under  $\kappa$ -small colimits. It follows from Lemma 4.3 that  $\mathcal{C}(\kappa)$  is a stable subcategory of  $\mathcal{C}$ . Choose a small  $\infty$ -category  $\mathcal{D}$  and an equivalence  $f : \mathcal{D} \rightarrow \mathcal{C}(\kappa)$ . According to Proposition T.5.3.5.11, we may suppose that  $f$  factors as a composition

$$\mathcal{D} \xrightarrow{j} \text{Ind}_{\kappa}(\mathcal{D}) \xrightarrow{F} \mathcal{C}$$

where  $j$  is the Yoneda embedding and  $F$  is a  $\kappa$ -continuous, fully faithful functor. We will complete the proof by showing that  $F$  is an equivalence.

Proposition T.5.5.1.9 implies that  $F$  preserves small colimits. It follows that  $F$  admits a right adjoint  $G : \mathcal{C} \rightarrow \text{Ind}_{\kappa}(\mathcal{D})$  (Remark T.5.5.2.10). We wish to show that the counit map  $u : F \circ G \rightarrow \text{id}_{\mathcal{C}}$  is an equivalence of functors. Choose an object  $Z \in \mathcal{C}$ , and let  $Y$  be a cokernel for the induced map  $u_Z : (F \circ G)(Z) \rightarrow Z$ . Since  $F$  is fully faithful,  $G(u_Z)$  is an equivalence. Because  $G$  is an exact functor, we deduce that  $G(Y) = 0$ . It follows that  $\text{Map}_{\mathcal{C}}(F(D), Y) \simeq \text{Map}_{\text{Ind}_{\kappa}(\mathcal{D})}(D, G(Y)) \simeq *$  for all  $D \in \text{Ind}_{\kappa}(\mathcal{D})$ . In particular, we conclude that  $\pi_0 \text{Map}_{\mathcal{C}}(X, Y) \simeq *$ . Since  $X$  generates  $\mathcal{C}$ , we deduce that  $Y \simeq 0$ . Thus  $u_Z$  is an equivalence as desired.  $\square$

**Remark 15.3.** In view of Proposition 15.1 and Corollary 15.2, the hypothesis that a stable  $\infty$ -category  $\mathcal{C}$  be compactly generated can be formulated entirely in terms of the homotopy category  $\mathfrak{h}\mathcal{C}$ . Consequently, one can study this condition entirely in the setting of triangulated categories, without making reference to (or assuming the existence of) an underlying stable  $\infty$ -category. We refer the reader to [55] for further discussion.

The following result gives a good class of examples of presentable  $\infty$ -categories.

**Proposition 15.4.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be presentable  $\infty$ -categories, and suppose that  $\mathcal{D}$  is stable.*

- (1) *The  $\infty$ -category  $\text{Stab}(\mathcal{C})$  is presentable.*
- (2) *The functor  $\Omega^{\infty} : \text{Stab}(\mathcal{C}) \rightarrow \mathcal{C}$  admits a left adjoint  $\Sigma^{\infty} : \mathcal{C} \rightarrow \text{Stab}(\mathcal{C})$ .*

(3) An exact functor  $G : \mathcal{D} \rightarrow \text{Stab}(\mathcal{C})$  admits a left adjoint if and only if  $\Omega^\infty \circ G : \mathcal{D} \rightarrow \mathcal{C}$  admits a left adjoint.

*Proof.* We first prove (1). Assume that  $\mathcal{C}$  is presentable, and let  $1$  be a final object of  $\mathcal{C}$ . Then  $\mathcal{C}_*$  is equivalent to  $\mathcal{C}_{1/}$ , and therefore presentable (Proposition T.5.5.3.11). The loop functor  $\Omega : \mathcal{C}_* \rightarrow \mathcal{C}_*$  admits a left adjoint  $\Sigma : \mathcal{C}_* \rightarrow \mathcal{C}_*$ . Consequently, we may view the tower

$$\dots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*$$

as a diagram in the  $\infty$ -category  $\mathcal{P}\text{r}^{\text{R}}$ . Invoking Theorem T.5.5.3.18, we deduce (1) and the following modified versions of (2) and (3):

(2') The functor  $\Omega_*^\infty : \text{Stab}(\mathcal{C}) \rightarrow \mathcal{C}_*$  admits a left adjoint  $\Sigma_*^\infty : \mathcal{C}_* \rightarrow \text{Stab}(\mathcal{C})$ .

(3') An exact functor  $G : \mathcal{D} \rightarrow \text{Stab}(\mathcal{C})$  admits a left adjoint if and only if  $\Omega_*^\infty \circ G : \mathcal{D} \rightarrow \mathcal{C}_*$  admits a left adjoint.

To complete the proof, it will suffice to verify the following:

(2'') The forgetful functor  $\mathcal{C}_* \rightarrow \mathcal{C}$  admits a left adjoint  $\mathcal{C} \rightarrow \mathcal{C}_*$ .

(3'') A functor  $G : \mathcal{D} \rightarrow \mathcal{C}_*$  admits a left adjoint if and only if the composition  $\mathcal{D} \xrightarrow{G} \mathcal{C}_* \rightarrow \mathcal{C}$  admits a left adjoint.

To prove (2'') and (3''), we recall that a functor  $G$  between presentable  $\infty$ -categories admits a left adjoint if and only if  $G$  preserves small limits and small,  $\kappa$ -filtered colimits, for some regular cardinal  $\kappa$  (Corollary T.5.5.2.9). The desired results now follow from Propositions T.4.4.2.9 and T.1.2.13.8.  $\square$

**Corollary 15.5.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be presentable  $\infty$ -categories, and suppose that  $\mathcal{D}$  is stable. Then composition with  $\Sigma^\infty : \mathcal{C} \rightarrow \text{Stab}(\mathcal{C})$  induces an equivalence*

$$\mathcal{P}\text{r}^{\text{L}}(\text{Stab}(\mathcal{C}), \mathcal{D}) \rightarrow \mathcal{P}\text{r}^{\text{L}}(\mathcal{C}, \mathcal{D}).$$

*Proof.* This is equivalent to the assertion that composition with  $\Omega^\infty$  induces an equivalence

$$\mathcal{P}\text{r}^{\text{R}}(\mathcal{D}, \text{Stab}(\mathcal{C})) \rightarrow \mathcal{P}\text{r}^{\text{R}}(\mathcal{D}, \mathcal{C}),$$

which follows from Propositions 10.12 and 15.4.  $\square$

Using Corollary 15.5, we obtain another characterization of the  $\infty$ -category of spectra. Let  $S \in \text{Sp}$  denote the image under  $\Sigma^\infty : \mathcal{S} \rightarrow \text{Sp}$  of the final object  $* \in \mathcal{S}$ . We will refer to  $S$  as the *sphere spectrum*.

