The Krull Filtration (Lecture 37)

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Let $A$ be a commutative Noetherian ring. Recall that the Zariski spectrum $\text{Spec } A$ is defined to be the set of all prime ideals $\{p \subseteq A\}$. Let $\text{Mod}_A$ denote the category of $A$-modules. It is possible to recover $\text{Spec } A$ directly from the category $\text{Mod}_A$. For this, we need to recall a few definitions and facts:

**Definition 1.** Let $\mathcal{C}$ be a Grothendieck abelian category. An object $X \in \mathcal{C}$ is **Noetherian** if every ascending chain of subobjects of $X$ eventually stabilizes. We say that $\mathcal{C}$ is **locally Noetherian** if every object of $\mathcal{C}$ is the direct limit of its Noetherian subobjects.

An object $I \in \mathcal{C}$ is **injective** if the functor $M \mapsto \text{Hom}_\mathcal{C}(M, I)$ is exact. We say that an injective object $I$ is **indecomposable** if, whenever $I$ is written as a direct sum $I \simeq I' \oplus I''$, either $I'$ or $I''$ is zero.

Let $X \in \mathcal{C}$ be an object. An **injective hull** of $X$ is a monomorphism $X \rightarrow I$ such that $I$ is injective, and every nonzero subobject $I' \subseteq I$ satisfies $I' \times_X X \neq 0$.

**Proposition 2.** Let $\mathcal{C}$ be a locally Noetherian abelian category. Then:

1. Every object $M \in \mathcal{C}$ admits an injective hull $M \rightarrow I$. Moreover, $I$ is uniquely determined up to (noncanonical) isomorphism. If $M$ is simple, then $I$ is indecomposable.

2. Every direct sum $\oplus_{\alpha} I_\alpha$ of injective objects is injective.

3. Every injective object $I \in \mathcal{C}$ can be obtained as a direct sum $\oplus_{\alpha} I_\alpha$, where each summand $I_\alpha$ is an indecomposable injective.

This motivates the following definition:

**Definition 3.** Let $\mathcal{C}$ be a locally Noetherian abelian category. Then we let $\text{Spec } \mathcal{C}$ denote the collection of all isomorphism classes of indecomposable injective objects of $\mathcal{C}$.

**Remark 4.** A priori, the collection $\text{Spec } \mathcal{C}$ might be very large, since $\mathcal{C}$ has a proper class of injective objects. However, if $I$ is an indecomposable injective object of $\mathcal{C}$, then $I$ can be regarded as the injective hull of any nonzero submodule $I_0 \subseteq I$. In particular, $I$ can be regarded as the injective hull of a Noetherian object of $\mathcal{C}$. It follows that $\text{Spec } \mathcal{C}$ is actually a set.

**Example 5.** Let $A$ be a Noetherian ring. Then there is a canonical bijection

$$\text{Spec } A \rightarrow \text{Spec } \text{Mod}_A$$

which carries a prime ideal $p \subseteq A$ to the injective hull of the $A$-module $A/p$.

For example, if $A = \mathbb{Z}$, then the indecomposable injective objects of $\text{Mod}_A$ are precisely the abelian groups $\mathbb{Q}$ and $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, where $p$ is a prime number.

**Example 6.** Let $\mathcal{U}$ denote the category of unstable Steenrod modules. The simple objects of $\mathcal{U}$ are precisely the modules $\Sigma^k \mathbb{F}_2$, where $k \geq 0$. The injective hull of $\Sigma^k \mathbb{F}_2$ can be identified with the Brown-Gitler module $J(k)$. 

If \( A \) is a Noetherian ring, then \( \text{Spec } A \) has a good deal more structure than just that of a set. For example, we can (at least in good cases) assign a \textit{Krull dimension} to every point of \( \text{Spec } A \). The points of Krull dimension zero correspond to the maximal ideals of \( A \). Note that the collection of maximal ideals of \( A \) can be described very simply in terms of \( \text{Mod}_A \): they are isomorphism classes of simple objects of \( \text{Mod}_A \) (more precisely, an \( A \)-module \( M \) is simple if and only if it is isomorphic to a quotient \( A/\mathfrak{m} \), where \( \mathfrak{m} \) is a maximal ideal of \( A \)). Therefore, the corresponding points of \( \text{Spec } \text{Mod}_A \) are precisely the injective hulls of the simple objects of \( A \). We now wish to generalize this picture to more general categories.

\textbf{Definition 7.} Let \( \mathcal{C} \) be a locally Noetherian abelian category. Then \( \text{Krull}^0(\mathcal{C}) \) is the smallest Serre class in \( \mathcal{C} \) which contains every simple object in \( \mathcal{C} \).

