Let $p$ be a prime number. In this lecture we will introduce the category of $p$-profinite spaces. We begin by reviewing an example from classical algebra.

Let $\mathcal{C}$ be the category of abelian groups, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory consisting of finitely generated abelian groups. Every abelian group $A$ is the union of its finitely generated subgroups. Consequently, every object of $\mathcal{C}$ can be obtained as a (filtered) direct limit of objects in $\mathcal{C}_0$. Moreover, the morphisms in $\mathcal{C}$ are determined by the morphisms in $\mathcal{C}_0$. If $A$ is a finitely generated abelian group and $\{B_\beta\}$ is any filtered system of abelian groups, then we have a bijection $\lim_{\rightarrow} \hom(A, B_\beta) \cong \hom(A, \lim_{\rightarrow} B_\beta)$.

More generally, if $A$ is given as a filtered colimit of abelian groups, then we get a bijection $\hom(\lim_{\rightarrow} A_\alpha, \lim_{\rightarrow} B_\beta) \cong \lim_{\leftarrow} \lim_{\rightarrow} \hom(A_\alpha, B_\beta)$.

We can summarize the situation by saying that $\mathcal{C}$ is equivalent to the category of Ind-objects of $\mathcal{C}_0$:

**Definition 1.** Let $\mathcal{C}_0$ be a category. The category $\text{Ind}(\mathcal{C}_0)$ of Ind-objects of $\mathcal{C}_0$ is defined as follows:

1. The objects of $\text{Ind}(\mathcal{C}_0)$ are formal direct limits $\lim_{\rightarrow} C_\alpha$, where $\{C_\alpha\}$ is a filtered diagram in $\mathcal{C}_0$.
2. Morphisms in $\text{Ind}(\mathcal{C}_0)$ are given by the formula $\hom(\lim_{\rightarrow} A_\alpha, \lim_{\rightarrow} B_\beta) \cong \lim_{\alpha} \lim_{\beta} \hom(A_\alpha, B_\beta)$.

**Remark 2.** There is a fully faithful embedding from $\mathcal{C}_0$ into $\text{Ind}(\mathcal{C}_0)$, which carries an object $C \in \mathcal{C}_0$ to the constant diagram consisting of the single object $C$. We will generally abuse notation and identify $\mathcal{C}_0$ with its image under this embedding.

The category $\text{Ind}(\mathcal{C}_0)$ admits filtered colimits. Moreover, an object $\lim_{\rightarrow} C_\alpha$ in $\text{Ind}(\mathcal{C}_0)$ actually does coincide with the colimit of the diagram $\{C_\alpha\}$ in $\text{Ind}(\mathcal{C}_0)$.

**Remark 3.** The category $\text{Ind}(\mathcal{C}_0)$ can be characterized by the following universal property: for any category $\mathcal{D}$ which admits filtered colimits, the restriction functor $\text{Fun}_0(\text{Ind}(\mathcal{C}_0), \mathcal{D}) \to \text{Fun}(\mathcal{C}_0, \mathcal{D})$ is an equivalence of categories, where the left side is the category of functors from $\text{Ind}(\mathcal{C}_0)$ to $\mathcal{D}$ which preserve filtered colimits.

**Example 4.** Let $\mathcal{C}$ be the category of groups (or rings, or any other type of algebraic structure). Then $\mathcal{C}$ is equivalent to $\text{Ind}(\mathcal{C}_0)$, where $\mathcal{C}_0 \subseteq \mathcal{C}$ is the full subcategory spanned by the finitely presented groups (or rings, etcetera).
There is a dual construction, which replaces a category $\mathcal{C}_0$ by the category $\text{Pro}(\mathcal{C}_0)$ of pro-objects in $\mathcal{C}_0$: that is, formal inverse limits $\lim_{\alpha} C_{\alpha}$ of filtered diagrams in $\mathcal{C}_0$.

**Example 5.** Let $\mathcal{C}_0$ be the category of finite groups. Then $\text{Pro}(\mathcal{C}_0)$ is equivalent to the category of profinite groups: that is, topological groups which are compact, Hausdorff, and totally disconnected.

The construction $\mathcal{C}_0 \mapsto \text{Pro}(\mathcal{C}_0)$ makes sense not only for ordinary categories, but also for homotopy theories. In other words, suppose that $\mathcal{C}_0$ is a category enriched over topological spaces (so that for every pair of objects $X, Y \in \mathcal{C}_0$, we have a mapping space $\text{Map}_{\mathcal{C}_0}(X, Y)$). Then we can define a new topological category $\text{Pro}(\mathcal{C}_0)$. Roughly speaking, the objects of $\text{Pro}(\mathcal{C}_0)$ are given by formal filtered limits $\lim_{\alpha} C_{\alpha}$ in $\mathcal{C}_0$, and the morphisms are described by the formula

$$\text{Map}(\lim_{\alpha} C_{\alpha}, \lim_{\beta} D_{\beta}) = \text{holim}_{\alpha} \text{hocolim}_{\beta} \text{Map}(C_{\alpha}, D_{\beta}).$$

To really make this idea precise requires the machinery of higher category theory; we will be content to work with this construction in an informal way.

We now specialize this construction to the case of interest. Let $S$ denote the category of spaces, $S_p$ the category of $p$-finite spaces, and $S_p^\vee$ the category $\text{Pro}(S_p)$ of pro-objects in $S_p$. We will refer to $S_p^\vee$ as the category of $p$-profinite spaces.

There is a canonical functor $G : S_p^\vee \to S$, which carries a formal inverse limit $\lim_{\alpha} C_{\alpha}$ to the space $\text{holim}_{\alpha} C_{\alpha}$. If we restrict to a suitable subcategory of $S_p^\vee$ by imposing finiteness and connectivity conditions, then the functor $G$ is fully faithful; its essential image being (a suitable subcategory of) the category of $p$-complete spaces. We will discuss this point in more detail in a future lecture.