**Corollary 15.6.** *Let  $\mathcal{D}$  be a stable, presentable  $\infty$ -category. Then evaluation on the sphere spectrum induces an equivalence of  $\infty$ -categories*

$$\theta : \mathcal{P}\text{r}^{\text{L}}(\text{Sp}, \mathcal{D}) \rightarrow \mathcal{D}.$$

In other words, we may regard the  $\infty$ -category  $\text{Sp}$  as the stable  $\infty$ -category which is freely generated, under colimits, by a single object.

*Proof.* We can factor the evaluation map  $\theta$  as a composition

$$\mathcal{P}\text{r}^{\text{L}}(\text{Stab}(\mathcal{S}), \mathcal{D}) \xrightarrow{\theta'} \mathcal{P}\text{r}^{\text{L}}(\mathcal{S}, \mathcal{D}) \xrightarrow{\theta''} \mathcal{D}$$

where  $\theta'$  is given by composition with  $\Sigma^\infty$  and  $\theta''$  by evaluation at the final object of  $\mathcal{S}$ . We now observe that  $\theta'$  and  $\theta''$  are both equivalences of  $\infty$ -categories (Corollary 15.5 and Theorem T.5.1.5.6).  $\square$



We conclude this section by establishing a characterization of the class of stable, presentable  $\infty$ -categories.

**Lemma 15.7.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category, and let  $\mathcal{C}' \subseteq \mathcal{C}$  be a localization of  $\mathcal{C}$ . Let  $L : \mathcal{C} \rightarrow \mathcal{C}'$  be a left adjoint to the inclusion. Then  $L$  is left exact if and only if  $\mathcal{C}'$  is stable.*

*Proof.* The “if” direction follows from Proposition 5.1, since  $L$  is right exact. Conversely, suppose that  $L$  is left exact. Since  $\mathcal{C}'$  is a localization of  $\mathcal{C}$ , it is closed under finite limits. In particular, it is closed under the formation of kernels and contains a zero object of  $\mathcal{C}$ . To complete the proof, it will suffice to show that  $\mathcal{C}'$  is stable under the formation of pushouts in  $\mathcal{C}$ . Choose a pushout diagram  $\sigma : \Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

in  $\mathcal{C}$ , where  $X, X', Y \in \mathcal{C}'$ . Proposition 4.4 implies that  $\sigma$  is also a pullback square. Let  $L : \mathcal{C} \rightarrow \mathcal{C}'$  be a left adjoint to the inclusion. Since  $L$  is left exact, we obtain a pullback square  $L(\sigma)$ :

$$\begin{array}{ccc} LX & \longrightarrow & LX' \\ \downarrow & & \downarrow \\ LY & \longrightarrow & LY'. \end{array}$$

Applying Proposition 4.4 again, we deduce that  $L(\sigma)$  is a pushout square in  $\mathcal{C}$ . The natural transformation  $\sigma \rightarrow L(\sigma)$  is an equivalence when restricted to  $\Lambda_0^2$ , and therefore induces an equivalence  $Y' \rightarrow LY'$ . It follows that  $Y'$  belongs to the essential image of  $\mathcal{C}'$ , as desired.  $\square$

**Lemma 15.8.** *Let  $\mathcal{C}$  be a stable  $\infty$ -category,  $\mathcal{D}$  an  $\infty$ -category which admits finite limits, and  $G : \mathcal{C} \rightarrow \text{Stab}(\mathcal{D})$  an exact functor. Suppose that  $g = \Omega^\infty \circ G : \mathcal{C} \rightarrow \mathcal{D}$  is fully faithful. Then  $G$  is fully faithful.*

*Proof.* It will suffice to show that each of the composite maps

$$g_n : \mathcal{C} \rightarrow \text{Stab}(\mathcal{D}) \xrightarrow{\Omega_*^{\infty-n}} \mathcal{D}_*$$

is fully faithful. Since  $g_n$  can be identified with  $g_{n+1} \circ \Omega$ , where  $\Omega : \mathcal{C} \rightarrow \mathcal{C}$  denotes the loop functor, we can reduce to the case  $n = 0$ . Fix objects  $C, C' \in \mathcal{C}$ ; we will show that the map  $\text{Map}_{\mathcal{C}}(C, C') \rightarrow \text{Map}_{\mathcal{D}_*}(g_0(C), g_0(C'))$  is a homotopy equivalence. We have a homotopy fiber sequence

$$\text{Map}_{\mathcal{D}_*}(g_0(C), g_0(C')) \xrightarrow{\theta} \text{Map}_{\mathcal{D}}(g(C), g(C')) \rightarrow \text{Map}_{\mathcal{D}}(*, g(C')).$$

Here  $*$  denotes a final object of  $\mathcal{D}$ . Since  $g$  is fully faithful, it will suffice to prove that  $\theta$  is a homotopy equivalence. For this, it suffices to show that  $\text{Map}_{\mathcal{D}}(*, g(C'))$  is contractible. Since  $g$  is left exact, this space can be identified with  $\text{Map}_{\mathcal{D}}(g(*), g(C'))$ , where  $*$  is the final object of  $\mathcal{C}$ . Invoking once again our assumption that  $g$  is fully faithful, we are reduced to proving that  $\text{Map}_{\mathcal{C}}(*, C')$  is contractible. This follows from the assumption that  $\mathcal{C}$  is pointed (since  $*$  is also an initial object of  $\mathcal{C}$ ).  $\square$

**Proposition 15.9.** *Let  $\mathcal{C}$  be an  $\infty$ -category. The following conditions are equivalent:*

- (1) *The  $\infty$ -category  $\mathcal{C}$  is presentable and stable.*
- (2) *There exists a presentable, stable  $\infty$ -category  $\mathcal{D}$  and an accessible left-exact localization  $L : \mathcal{D} \rightarrow \mathcal{C}$ .*
- (3) *There exists a small  $\infty$ -category  $\mathcal{E}$  such that  $\mathcal{C}$  is equivalent to an accessible left-exact localization of  $\text{Fun}(\mathcal{E}, \text{Sp})$ .*

*Proof.* The  $\infty$ -category  $\mathrm{Sp}$  is stable and presentable, so for every small  $\infty$ -category  $\mathcal{E}$ , the functor  $\infty$ -category  $\mathrm{Fun}(\mathcal{E}, \mathrm{Sp})$  is also stable (Proposition 4.1) and presentable (Proposition T.5.5.3.6). This proves (3)  $\Rightarrow$  (2). The implication (2)  $\Rightarrow$  (1) follows from Lemma 15.7. We will complete the proof by showing that (1)  $\Rightarrow$  (3).

