Whitehead Triangulations (Lecture 3)

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In the last lecture, we cited the theorem of Cannon-Edwards which shows that the 5-sphere S^5 admits "bad" triangulations: that is, S^5 can be realized as the underlying topological space of polyhedra which are not piecewise linear manifolds. In this lecture, we will see that such a triangulation is necessarily "wild" in the sense that the simplices are not smoothly embedded in S^5. To be more precise, we need to introduce some terminology.

Definition 1. Let K be a polyhedron and M a smooth manifold. We say that a map $f : K \to M$ *piecewise differentiable* (PD) if there exists a triangulation of K such that the restriction of f to each simplex is smooth. We will say that f is a *PD homeomorphism* if f is piecewise differentiable, a homeomorphism, and the restriction of f to each simplex has injective differential at each point.

The problems of smoothing and triangulating manifolds can now be formulated as follows:

(i) Given a smooth manifold M, does there exist a piecewise linear manifold N and a PD homeomorphism $N \to M$?

(ii) Given a piecewise linear manifold N, does there exist a smooth manifold M and a PD homeomorphism $N \to M$?

Question (i) is much easier, and was addressed by Whitehead in the first half of the last century. More precisely, Whitehead proved the following:

(1) Given a smooth manifold M, there exists a polyhedron K and a PD homeomorphism $K \to M$.

(2) Any such polyhedron K is automatically a piecewise linear manifold.

(3) The polyhedron K is unique up to PL homeomorphism.

Remark 2. Whitehead actually worked in the context of C^1 maps, rather than the infinitely differentiable maps considered here. The distinction will not be important. However, the difference between C^1 maps and continuous maps is vital: as the Cannon-Edwards theorem shows, assertions (2) and (3) fail if we do not assume that our triangulations have some degree of smoothness.

Question (ii) is more difficult, and does not always have an affirmative answer. It is true provided that N has dimension ≤ 7, but false in general. Moreover, if N has dimension 7 then M need not be unique (Milnor's exotic 7-spheres provide examples). Our eventual goal is to show that if N has dimension ≤ 3, then M is unique in a very strong homotopy-theoretic sense.

Our goal for this week is to prove Whitehead's theorems. We will begin with part (2), which asserts that the existence of a piecewise differentiable homeomorphism $f : K \to M$ implies that K is a PL manifold. This question is local, so we may replace M by a Euclidean space $\mathbb{R}^n$. To prove that K is a piecewise linear manifold, it will suffice to show that near every point $x \in K$, we can choose a PD map $f' : K \to \mathbb{R}^n$ which is piecewise linear in a neighborhood of x. We will prove this in two steps:

Proposition 3. *Let K be a polyhedron and let $f : K \to \mathbb{R}^n$ be a PD map. Let K_0 be a finite subpolyhedron. Then there exists another map $f' : K \to \mathbb{R}^n$ with the following properties:*

(1) *The map f' is an arbitrarily good approximation to f in the C^1-sense: that is, we may assume that there is a triangulation S of K such that for each simplex σ of S, both $f|\sigma$ and $f'|\sigma$ are smooth, and $(f - f')|\sigma$ can be chosen to have arbitrarily small values and arbitrarily small first derivatives.*

(2) *The restriction $f'|K_0$ is piecewise linear.*

(3) *The maps f and f' coincide outside of a compact subset of K.*

Proposition 4. *Let $f, f' : K \to \mathbb{R}^n$ be PD maps. Suppose that f is a PD homeomorphism and that f' is a sufficiently good approximation to f in the C^1-sense. Then f is a homeomorphism onto an open subset of $\mathbb{R}^n$.*

We begin with the proof of Proposition 4, since it is easier. The proof is based on the following classical result from point-set topology:

Theorem 5 (Brouwer)**.** *Let $g : M \to N$ be a continuous injective map between topological n-manifolds. Then g is a homeomorphism from M onto some open subset of N.*

Proof of Proposition 4. Since f is a homeomorphism, K is a topological manifold. Consequently, by Theorem 5, it will suffice to show that f' is injective. This is equivalent to the assertion that the map $g = f' \circ f^{-1}$ is injective. Choose a triangulation S of K such that f and f' are smooth on each simplex of S. For each simplex σ of S, the map g is smooth when restricted to the simplex $\sigma' = f(\sigma)$. We will assume that f' is a sufficiently good approximation to f that for each $x \in \sigma'$, the derivative $D_x(g|\sigma) = \mathrm{id}_{\mathbb{R}^n} + A_{x,\sigma}$ for some linear map $A_{x,\sigma} : \mathbb{R}^n \to \mathbb{R}^n$ having operator norm $\leq \frac{1}{2}$. To show that g is injective, it will suffice to prove the following estimate:

$$(g(x) - g(y), x - y) \geq \frac{(x - y, x - y)}{2}.$$

The collection of pairs $x, y \in \mathbb{R}^n$ which satisfy this condition is closed. It will therefore suffice to prove that this condition holds for a dense set of pairs $x, y \in \mathbb{R}^n$.