\textbf{Remark 8.} If \( \mathcal{C} \neq 0 \), then \( \text{Krull}^0(\mathcal{C}) \neq 0 \). In other words, \( \mathcal{C} \) contains a simple object. To prove this, choose a nonzero object \( M \in \mathcal{C} \). Since \( \mathcal{C} \) is locally Noetherian, \( M \) is the union of its Noetherian subobjects. We may therefore assume that \( M \) is Noetherian. Let \( M_0 \) be a maximal proper submodule of \( M \). Then \( M/M_0 \) is a simple object of \( \mathcal{C} \).

\textbf{Proposition 9.} Let \( \mathcal{C} \) be a locally Noetherian abelian category, and let \( I \) be an injective object of \( \mathcal{C} \). Then exactly one of the following statements holds:

1. The object \( I \) is the injective hull of a simple object \( C \in \mathcal{C} \) (which is then determined up to isomorphism).

2. The object \( I \) belongs to \( \mathcal{C}/\text{Krull}^0(\mathcal{C}) \) (and is injective as an object of \( \mathcal{C}/\text{Krull}^0(\mathcal{C}) \)).

\textit{Proof.} Let \( \mathcal{C}_0 = \{ C \in \mathcal{C} : \text{Hom}_\mathcal{C}(C, I) = 0 \} \). Since \( I \) is injective, \( \mathcal{C}_0 \) is a Serre class in \( \mathcal{C} \).

By definition, \( I \) belongs to \( \mathcal{C}/\text{Krull}^0(\mathcal{C}) \) if and only if, for every object \( C \in \text{Krull}^0(\mathcal{C}) \), we have \( \text{Hom}_\mathcal{C}(C, I) = \text{Ext}_\mathcal{C}(C, I) = 0 \). The second equality is automatic, since \( I \) is injective, and the first is equivalent to the assertion that \( C \in \mathcal{C}_0 \). In other words, \( I \in \mathcal{C}/\text{Krull}^0(\mathcal{C}) \) if and only if \( \text{Krull}^0(\mathcal{C}) \subseteq \mathcal{C}_0 \). Consequently, (2) holds if and only if \( \text{Hom}_\mathcal{C}(C, I) = 0 \) for every simple object \( C \in \mathcal{C} \).

Suppose that (2) does not hold, and choose a nonzero map \( f : C \to I \) where \( C \) is simple. Then \( f \) must be a monomorphism. Choose an injective hull \( C \subseteq I' \). Since \( I \) is injective, we can extend \( f \) to a map \( \overline{f} : I' \to I \). Since \( \text{ker}(\overline{f}) \cap C \cong \text{ker}(f) \cong 0 \), we deduce that \( \overline{f} \) is injective. Since \( I' \) is injective, the injective map \( \overline{f} \) splits and we get an isomorphism \( I \cong I' \oplus I'' \). Since \( I \) is indecomposable, \( I'' \cong 0 \) so that \( \overline{f} \) is an isomorphism. This proves (1), except for the uniqueness of \( C \). To establish the uniqueness, we note that given injective maps

\[ C \hookrightarrow I \hookrightarrow D, \]

the intersection \( C \times_I D \) can be regarded as a nonzero submodule of both \( C \) and \( D \). If \( C \) and \( D \) are simple, this gives isomorphisms

\[ C \hookrightarrow C \times_I D \hookrightarrow D. \]

This motivates the following definition:

\textbf{Definition 10.} Let \( \mathcal{C} \) be a Grothendieck abelian category. For each \( n > 0 \), we let \( \text{Krull}^n(\mathcal{C}) \) denote the inverse image of \( \text{Krull}^0(\mathcal{C}/\text{Krull}^{n-1}(\mathcal{C})) \) under the localization functor

\[ L : \mathcal{C} \to \mathcal{C}/\text{Krull}^{n-1}(\mathcal{C}). \]

We will say that an indecomposable injective \( I \in \text{Spec } \mathcal{C} \) has \textit{Krull dimension} \( > n \) if \( I \) belongs to \( \mathcal{C}/\text{Krull}^n(\mathcal{C}) \).

We have a filtration of \( \mathcal{C} \) by Serre classes

\[ \text{Krull}^0(\mathcal{C}) \subseteq \text{Krull}^1(\mathcal{C}) \subseteq \text{Krull}^2(\mathcal{C}) \subseteq \ldots \]

By construction, each of the successive quotients \( \text{Krull}^{n+1}(\mathcal{C})/\text{Krull}^n(\mathcal{C}) \) is generated by simple objects.
Remark 11. If $A$ is a well-behaved commutative ring (such as a finitely generated algebra over a field), then the Krull filtration above is finite: we have $\text{Krull}^n(M) = M$ as soon as $n \geq \dim(A)$. In general, the filtration need not terminate nor exhaust $\mathcal{C}$ (to obtain the whole of $\mathcal{C}$, one needs to define an analogous filtration indexed by the ordinals).