Since  $\mathcal{C}$  is presentable, there exists a small  $\infty$ -category  $\mathcal{E}$  and a fully faithful embedding  $g : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{E})$ , which admits a left adjoint (Theorem T.5.5.1.1). Propositions 10.12 and 15.4 implies that  $g$  is equivalent to a composition

$$\mathcal{C} \xrightarrow{G} \mathrm{Stab}(\mathcal{P}(\mathcal{E})) \xrightarrow{\Omega^\infty} \mathcal{P}(\mathcal{E}),$$

where the functor  $G$  admits a left adjoint. Lemma 15.8 implies that  $G$  is fully faithful. It follows that  $\mathcal{C}$  is an (accessible) left exact localization of  $\mathrm{Stab}(\mathcal{P}(\mathcal{E}))$ . We now invoke Example 10.13 to identify  $\mathrm{Stab}(\mathcal{P}(\mathcal{E}))$  with  $\mathrm{Fun}(\mathcal{E}, \mathrm{Sp})$ .  $\square$

**Remark 15.10.** Proposition 15.9 can be regarded as an analogue of Giraud’s characterization of topoi as left exact localizations of presheaf categories ([2]). Other variations on this theme include the  $\infty$ -categorical version of Giraud’s theorem (Theorem T.6.1.0.6) and the Gabriel-Popescu theorem for abelian categories (see [54]).

We conclude this section by describing an abstract version of the Brown representability theorem, which is valid in any compactly generated stable  $\infty$ -category (as well as many other  $\infty$ -categories of interest).

**Theorem 15.11** (Brown Representability). *Let  $\mathcal{C}$  be a presentable  $\infty$ -category containing a set of objects  $\{S_\alpha\}_{\alpha \in A}$  with the following properties:*

- (i) *Each object  $S_\alpha$  is a cogroup object of the homotopy category  $\mathrm{h}\mathcal{C}$ .*
- (ii) *Each object  $S_\alpha \in \mathcal{C}$  is compact.*
- (iii) *The  $\infty$ -category  $\mathcal{C}$  is generated by the objects  $S_\alpha$  under small colimits.*

*Then a functor  $F : \mathrm{h}\mathcal{C}^{\mathrm{op}} \rightarrow \mathrm{Set}$  is representable if and only if it satisfies the following conditions:*

- (a) *For every collection of objects  $C_\beta$  in  $\mathcal{C}$ , the map  $F(\coprod_\beta C_\beta) \rightarrow \prod_\beta F(C_\beta)$  is a bijection.*
- (b) *For every pushout square*

$$\begin{array}{ccc} C & \longrightarrow & C' \\ \downarrow & & \downarrow \\ D & \longrightarrow & D' \end{array}$$

*in  $\mathcal{C}$ , the induced map  $F(D') \rightarrow F(C') \times_{F(C)} F(D)$  is surjective.*

*Proof of Theorem 15.11.* The necessity of conditions (a) and (b) is obvious. We will prove that these conditions are sufficient. Let  $\emptyset$  denote an initial object of  $\mathcal{C}$ . If  $S$  is an object of  $\mathcal{C}$  equipped with a map  $\epsilon : S \rightarrow \emptyset$ , we define the suspension  $\Sigma(S)$  to be the pushout  $\emptyset \coprod_S \emptyset$ . Note that  $\Sigma(S)$  always has the structure of a cogroup object of  $\mathcal{C}$ , and therefore a cogroup object of the homotopy category  $\mathrm{h}\mathcal{C}$ . Each of the objects  $S_\alpha$  is equipped with a counit map  $S_\alpha \rightarrow \emptyset$  (by virtue of (i)), so that the suspension  $\Sigma S_\alpha$  is well-defined. Enlarging the collection  $\{S_\alpha\}$  if necessary, we may assume that this collection is stable under the formation of suspensions.

We first prove the following:

- (\*) Let  $f : C \rightarrow C'$  be a morphism in  $\mathcal{C}$  such that the induced map  $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(S_\alpha, C) \rightarrow \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(S_\alpha, C')$  is an isomorphism, for every index  $\alpha$ . Then  $f$  is an equivalence in  $\mathcal{C}$ .

To prove (\*), it will suffice to show that for every object  $X \in \mathcal{C}$ , the map  $f$  induces a homotopy equivalence  $\phi : \text{Map}_{\mathcal{C}}(X, C) \rightarrow \text{Map}_{\mathcal{C}}(X, C')$ . The collection of  $X \in \mathcal{C}$  is stable under colimits in  $\mathcal{C}$ . By virtue of assumption (iii), it suffices to consider the case where  $X$  is an object of the form  $S_\alpha$ . Since  $S_\alpha$  is a cogroup object of  $\text{h}\mathcal{C}$ , we can regard  $\phi$  as a map between homotopy associative  $H$ -spaces. It follows that  $\phi$  is a homotopy equivalence if and only if it induces an isomorphism of groups  $\pi_n \text{Map}_{\mathcal{C}}(X, C) \rightarrow \pi_n \text{Map}_{\mathcal{C}}(X, C')$  for each  $n \geq 0$  (here the homotopy groups are taken with respect to the base point given by the  $H$ -space structure). Replacing  $S_\alpha$  by  $\Sigma^n S_\alpha$ , we can reduce to the case  $n = 0$ , in which case we are reduced to the bijectivity of the map  $\text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, C) \rightarrow \text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, C')$ .

Now suppose that  $F$  is a functor satisfying conditions (a) and (b). We will prove the following:

(\*') Let  $X \in \mathcal{C}$  and let  $\eta \in F(X)$ . Then there exists a map  $f : X \rightarrow X'$  in  $\mathcal{C}$  and an object  $\eta' \in F(X')$  lifting  $\eta$  with the following property: for every index  $\alpha \in A$ ,  $\eta'$  induces a bijection  $\text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, X') \rightarrow F(S_\alpha)$ .

To prove (\*'), we begin by defining  $X_0$  to be the coproduct of  $X$  with  $\coprod_{\alpha \in A, \gamma \in F(S_\alpha)} S_\alpha$ . Using (a), we deduce the existence element  $\eta_0 \in F(X_0)$  lifting  $\eta$ . By construction,  $\eta_0$  induces a *surjection*  $\text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, X_0) \rightarrow F(S_\alpha)$  for each index  $\alpha$ .

We now define a sequence of morphisms  $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \dots$  and a compatible family of elements  $\eta_n \in F(X_n)$  using induction on  $n$ . Suppose that  $X_n$  and  $\eta_n$  have already been constructed. For each index  $\alpha \in A$ , let  $K_\alpha$  be the kernel of the group homomorphism  $\text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, X_n) \rightarrow F(S_\alpha)$ , and define  $X_{n+1}$  to fit into a pushout diagram

$$\begin{array}{ccc} \coprod_{\alpha \in A, \gamma \in K_\alpha} S_\alpha & \longrightarrow & \emptyset \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_{n+1} \end{array}$$

where the upper horizontal map is given by the counit on each  $S_\alpha$ . The existence of a point  $\eta_{n+1} \in F(X_{n+1})$  lifting  $\eta_n$  follows from assumption (a).