Let us say that a pair of elements $x, y \in \mathbb{R}^n$ is *good* if the closed interval $\overline{xy}$ is transverse to the PD triangulation of $\mathbb{R}^n$ provided by the map f. We note that the collection of pairs (x, y) which are not good is the image in $\mathbb{R}^{2n}$ of a countably many smooth maps whose domains are manifolds of dimension $2n - 1$, and therefore has measure zero (by Sard's theorem). It follows that the collection of good pairs is dense in $\mathbb{R}^{2n}$.

Suppose now that (x, y) is good, and let $h : [0, 1] \to \mathbb{R}$ be the map defined

$$h(t) = (g(x) - g((1 - t)x + ty), x - y).$$

Then $h(t)$ is a piecewise differentiable function of t, and $h(0) = 0$. We wish to prove that $h(1) \geq \frac{(x-y,x-y)}{2}$. It will suffice to show that the derivative h' (which is defined at all but finitely many points) satisfies the inequality

$$h'(t) \geq \frac{(x - y, x - y)}{2}.$$

Choose a simplex σ such that $z = (1 - t)x + ty \in \sigma'$; then we can write

$$h'(t) = (D_z(g)(x - y), x - y) = (x - y, x - y) + (A_{z,\sigma}(x - y), x - y) \geq \frac{(x - y, x - y)}{2}$$

as desired. □

The proof of Proposition 3 is more difficult. First, choose a PL map $\chi : K \to [0, 1]$ supported in a compact subset $K_1 \subseteq K$ such that $\chi(x) = 1$ for $x \in K_0$. If $f'' : K_1 \to \mathbb{R}^n$ is a PL map which closely approximates $f|K_1$, then the map $f' = \chi f'' + (1 - \chi)f$ satisfies the conditions of Proposition 3. It will therefore suffice to prove the following:

Proposition 6. *Let K be a finite polyhedron and let $f : K \to \mathbb{R}^n$ be a PD map. Then there exists a piecewise linear map $f' : K \to \mathbb{R}^n$ which is an arbitrarily good approximation to f (in the C^1-sense).*

To prove Proposition 6, we need a way of producing piecewise linear maps.

Definition 7. Let K be a polyhedron equipped with a triangulation $S = \{\sigma_i\}$ and let $f : K \to \mathbb{R}^n$ be a map. We define the map $L_f^S : K \to \mathbb{R}^n$ so that the following conditions are satisfied:

(1) For every point $x \in K$ which is a vertex of the triangulation S, we have $L_f^S(x) = f(x)$.

(2) The restriction of L_f^S to each simplex σ of the triangulation S is a linear map $\sigma \to \mathbb{R}^n$.

It is easy to see that for any map $f : K \to \mathbb{R}^n$, the map L_f^S is well-defined and piecewise linear. To prove Proposition 6, we need to show that we can choose the triangulation S such that L_f^S is a good approximation to f (in the C^1-sense). First, fix a triangulation S_0 of K such that the restriction of f to each simplex of S_0 is smooth. Fix $\epsilon > 0$. Refining the triangulation S_0 if necessary, we may assume that f carries each simplex σ of S_0 into an open ball U_σ of radius ϵ. If S refines S_0, then a convexity argument shows that L_f^S carries σ into U_σ, so that $|L_f^S(x) - f(x)|$ is bounded above by 2ϵ. Thus, L_f^S is a good approximation to f in the C^0-sense for any sufficiently fine triangulation.

To guarantee that L_f^S is a good approximation to f in the C^1-sense, we need to work a bit harder. Let us identify K with a finite polyhedron embedded in Euclidean space $\mathbb{R}^m$. For each simplex σ of K (which we will assume is contained in a simplex of S_0), we define the *diameter* $d(\sigma)$ of σ to be the supremum of the distance between any two points of σ (by a convexity argument, this coincides with the length of the longest side of σ). We define the *radius* $r(\sigma)$ to be the distance from the barycenter of σ to the boundary of σ. We define the *thickness* $t(\sigma)$ to be the ratio $\frac{r(\sigma)}{d(\sigma)}$.

We will need the following fact:

Lemma 8. *Let $K \subseteq \mathbb{R}^m$ be a finite polyhedron equipped with a triangulation S_0. Then there exists a positive constant $\delta \leq 1$ such that K has arbitrarily fine triangulations S (in other words, triangulations such that the each simplex has diameter $\leq \epsilon$, for any $\epsilon > 0$) refining S_0 such that each simplex of S has thickness $\geq \delta$.*

Proof. We first note that the claim is independent of the choice of embedding $K \to \mathbb{R}^m$: an embedding $K \to \mathbb{R}^{m'}$ which is linear on each simplex of S_0 (or any triangulation refining S_0) can change the widths of simplices contained in simplices of S_0 by at most a bounded factor.