We wish to study the Krull filtration on the abelian category $\mathcal{U}$ of unstable $A$-modules. We begin by determining $\text{Krull}^0(A)$.

Definition 12. An unstable $A$-module $M$ is locally finite if, for each $x \in M$, the cyclic submodule $A \cdot x \subseteq M$ has finite dimension over $F$.

Proposition 13. An unstable $A$-module $M$ belongs to $\text{Krull}^0(\mathcal{U})$ if and only if $M$ is locally finite.

Proof. We first observe that the collection of locally finite $A$-modules forms a Serre class in $\mathcal{U}$. Consequently, to prove the “only if” direction it will suffice to show that every simple $A$-module is locally finite. This follows from the characterization of simple objects given in Remark 11.

For the converse, let us suppose that $M$ is locally finite. We wish to prove that $M \in \text{Krull}^0(\mathcal{U})$. Write $M$ as the union of its finitely generated submodules $M_\alpha$. Since $\text{Krull}^0(\mathcal{U})$ is a Serre class, it will suffice to show that each $M_\alpha$ belongs to $\text{Krull}^0(\mathcal{U})$. Since $M$ is locally finite, each $M_\alpha$ is finite dimensional over $F_2$. We may therefore assume that $M$ has finite dimension over $F_2$. We now work by induction on the dimension of $M$. Let $x$ be a nonzero element of $M$ of maximal degree $k$. Then $x$ determines an exact sequence

$$0 \to \Sigma^k F_2 \to M \to M' \to 0.$$ 

By construction, we have $\Sigma^k F_2 \in \text{Krull}^0(\mathcal{U})$, and $M' \in \text{Krull}^0(\mathcal{U})$ by the inductive hypothesis. It follows that $M \in \text{Krull}^0(\mathcal{U})$, as desired.

We now wish to give another characterization of $\text{Krull}^0(\mathcal{U})$, this time using Lannes’ $T$-functor. We first observe that $H^*(BF_2)$ canonically decomposes as a direct sum $F_2 \oplus H^*_{\text{red}}(BF_2)$. Consequently, we get a canonical isomorphism of functors

$$(\bullet \otimes H^*(BF_2)) \simeq \bullet \oplus (\bullet \otimes H^*_{\text{red}}(BF_2)).$$

Passing to adjoints, we get a decomposition of functors

$$T \simeq \text{id} \oplus T$$

from the category $\mathcal{U}$ to itself. Moreover, formal properties of $T$ are inherited by $T$: for example, since $T$ is exact and commutes with suspension and $\Phi$, we deduce that $T$ is exact and commutes with suspension and $\Phi$.

Proposition 14. Let $M$ be an unstable $A$-module. Then $M \in \text{Krull}^0(\mathcal{U})$ if and only if $TM = 0$.

Proof. The “only if” direction is easy: let $\mathcal{E} = \{ M \in \mathcal{U} : TM = 0 \}$. Then $\mathcal{E}$ is a Serre class in $\mathcal{U}$. To show that $\text{Krull}^0(\mathcal{U}) \subseteq \mathcal{E}$, it suffices to show that every simple object $\Sigma^k F_2$ belongs to $\mathcal{E}$. Since $T$ commutes with suspensions, it suffices to show that $T F_2$ vanishes. This is equivalent to the assertion that $T F_2 \simeq F_2$, which was established in an earlier lecture.

The converse is much more difficult to prove. It relies on the following classification of the injective objects of $\mathcal{U}$:

Theorem 15. Every indecomposable injective object of $\mathcal{U}$ appears as a summand of $J(m) \otimes (H^*_{\text{red}}(BF_2))^\otimes n$ for some integers $m$ and $n$.

Let us assume Theorem 15 and complete the proof. Let $M \in \mathcal{U}$ be such that $TM = 0$. We wish to show that $M \in \text{Krull}^0(\mathcal{U})$. Equivalently, we wish to show that the localization functor $L : \mathcal{U} \to \mathcal{U}/\text{Krull}^0(\mathcal{U})$ annihilates $M$. If not, there exists a nonzero map $\eta \in \text{Hom}(LM, I) \simeq \text{Hom}(M, I)$, where $I$ is an indecomposable
injective of $\mathcal{U}/\text{Krull}^0(\mathcal{U})$. According to Proposition 9, we can identify $I$ with an indecomposable injective of $\mathcal{U}$ which is not the injective hull of a simple object (in other words, $I$ is not isomorphic to a Brown-Gitler module $J(m)$). Invoking Theorem 15, we get a nonzero map

$$M \to J(m) \otimes H^*_\text{red}(BF_2)^{\otimes n}$$

for some $n > 0$. This is adjoint to a nonzero map $T^n M \to J(m)$, so that $T M \neq 0$. \hfill \square

We now extend the previous result to describe each step of the Krull filtration.