Let  $X' = \varinjlim_n X_n$ . We have a pushout diagram

$$\begin{array}{ccc} \coprod_n X_n & \longrightarrow & \coprod_n X_{2n} \\ \downarrow & & \downarrow \\ \coprod_n X_{2n+1} & \longrightarrow & X' \end{array}$$

Using (a) and (b), we deduce the existence of a point  $\eta' \in F(X')$  lifting the sequence  $\{\eta_n \in F(X_n)\}$ . We claim that  $\eta'$  satisfies the condition described in (\*). Fix an index  $\alpha$ ; we wish to prove that the map  $\psi : \text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, X') \rightarrow F(S_\alpha)$  is bijective. It is clear that  $\psi$  is surjective (since the composite map  $\text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, X_0) \rightarrow \text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, X') \rightarrow F(S_\alpha)$  is surjective by construction). To prove that  $\psi$  is injective, it will suffice to show that the kernel of  $\psi$  is trivial (since  $\psi$  is a group homomorphism, using the cogroup structure on  $S_\alpha$  given by (i)). Fix an element  $\gamma \in \ker(\psi)$ , represented by a map  $f : S_\alpha \rightarrow X'$ . Assumption (ii) guarantees that  $S_\alpha$  is compact, so that  $f$  factors through some map  $\bar{f} : S_\alpha \rightarrow X_n$ , which determines an element of the kernel  $K$  of the map  $\text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, X_n) \rightarrow F(S_\alpha)$ . It follows from our construction that the composite map  $S_\alpha \xrightarrow{\bar{f}} X_n \rightarrow X_{n+1}$  factors through the counit of  $S_\alpha$ , so that  $f$  is the unit element of  $\ker(\psi)$ . This completes the proof of (\*').

Assertion (b) guarantees that  $F(\emptyset)$  consists of a single element. Applying (\*') in the case  $X = \emptyset$ , we obtain an element  $\eta' \in F(X')$  which induces isomorphisms  $\text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, X') \rightarrow F(S_\alpha)$  for each index  $\alpha$ . We will complete the proof by showing that  $\eta'$  exhibits  $F$  as the functor on  $\text{h}\mathcal{C}$  represented by the object  $X'$ . In other words, we claim that for every object  $Y \in \mathcal{C}$ , the element  $\eta'$  induces a bijection  $\theta : \text{Hom}_{\text{h}\mathcal{C}}(Y, X') \rightarrow F(Y)$ .

We begin by showing that  $\theta$  is surjective. Fix an element  $\eta'' \in F(Y)$ . Assumption (b) guarantees that  $(\eta', \eta'')$  determines an element of  $F(Y \coprod X')$ . Applying assertion (\*') to this element, we deduce the

existence of a map  $X' \amalg Y \rightarrow Z$  and an element  $\bar{\eta} \in F(Z)$  lifting the pair  $(\eta', \eta'')$  which induces isomorphisms  $\text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, Z) \rightarrow F(S_\alpha)$  for each index  $\alpha$ . We have a commutative diagram

$$\begin{array}{ccc} \text{Map}_{\text{h}\mathcal{C}}(S_\alpha, Z) & \xrightarrow{\quad\quad\quad} & \text{Map}_{\text{h}\mathcal{C}}(S_\alpha, X') \\ & \searrow & \swarrow \\ & F(S_\alpha) & \end{array}$$

for each index  $\alpha$ , in which the vertical maps are bijective. It follows that the horizontal map is also bijective. Invoking  $(*)'$ , we deduce that  $X' \rightarrow Z$  is an equivalence. The composite map  $Y \rightarrow Z \simeq X'$  is then a preimage of  $\eta''$  in the set  $\text{Hom}_{\text{h}\mathcal{C}}(Y, X')$ .

We now complete the proof by showing that  $\theta$  is injective. Fix a pair of maps  $f, g : Y \rightarrow X'$  which determine the same element of  $F(Y)$ . Form a pushout diagram

$$\begin{array}{ccc} Y \amalg Y & \xrightarrow{(f,g)} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\quad\quad\quad} & Z. \end{array}$$

Using assumption  $(b)$ , we deduce that  $\eta' \in F(X')$  can be lifted to an element  $\bar{\eta} \in F(Z)$ . Applying  $(*)'$ , we deduce the existence of a map  $Z \rightarrow Z'$  and an element  $\bar{\eta}' \in F(Z')$  lifting  $\bar{\eta}$  and inducing bijections  $\text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, Z') \rightarrow F(S_\alpha)$  for each index  $\alpha$ . We have commutative diagrams

$$\begin{array}{ccc} \text{Map}_{\text{h}\mathcal{C}}(S_\alpha, Z') & \xrightarrow{\quad\quad\quad} & \text{Map}_{\text{h}\mathcal{C}}(S_\alpha, X') \\ & \searrow & \swarrow \\ & F(S_\alpha) & \end{array}$$

in which the vertical maps are bijective. It follows that the horizontal maps are also bijective, so that  $(*)$  guarantees that the map  $h : X' \rightarrow Z'$  is an equivalence in  $\mathcal{C}$ . Since the compositions  $h \circ f$  and  $h \circ g$  are homotopic, we deduce that  $f$  and  $g$  are homotopic and therefore represent the same element of  $\text{Hom}_{\text{h}\mathcal{C}}(Y, X')$ , as desired.  $\square$

## 16 Accessible t-Structures

Let  $\mathcal{C}$  be a stable  $\infty$ -category. If  $\mathcal{C}$  is presentable, then it is reasonably easy to construct t-structures on  $\mathcal{C}$ : for any small collection of objects  $\{X_\alpha\}$  of  $\mathcal{C}$ , there exists a t-structure *generated* by the objects  $X_\alpha$ . More precisely, we have the following result:

**Proposition 16.1.** *Let  $\mathcal{C}$  be a presentable stable  $\infty$ -category.*

- (1) *If  $\mathcal{C}' \subseteq \mathcal{C}$  is a full subcategory which is presentable, closed under small colimits, and closed under extensions, then there exists a t-structure on  $\mathcal{C}$  such that  $\mathcal{C}' = \mathcal{C}_{\geq 0}$ .*
- (2) *Let  $\{X_\alpha\}$  be a small collection of objects of  $\mathcal{C}$ , and let  $\mathcal{C}'$  be the smallest full subcategory of  $\mathcal{C}$  which contains each  $X_\alpha$  and is closed under extensions and small colimits. Then  $\mathcal{C}'$  is presentable.*