Let $\{x_1, \ldots, x_k\}$ be the set of vertices of the triangulation S_0, and let $\{y_1, \ldots, y_k\}$ be a linearly independent set in $\mathbb{R}^k$. Then there exists a unique map $K \to \mathbb{R}^k$ which is linear on each simplex of S_0 and carries each x_i to y_i. This map is a PL embedding and its image is a union of faces of the simplex spanned by $\{y_1, \ldots, y_k\}$. Replacing m by k and K by its image in $\mathbb{R}^k$, we may assume that K is a union of faces of some linearly embedded simplex (with its standard triangulation). Enlarging K if necessary, we may suppose that K is itself a simplex Δ^n, where $n = k - 1$.

The existence of the desired triangulations is now a consequence of the following assertion:

($*$) For each $n \geq 0$, there exists a tesselation of Euclidean space $\mathbb{R}^n$ by n-simplices, all congruent to one another, such that multiplication by any integer gives a map $\mathbb{R}^n \to \mathbb{R}^n$ which is a refinement of tesselations.

This proves that the n-simplex admits arbitrarily fine subdivisions into pieces which are similar to itself, thereby providing fine triangulations of Δ^n whose simplices have their width bounded below.

One way to prove ($*$) is to identify $\mathbb{R}^n$ with the Lie algebra of a maximal torus of any compact simplex Lie group G of rank n, and choose the tesselation of $\mathbb{R}^n$ by Weyl alcoves. □

We now continue to fix $\epsilon > 0$, and let δ be as in Lemma 8. Let σ be a k-simplex of K which is contained in a simplex of S_0. Then σ has a tangent plane which we may identify with a subspace $V_\sigma \subseteq \mathbb{R}^m$ of dimension k. The restriction $f|\sigma$ is smooth, and therefore has a differential $D(f|\sigma) : \sigma \to \mathrm{Hom}(V_\sigma, \mathbb{R}^n)$. Choose a triangulation S of K refining S_0 with the following properties:

(*i*) The triangulation S satisfies the hypothesis of Lemma 8.

(*ii*) For each simplex σ of S and each pair of elements $x, y \in \sigma$, we have $|D_x(f|\sigma) - D_y(f|\sigma)| \leq \frac{\epsilon\delta}{4m}$.

(Since the functions $D(f|\sigma)$ are continuous on each simplex σ of S_0, assertion (*ii*) holds for any sufficiently fine refinement of S_0). We will prove that $|D_x(f|\sigma) - D_x(L_f^S|\sigma)| \leq \epsilon$ for each $x \in \sigma \in S$.

Let σ be a k-dimensional simplex given as the convex hull of a set of points $\{v_0, \ldots, v_k\} \in \mathbb{R}^m$. The proof proceeds in several steps:

(*a*) Since L_f^S is linear on σ, we have $D_x(L_f^S|\sigma) = D_{v_0}(L_f^S|\sigma)$. It will therefore suffice to prove the inequalities

$$|D_x(f|\sigma) - D_{v_0}(f|\sigma)| \leq \frac{\epsilon}{2}$$

$$|D_{v_0}(f|\sigma) - D_{v_0}(L_f^S|\sigma)| \leq \frac{\epsilon}{2}.$$

The first of these follows immediately from assumption (2).

(*b*) Let $A = D_{v_0}(f|\sigma) - D_{v_0}(L_f^S|\sigma)$. It will suffice to prove that if $q \in V_\sigma$ is a vector of length $\leq r(\sigma)$, then $|A(q)| \leq \frac{r(\sigma)\epsilon}{2}$.

(*c*) Since $r(\sigma) \geq d(\sigma)\delta$, it will suffice to show that $|A(q)| \leq \frac{d(\sigma)\delta\epsilon}{2}$.

(*d*) Let v be the barycenter of σ. Since $|q| \leq r(\sigma)$, we have $v, v+q \in \sigma$, so that $|A(q)| \leq |A(v - v_0)| + A(v + q - v_0)|$. It will therefore suffice to show that if $v_0 + w \in \sigma$, then $|A(w)| \leq \frac{d(\sigma)\delta\epsilon}{4}$.

(*e*) If $v_0 + w \in \sigma$, then we can write $w = \sum_i c_i(v_i - v_0)$ where $0 \leq c_i \leq 1$. It will therefore suffice to show that $|A(v_i - v_0)| \leq \frac{d(\sigma)\delta\epsilon}{4m}$ (since $k \leq m$).

(*f*) We have

$$\begin{aligned} A(v_i - v_0) &= D_{v_0}(f|\sigma)(v_i - v_0) - D_{v_0}(L_f^S|\sigma)(v_i - v_0) \\ &= D_{v_0}(f|\sigma)(v_i - v_0) + f(v_0) - f(v_i) \\ &= \int_0^1 (D_{v_0}(f|\sigma) - D_{tv_0+(1-t)v_i}(f|\sigma))(v_i - v_0)dt. \end{aligned}$$

Since $|v_i - v_0| \leq d(\sigma)$, we can apply (*ii*) to deduce that $|A(v_i - v_0)| \leq \frac{d(\sigma)\delta\epsilon}{4m}$ which completes the proof.