**Proposition 16.** Let $M$ be an unstable $A$-module. Then $M \in \text{Krull}^n(\mathcal{U})$ if and only if $T^{n+1} M \simeq 0$.

**Proof.** The proof goes by induction on $n$, the case $n = 0$ being Proposition 14. Suppose first that $T^{n+1} M \simeq 0$. We wish to prove that $M \in \text{Krull}^n(\mathcal{U})$. Writing $M$ as the union of its finitely generated submodules, we may reduce to the case where $M$ is finitely generated. Let $L : \mathcal{U} \to \mathcal{U}/\text{Krull}^{n-1}(\mathcal{U})$ be the localization functor. We wish to show that $LM$ belongs to $\text{Krull}^0(\mathcal{U}/\text{Krull}^{n-1}(\mathcal{U}))$. For this, we will show that $LM$ has finite length in $\mathcal{U}/\text{Krull}^{n-1}(\mathcal{U})$.

By the inductive hypothesis, the functor $T^n$ factors as a composition

$$\mathcal{U} \xrightarrow{L} \mathcal{U}/\text{Krull}^{n-1}(\mathcal{U}) \xrightarrow{F} \mathcal{U}.$$

Consequently, for any subobject $N \subseteq LM$, we can identify $FN$ with a subobject of $T^n M$. Note that $T^n M$ is locally finite (by Proposition 14) and finitely generated (since $T$ preserves finitely generated objects), and therefore finite dimensional. Thus there are only finitely many possibilities for the subobject $FN \subseteq T^n M$.

But if $FN = FN' \subseteq T^n M$, then the inclusions

$$N \hookrightarrow N \cap N' \hookrightarrow N'$$

induce isomorphisms

$$FN \hookrightarrow F(N \cap N') \hookrightarrow FN'.$$

Using the inductive hypothesis, we deduce that $N = N \cap N' = N'$. Thus, there are only finitely many subobjects of $LM \in \mathcal{U}/\text{Krull}^{n-1}(\mathcal{U})$, so that $LM$ has finite length.

We now prove the reverse inclusion: $\text{Krull}^n(\mathcal{U}) \subseteq \{M : T^{n+1} M \simeq 0\}$. As before, the right side is a Serre class, to it will suffice to show that $T^{n+1} M = 0$ whenever $LM$ is a simple object of $\mathcal{U}/\text{Krull}^{n-1}(\mathcal{U})$. We have a sequence of surjective maps

$$M \to \Sigma \Omega M \to \Sigma^2 \Omega^2 M \to \ldots$$

whose colimit is zero. Since $LM$ is simple, we conclude that there exists an integer $k$ such that the map

$$LM \to \Sigma^k \Omega^k M$$

is an isomorphism and $L \Sigma^{k+1} \Omega^{k+1} M = 0$. We then have isomorphisms

$$T^n M \to T^n \Sigma^k \Omega^k M \simeq \Sigma^k T^n \Omega^k M.$$

Moreover, the inductive hypothesis implies that $\Sigma$ and $\Omega$ induce adjoint functors on the localized category $\mathcal{U}/\text{Krull}^{n-1}(\mathcal{U})$; it is not difficult to deduce from this that $L \Omega^k M$ is again simple. We may therefore replace $M$ by $\Omega^k M$, and thereby assume that $L \Sigma \Omega M \simeq 0$.

Consider the exact sequence

$$\Phi M \to M \to \Sigma \Omega M \to 0.$$

This gives rise to an exact sequence of localizations

$$L\Phi M \xrightarrow{\alpha} LM \to L\Sigma \Omega M \to 0.$$
in the category \( \mathcal{U}/\text{Krull}^{n-1}(\mathcal{U}) \). Since \( LM \) is simple and the last term vanishes, we conclude that \( \alpha \) is an epimorphism.

Applying the functor \( F \), we get an epimorphism \( T^n \Phi M \to T^n M \). Let \( N = T^n M \). Since \( \Phi \) commutes with \( T^n \), we deduce that the canonical map \( \Phi N \to N \) is surjective. It then follows by induction on \( m \) that \( N^m \simeq 0 \) for \( m > 0 \). In other words, \( N \) is concentrated in degree zero, and is a direct sum of copies of \( F_2 \). It follows that \( 0 \simeq TN \simeq T^{n+1} M \), as desired. \( \square \)