*Proof.* We will give the proof of (1) and defer the (somewhat technical) proof of (2) until the end of this section. Fix  $X \in \mathcal{C}$ , and let  $\mathcal{C}'_X$  denote the fiber product  $\mathcal{C}'_X \times_{\mathcal{C}} \mathcal{C}'$ . Using Proposition T.5.5.3.12, we deduce that  $\mathcal{C}'_X$  is presentable, so that it admits a final object  $f : Y \rightarrow X$ . It follows that composition with  $f$  induces a homotopy equivalence

$$\text{Map}_{\mathcal{C}}(Z, Y) \rightarrow \text{Map}_{\mathcal{C}}(Z, X)$$

for each  $Z \in \mathcal{C}'$ . Proposition T.5.2.7.8 implies that  $\mathcal{C}'$  is a colocalization of  $\mathcal{C}$ . Since  $\mathcal{C}'$  is stable under extensions, Proposition 6.15 implies the existence of a (uniquely determined) t-structure such that  $\mathcal{C}' = \mathcal{C}_{\geq 0}$ .  $\square$

**Definition 16.2.** Let  $\mathcal{C}$  be a presentable stable  $\infty$ -category. We will say that a t-structure on  $\mathcal{C}$  is *accessible* if the subcategory  $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$  is presentable.

Proposition 16.1 can be summarized as follows: any small collection of objects  $\{X_\alpha\}$  of a presentable stable  $\infty$ -category  $\mathcal{C}$  determines an accessible t-structure on  $\mathcal{C}$ , which is minimal among t-structures such that each  $X_\alpha$  belongs to  $\mathcal{C}_{\geq 0}$ .

Definition 16.2 has a number of reformulations:

**Proposition 16.3.** *Let  $\mathcal{C}$  be a presentable stable  $\infty$ -category equipped with a t-structure. The following conditions are equivalent:*

- (1) *The  $\infty$ -category  $\mathcal{C}_{\geq 0}$  is presentable (equivalently: the t-structure on  $\mathcal{C}$  is accessible).*
- (2) *The  $\infty$ -category  $\mathcal{C}_{\geq 0}$  is accessible.*
- (3) *The  $\infty$ -category  $\mathcal{C}_{\leq 0}$  is presentable.*
- (4) *The  $\infty$ -category  $\mathcal{C}_{\leq 0}$  is accessible.*
- (5) *The truncation functor  $\tau_{\leq 0} : \mathcal{C} \rightarrow \mathcal{C}$  is accessible.*
- (6) *The truncation functors  $\tau_{\geq 0} : \mathcal{C} \rightarrow \mathcal{C}$  is accessible.*

*Proof.* We observe that  $\mathcal{C}_{\geq 0}$  is stable under all colimits which exist in  $\mathcal{C}$ , and that  $\mathcal{C}_{\leq 0}$  is a localization of  $\mathcal{C}$ . It follows that  $\mathcal{C}_{\geq 0}$  and  $\mathcal{C}_{\leq 0}$  admit small colimits, so that (1)  $\Leftrightarrow$  (2) and (3)  $\Leftrightarrow$  (4). We have a distinguished triangle of functors

$$\tau_{\geq 0} \xrightarrow{\alpha} \text{id}_{\mathcal{C}} \xrightarrow{\beta} \tau_{\leq -1} \rightarrow \tau_{\geq 0}[1]$$

in the homotopy category  $\text{hFun}(\mathcal{C}, \mathcal{C})$ . The collection of accessible functors from  $\mathcal{C}$  to itself is stable under shifts and under small colimits. Since  $\tau_{\leq 0} \simeq \text{coker}(\alpha)[1]$  and  $\tau_{\geq 0} \simeq \text{coker}(\beta)[-1]$ , we conclude that (5)  $\Leftrightarrow$  (6). The equivalence (1)  $\Leftrightarrow$  (5) follows from Proposition T.5.5.1.2. We will complete the proof by showing that (1)  $\Leftrightarrow$  (3).

Suppose first that (1) is satisfied. Then  $\mathcal{C}_{\geq 1} = \mathcal{C}_{\geq 0}[1]$  is generated under colimits by a set of objects  $\{X_\alpha\}$ . Let  $S$  be the collection of all morphisms  $f$  in  $\mathcal{C}$  such that  $\tau_{\leq 0}(f)$  is an equivalence. Using Proposition 6.15, we conclude that  $S$  is generated by  $\{0 \rightarrow X_\alpha\}$  as a quas saturated class of morphisms, and therefore also as a strongly saturated class of morphisms (Definition T.5.5.4.5). We now apply Proposition T.5.5.4.15 to conclude that  $\mathcal{C}_{\leq 0} = S^{-1}\mathcal{C}$  is presentable; this proves (3).

We now complete the proof by showing that (3)  $\Rightarrow$  (1). If  $\mathcal{C}_{\leq -1} = \mathcal{C}_{\leq 0}[-1]$  is presentable, then Proposition T.5.5.4.16 implies that  $S$  is of small generation (as a strongly saturated class of morphisms). Proposition 6.15 implies that  $S$  is generated (as a strongly saturated class) by the morphisms  $\{0 \rightarrow X_\alpha\}_{\alpha \in A}$ , where  $X_\alpha$  ranges over the collection of all objects of  $\mathcal{C}_{\geq 0}$ . It follows that there is a small subcollection  $A_0 \subseteq A$  such that  $S$  is generated by the morphisms  $\{0 \rightarrow X_\alpha\}_{\alpha \in A_0}$ . Let  $\mathcal{D}$  be the smallest full subcategory of  $\mathcal{C}$  which contains the objects  $\{X_\alpha\}_{\alpha \in A_0}$  and is closed under colimits and extensions. Since  $\mathcal{C}_{\geq 0}$  is closed under colimits and extensions, we have  $\mathcal{D} \subseteq \mathcal{C}_{\geq 0}$ . Consequently,  $\mathcal{C}_{\leq -1}$  can be characterized as full subcategory of  $\mathcal{C}$  spanned by those objects  $Y \in \mathcal{C}$  such that  $\text{Ext}_{\mathcal{C}}^k(X, Y) = 0$  for all  $k \leq 0$  and  $X \in \mathcal{D}$ . Proposition 16.1 implies that  $\mathcal{D}$  is the collection of nonnegative objects for some accessible t-structure on  $\mathcal{C}$ . Since the negative objects of this new t-structure coincide with the negative objects of the original t-structure, we conclude that  $\mathcal{D} = \mathcal{C}_{\geq 0}$ , which proves (1).  $\square$

The following result provides a good source of examples of accessible t-structures:

**Proposition 16.4.** *Let  $\mathcal{C}$  be a presentable  $\infty$ -category, and let  $\text{Stab}(\mathcal{C})_{\leq -1}$  be the full subcategory of  $\text{Stab}(\mathcal{C})$  spanned by those objects  $X$  such that  $\Omega^\infty(X)$  is a final object of  $\mathcal{C}$ . Then  $\text{Stab}(\mathcal{C})_{\leq -1}$  determines an accessible t-structure on  $\text{Stab}(\mathcal{C})$ .*

*Proof.* Choose a small collection of objects  $\{C_\alpha\}$  which generate  $\mathcal{C}$  under colimits. We observe that an object  $X \in \text{Stab}(\mathcal{C})$  belongs to  $\text{Stab}(\mathcal{C})_{\leq -1}$  if and only if each of the spaces

$$\text{Map}_{\mathcal{C}}(C_\alpha, \Omega^\infty(X)) \simeq \text{Map}_{\text{Stab}(\mathcal{C})}(\Sigma^\infty(C_\alpha), X)$$

is contractible. Let  $\text{Stab}(\mathcal{C})_{\geq 0}$  be the smallest full subcategory of  $\text{Stab}(\mathcal{C})$  which is stable under colimits and extensions, and contains each  $\Sigma^\infty(C_\alpha)$ . Proposition 16.1 implies that  $\text{Stab}(\mathcal{C})_{\geq 0}$  is the collection of nonnegative objects of the desired t-structure on  $\text{Stab}(\mathcal{C})$ .  $\square$

**Remark 16.5.** The proof of Proposition 16.4 gives another characterization of the t-structure on  $\text{Stab}(\mathcal{C})$ : the full subcategory  $\text{Stab}(\mathcal{C})_{\geq 0}$  is generated, under extensions and colimits, by the essential image of the functor  $\Sigma^\infty : \mathcal{C} \rightarrow \text{Stab}(\mathcal{C})$ .

We conclude this section by completing the proof of Proposition 16.1.

*Proof of part (2) of Proposition 16.1.* Choose a regular cardinal  $\kappa$  such that every object of  $X_\alpha$  is  $\kappa$ -compact, and let  $\mathcal{C}^\kappa$  denote the full subcategory of  $\mathcal{C}$  spanned by the  $\kappa$ -compact objects. Let  $\mathcal{C}'^\kappa = \mathcal{C}' \cap \mathcal{C}^\kappa$ , and let  $\mathcal{C}''$  be the smallest full subcategory of  $\mathcal{C}'$  which contains  $\mathcal{C}'^\kappa$  and is closed under small colimits. The  $\infty$ -category  $\mathcal{C}''$  is  $\kappa$ -accessible, and therefore presentable. To complete the proof, we will show that  $\mathcal{C}' \subseteq \mathcal{C}''$ . For this, it will suffice to show that  $\mathcal{C}''$  is stable under extensions.

Let  $\mathcal{D}$  be the full subcategory of  $\text{Fun}(\Delta^1, \mathcal{C})$  spanned by those morphisms  $f : X \rightarrow Y$  where  $Y \in \mathcal{C}''$ ,  $X \in \mathcal{C}''[-1]$ . We wish to prove that the cokernel functor  $\text{coker} : \mathcal{D} \rightarrow \mathcal{C}$  factors through  $\mathcal{C}''$ . Let  $\mathcal{D}^\kappa$  be the full subcategory of  $\mathcal{D}$  spanned by those morphisms  $f : X \rightarrow Y$  where both  $X$  and  $Y$  are  $\kappa$ -compact objects of  $\mathcal{C}$ . By construction,  $\text{coker}|_{\mathcal{D}^\kappa}$  factors through  $\mathcal{C}''$ . Since  $\text{coker} : \mathcal{D} \rightarrow \mathcal{C}$  preserves small colimits, it will suffice to show that  $\mathcal{D}$  is generated (under small colimits) by  $\mathcal{D}^\kappa$ .

Fix an object  $f : X \rightarrow Y$  in  $\mathcal{D}$ . To complete the proof, it will suffice to show that the canonical map  $(\mathcal{D}_{/f}^\kappa)^\triangleright \rightarrow \mathcal{D}$  is a colimit diagram. Since  $\mathcal{D}$  is stable under colimits in  $\text{Fun}(\Delta^1, \mathcal{C})$  and colimits in  $\text{Fun}(\Delta^1, \mathcal{C})$  are computed pointwise (Proposition T.5.1.2.2), it will suffice to show that composition with the evaluation maps give colimit diagrams  $(\mathcal{D}_{/f}^\kappa)^\triangleright \rightarrow \mathcal{C}$ . Lemma T.5.3.5.8 implies that the maps  $(\mathcal{C}'^\kappa[-1])_{/X}^\triangleright \rightarrow \mathcal{C}$ ,  $(\mathcal{C}'^\kappa)_{/Y}^\triangleright \rightarrow \mathcal{C}$  are colimit diagrams. It will therefore suffice to show that the evaluation maps

$$(\mathcal{C}'^\kappa[-1])_{/X} \xleftarrow{\theta} (\mathcal{D}_{/f}^\kappa) \xrightarrow{\theta'} (\mathcal{C}'^\kappa)_{/Y}$$

are cofinal.

We first show that  $\theta$  is cofinal. According to Theorem T.4.1.3.1, it will suffice to show that for every morphism  $\alpha : X' \rightarrow X$  in  $\mathcal{C}'[-1]$ , where  $X'$  is  $\kappa$ -compact, the  $\infty$ -category

$$\mathcal{E}_\theta : \mathcal{D}_{/f}^\kappa \times_{\mathcal{C}'^\kappa[-1]_{/X}} (\mathcal{C}'^\kappa[-1]_{/X})_{X'}$$

is weakly contractible. For this, it is sufficient to show that  $\mathcal{E}_\theta$  is filtered (Lemma T.5.3.1.18).

We will show that  $\mathcal{E}_\theta$  is  $\kappa$ -filtered. Let  $K$  be a  $\kappa$ -small simplicial set, and  $p : K \rightarrow \mathcal{E}_\theta$  a diagram; we will extend  $p$  to a diagram  $\bar{p} : K^\triangleright \rightarrow \mathcal{E}_\theta$ . We can identify  $p$  with two pieces of data:

- (i) A map  $p' : K^\triangleleft \rightarrow \mathcal{C}'^\kappa[-1]_{/X}$ .
- (ii) A map  $p'' : (K \star \{\infty\}) \times \Delta^1 \rightarrow \mathcal{C}$ , with the properties that  $p''|(K \star \{\infty\}) \times \{0\}$  can be identified with  $p'$ ,  $p''|\{\infty\} \times \Delta^1$  can be identified with  $f$ , and  $p''|K \times \{1\}$  factors through  $\mathcal{C}'^\kappa$ .

Let  $\bar{p}' : (K^\triangleright)^\triangleright \rightarrow \mathcal{C}'^\kappa[-1]_{/X}$  be a colimit of  $p'$ . To complete the proof that  $\mathcal{E}_\theta$  is  $\kappa$ -filtered, it will suffice to show that we can find a compatible extension  $\bar{p}'' : (K^\triangleright \star \{\infty\}) \times \Delta^1 \rightarrow \mathcal{C}$  with the appropriate properties. Let  $L$  denote the full simplicial subset of  $(K^\triangleright \star \{\infty\}) \times \Delta^1$  spanned by every vertex except  $(v, 1)$ , where  $v$  denotes the cone point of  $K^\triangleright$ . We first choose a map  $q : L \rightarrow \mathcal{C}$  compatible with  $p''$  and  $\bar{p}'$ . This is equivalent to solving the lifting problem

$$\begin{array}{ccc} & & \mathcal{C}_{/f} \\ & \nearrow \text{dashed} & \downarrow \\ K^\triangleright & \longrightarrow & \mathcal{C}_{/X}, \end{array}$$

which is possible since the vertical arrow is a trivial fibration. Let  $L' = L \cap (K^\triangleright \times \Delta^1)$ . Then  $q$  determines a map  $q_0 : L' \rightarrow \mathcal{C}_{/Y}$ . Finding the desired extension  $\bar{p}''$  is equivalent to finding a map  $\bar{q}_0 : L'^\triangleright \rightarrow \mathcal{C}_{/Y}$ , which carries the cone point into  $\mathcal{C}'^\kappa$ .

Let  $g : Z \rightarrow Y$  be a colimit of  $q_0$  (in the  $\infty$ -category  $\mathcal{C}_{/Y}$ ). We observe that  $Z$  is a  $\kappa$ -small colimit of  $\kappa$ -compact objects of  $\mathcal{C}$ , and therefore  $\kappa$ -compact. Since  $Y \in \mathcal{C}''$ ,  $Y$  can be written as the colimit of a  $\kappa$ -filtered diagram  $\{Y_\alpha\}$ , taking values in  $\mathcal{C}'^\kappa$ . Since  $Z$  is  $\kappa$ -compact, the map  $g$  factors through some  $Y_\alpha$ ; it follows that there exists an extension  $\bar{q}_0$  as above, which carries the cone point to  $Y_\alpha$ . This completes the proof that  $\mathcal{E}_\theta$  is  $\kappa$ -filtered, and also the proof that  $\theta$  is cofinal.

The proof that  $\theta'$  is cofinal is similar but slightly easier: it suffices to show that for every map  $Y' \rightarrow Y$  in  $\mathcal{C}'$ , where  $Y'$  is  $\kappa$ -compact, the fiber product

$$\mathcal{E}_{\theta'} = \mathcal{D}_{/f}^\kappa \times_{\mathcal{C}'_{/Y}} (\mathcal{C}'^\kappa_{/Y})_{Y' /}$$

is filtered. For this, we can either argue as above, or simply observe that  $\mathcal{E}_{\theta'}$  admits  $\kappa$ -small colimits.  $\square$

## References

- [1] Arone, G. and M. Ching. *Operads and Chain Rules for the Calculus of Functors*. Preprint.
- [2] Artin, M. *Théorie des topos et cohomologie étale des schémas*. SGA 4. Lecture Notes in Mathematics 269, Springer-Verlag, Berlin and New York, 1972.
- [3] Artin, M. and B. Mazur. *Étale Homotopy*. Lecture Notes in Mathematics 100, Springer-Verlag, Berlin and New York, 1969.
- [4] Bastera, M. *André-Quillen cohomology of commutative  $S$ -algebras*. Journal of Pure and Applied Algebra 144 (1999) no. 2, 111-143.
- [5] Behrend, K. and B. Fantechi. *The intrinsic normal cone*. Inventiones Mathematicae 128 (1997) no. 1, 45-88.
- [6] Beilinson, A. , Bernstein, J. and P. Deligne. *Faisceaux pervers*. Asterisque 100, Volume 1, 1982.
- [7] Bergner, J.E. *A Model Category Structure on the Category of Simplicial Categories*. Transactions of the American Mathematical Society 359 (2007), 2043-2058.
- [8] Bergner, J.E. *A survey of  $(\infty, 1)$ -categories*. Available at math.AT/0610239
- [9] Bergner, J.E. *Rigidification of algebras over multi-sorted theories*. Algebraic and Geometric Topology 7, 2007.
- [10] Bergner, J.E. *Three models for the homotopy theory of homotopy theories*, Topology 46 (2007), 397-436.
- [11] Bosch, Guntzer, U., and R. Remmert, R. *Non-Archimedean Analysis: a Systematic Approach to Rigid Analytic Geometry*. Springer-Verlag, Berlin and Heidelberg, 1984.
- [12] Bousfield, A.K. and D.M. Kan. *Homotopy limits, completions, and localizations*. Lecture Notes in Mathematics 304, Springer-Verlag, 1972.
- [13] Cisinski, D-C and I. Moerdijk. *Dendroidal sets as models for homotopy operads*. Available for download as arXiv:0902.1954v1.
- [14] Dugger, D. *Combinatorial model categories have presentations*. Advances in Mathematics 164, 2001, 177-201.
- [15] Eilenberg, S. and N.E. Steenrod. *Axiomatic approach to homology theory*. Proc. Nat. Acad. Sci. U.S.A. 31, 1945, 117-120.
- [16] Eisenbud, D. *Commutative algebra*. Springer-Verlag, New York, 1995.
- [17] Elmendorf, A.D., Kriz, I. , Mandell, M.A., and J.P. May. *Rings, modules and algebras in stable homotopy theory*. Mathematical Surveys and Monographs 47, American Mathematical Society, 1997.
- [18] Fulton, W. *Algebraic curves*. W.A. Benjamin, Inc., New York, 1969.
- [19] Goerss, P. and J.F. Jardine. *Simplicial Homotopy Theory*. Progress in Mathematics, Birkhauser, Boston, 1999.
- [20] Goodwillie, T. *Calculus I: The first derivative of pseudoisotopy theory*. K-Theory 4 (1990), no. 1, 127.
- [21] Goodwillie, T. *Calculus. II. Analytic functors*. K-Theory 5 (1991/92), no. 4, 295-332.
- [22] Goodwillie, T. *Calculus III: Taylor Series*. Geometry and Topology, Volume 7 (2003) 645-711.



- [23] Grauert, H. and R. Remmert. *Theory of Stein Spaces*. Springer-Verlag, Berlin Heidelberg, 2004.
- [24] Gunning, R. and H. Rossi. *Analytic functions of several complex variables*. Prentice-Hall, Englewood Cliffs, N.J, 1965.
- [25] Hatcher, A. *Algebraic Topology*. Cambridge University Press, 2002.
- [26] Hook, E.C. *Equivariant cobordism and duality*. Transactions of the American Mathematical Society 178 (1973) 241-258.
- [27] Hovey, M. *Model Categories*. Mathematical Surveys and Monographs 63, AMS, Providence, RI, 1999.
- [28] Hovey, M., Shipley, B. and J. Smith. *Symmetric spectra*. Journal of the American Mathematical Society 13, 2000, no. 1, 149-208.
- [29] Illusie, L. *Complexe cotangent et déformations I*. Lecture Notes in Mathematics 239, Springer-Verlag, 1971.
- [30] Illusie, L. *Complexe cotangent et déformations II*. Lecture Notes in Mathematics 283, Springer-Verlag, 1972.
- [31] Joyal, A. *Notes on quasi-categories*.
- [32] Joyal, A. *Simplicial categories vs. quasi-categories*.
- [33] Joyal, A. and M. Tierney. *Quasi-categories vs. Segal Spaces*. Preprint available at math.AT/0607820.
- [34] Kerz, M. *The complex of words and Nakaoka stability*. Homology, Homotopy and Applications, volume 7(1), 2005, pp. 77-85.
- [35] Klein, J. and J. Rognes. *A chain rule in the calculus of homotopy functors*. Geom. Topol. 6 (2002), 853887.
- [36] Knutson, D. *Algebraic spaces*. Lecture Notes in Mathematics 203, Springer-Verlag, 1971.
- [37] Laplaza, M. *Coherence for distributivity*. Coherence in categories, 29-65. Lecture Notes in Mathematics 281, Springer-Verlag, 1972.
- [38] Laumon, G. and L. Moret-Bailly. *Champs algébriques*. Springer-Verlag, 2000.
- [39] Lazard, Daniel. *Sur les modules plats*. C.R. Acad. Sci. Paris 258, 1964, 6313-6316.
- [40] Lurie, J. *Higher Topos Theory*. Available for download at <http://www.math.harvard.edu/~lurie/> .
- [41] Lurie, J. *Derived Algebraic Geometry I: Stable  $\infty$ -Categories*. Available for download.
- [42] Lurie, J. *Derived Algebraic Geometry II: Noncommutative Algebra*. Available for download.
- [43] Lurie, J. *Derived Algebraic Geometry III: Commutative Algebra*. Available for download.
- [44] Lurie, J. *Derived Algebraic Geometry IV: Deformation Theory*. Available for download.
- [45] Lurie, J. *Derived Algebraic Geometry V: Structured Spaces*. Available for download.
- [46] Lurie, J. *Derived Algebraic Geometry VI:  $\mathbb{E}[k]$ -Algebras*. Available for download.
- [47] Lurie, J. *Derived Algebraic Geometry VII: Spectral Schemes*. In preparation.
- [48] Lurie, J.  *$(\infty, 2)$ -Categories and the Goodwillie Calculus I*. Available for download.

- [49] Lurie, J.  $(\infty, 2)$ -categories and the Goodwillie Calculus II. In preparation.
- [50] Lurie, J. *Elliptic curves in spectral algebraic geometry*. In preparation.
- [51] Lurie, J. *Toric varieties, elliptic cohomology at infinity, and loop group representations*. In preparation.
- [52] MacLane, S. *Categories for the Working Mathematician*. Second edition. Graduate Ttxts in Mathematics, 5. Springer-Verlag, New York, 1998.
- [53] May, J.P. *The Geometry of Iterated Loop Spaces*.
- [54] Mitchell, B. *A quick proof of the Gabriel-Popesco theorem*. Journal of Pure and Applied Algebra 20 (1981), 313-315.
- [55] Neeman, A. *Triangulated categories*. Annals of Mathematics Studies, 148. Princeton University Press, 2001.
- [56] Quillen, D. *Homotopical Algebra*. Lectures Notes in Mathematics 43, SpringerVerlag, Berlin, 1967.
- [57] Rezk, C. *A model for the homotopy theory of homotopy theory*. Transactions of the American Mathematical Society 35 (2001), no. 3, 973-1007.
- [58] Rosicky, J. *On Homotopy Varieties*. Advances in Mathematics 214, 2007 no. 2, 525-550.
- [59] Schwede, S. *Spectra in model categories and applications to the algebraic cotangent complex*. Journal of Pure and Applied Algebra 120 (1997), pp. 77-104.
- [60] Schwede, S. and B. Shipley. *Algebras and Modules in Monoidal Model Categories*. Proceedings of the London Mathematical Society (80) 2000, 491-511.
- [61] Schwede, S. and B. Shipley. *Stable model categories are categories of modules*. Topology 42, 2003, no. 1, 103-153.
- [62] Serre, Jean-Pierre. *Local algebra*. Springer-Verlag, 2000.
- [63] Shipley, B. *A Convenient Model Category for Commutative Ring Spectra*. Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic  $K$ -theory. Contemp. Math. volume 346 pp. 473-483, American Mathematical Society, Providence, RI, 2004.
- [64] Spivak, D. *Quasi-smooth Derived Manifolds*. PhD dissertation.
- [65] Srinivas, V. *Algebraic K-Theory*. Birkhauser, Boston, 1993.
- [66] Toën, B. *Champs affines*. Available for download: [math.AG/0012219](https://arxiv.org/abs/math/0612219).
- [67] Toën, B. *Vers une axiomatisation de la théorie des catégories supérieures*. K-theory 34 (2005), no. 3, 233-263.
- [68] Toën, B. and G. Vezzosi. *From HAG to DAG: derived moduli stacks*. Available for download: [math.AG/0210407](https://arxiv.org/abs/math/0612407).
- [69] Toën, B. and G. Vezzosi. *Algebraic geometry over model categories*. Available for download: [math.AG/0110109](https://arxiv.org/abs/math/0612109).
- [70] Toën, B. and G. Vezzosi. *"Brave New" Algebraic Geometry and global derived moduli spaces of ring spectra*. Available for download: [math.AT/0309145](https://arxiv.org/abs/math/0309145).
- [71] Toën, B. and G. Vezzosi. *Segal topoi and stacks over Segal categories*. Available for download: [math.AG/0212330](https://arxiv.org/abs/math/0212330).

- [72] Toën, B. and G. Vezzosi. *A remark on K-theory and S-categories*. *Topology* 43, No. 4 (2004), 765-791
- [73] Verity, D. *Weak complicial sets, a simplicial weak omega-category theory. Part I: basic homotopy theory*.
- [74] Verity, D. *Weak complicial sets, a simplicial weak omega-category theory. Part II: nerves of complicial Gray-categories*.
- [75] Weibel, C. *An Introduction to Homological Algebra*. Cambridge University Press, 1995.