The functor $G$ has a left adjoint $X \mapsto X^\vee$, which we will refer to as the functor of $p$-profinite completion. The functor $\vee$ carries a topological space $X$ to the formal inverse limit $X^\vee = \lim X_{\alpha}$, where $X_{\alpha}$ ranges over all $p$-finite spaces equipped with a map to $X$. If $X$ is itself $p$-finite, then we can identify this inverse limit with $X$ itself.

**Definition 6.** Let $X$ be a $p$-profinite space. We let $H^n(X) = H^n(X; F_p)$ denote the set of homotopy classes of maps from $X$ into an Eilenberg-MacLane space $K(F_p, n)$ in the $p$-profinite category $S_p^\vee$.

Since $K(F_p, n)$ is $p$-finite, we see that

$$H^n(\lim X_{\alpha}) \simeq \lim H^n(X_{\alpha}).$$

It follows that for any $p$-profinite space $X$, the cohomology $H^*(X) \simeq \bigoplus_n H^n(X)$ is a filter colimit of the cohomology rings of a collection of $p$-finite spaces, and therefore inherits the structure of an unstable algebra over the Steenrod algebra.

**Remark 7.** If $X$ is a topological space, then the cohomology $H^*(X; F_p)$ (in the usual sense) can be identified with the cohomology $H^*(X^\vee)$ of the $p$-profinite completion of $X$, defined as in Definition 6.

The process of extracting cohomology does not generally commute with the inverse limit functor $G : S_p^\vee \to S$, unless we make suitable finiteness assumptions.

We now discuss the existence of mapping objects in the $p$-profinite category.

**Proposition 8.** Let $X$ be a $p$-profinite space, and let $V$ be a finite dimensional vector space over $F_p$. Then there exists a $p$-profinite space $X^{BV}$ equipped with an evaluation map $X^{BV} \times BV \to X$ with the following universal property: for any $p$-profinite space $Y$, the induced map

$$\theta : \text{Map}(Y, X^{BV}) \to \text{Map}(Y \times BV, X)$$

is a weak homotopy equivalence.
Proof. If \( X = \varprojlim X_\alpha \), then we can take \( X^{BV} = \varprojlim X^{BV}_\alpha \) (here we are using the fact that each \( X^{BV}_\alpha \) is again \( p \)-finite). We claim that \( X^{BV} \) has the appropriate universal property. For any \( p \)-profinite space \( Y \), we can identify \( \theta \) with a map
\[ \text{holim} \text{Map}(Y, X^{BV}_\alpha) \simeq \text{Map}(Y \times BV, X) \simeq \text{holim} \text{Map}(Y \times BV, X_\alpha). \]
It will therefore suffice to prove the result after replacing \( X \) by \( X_\alpha \), so we may assume that \( X \) is \( p \)-finite.

Let \( Y = \varprojlim Y_\beta \). Then the map \( \theta \) can be identified with
\[ \text{hocolim} \text{Map}(Y_\beta, X^{BV}) \simeq \text{Map}(Y \times BV, X) \simeq \text{hocolim} \text{Map}(Y_\beta \times BV, X), \]
where the last equivalence follows from the observation that \( Y \times BV \simeq \varprojlim Y_\beta \times BV \) is a product for \( Y \) and \( BV \) in the \( p \)-profinite category. We may therefore assume that \( Y \) is \( p \)-finite as well, in which case the result is obvious.

Remark 9. Proposition 9 remains valid if we replace \( BV \) by an arbitrary \( p \)-finite space. However, it is not valid if \( BV \) is a general \( p \)-profinite space; the \( p \)-profinite category \( S^p \) does not have internal mapping objects in general.

Remark 10. Let \( X = \varprojlim X_\alpha \) and \( Y = \varprojlim Y_\beta \) be \( p \)-profinite spaces. Then \( \varprojlim X_\alpha \times Y_\beta \) is a product for \( X \) and \( Y \) in the category of \( p \)-profinite spaces. Applying the Kunneth theorem to the \( p \)-finite spaces \( X_\alpha \) and \( Y_\beta \), we deduce
\[ H^*(X \times Y) \simeq \varprojlim H^*(X_\alpha \times Y_\beta) \simeq \varprojlim H^* X_\alpha \otimes H^* Y_\beta \simeq H^* X \otimes H^* Y. \]

Let us now assume that \( p = 2 \). Let \( X \) be a \( p \)-profinite space. The evaluation map \( X^{BV} \times BV \to X \) induces a map on cohomology
\[ H^* X \to H^*(X^{BV} \times BV) \simeq H^*(X^{BV}) \otimes H^*(BV), \]
which is adjoint to a map \( \psi : T_V H^*(X) \to H^*(X^{BV}) \).

Theorem 11. The map \( \psi \) is an isomorphism, for every \( 2 \)-profinite space \( X \).

Proof. The proof when \( X \) is 2-finite was given in the previous lecture. In general, write \( X = \varprojlim X_\alpha \). Then we have
\[ T_V H^*(X) \simeq T_V \varprojlim H^*(X_\alpha) \]
\[ \simeq \varprojlim T_V H^*(X_\alpha) \]
\[ \simeq \varprojlim H^*(X^{BV}_\alpha) \]
\[ \simeq H^*(X^{BV}). \]

Using this result, we get a measure of exactly how the \( \psi \) might fail to be an isomorphism when we work in the usual category of spaces. For any space \( X \), we have
\[ T_V H^*(X) \simeq T_V H^*(X^\vee) \simeq H^*(X^{BV^\vee}) \to H^*(X^{BV})^\vee. \]
In other words, the failure of \( T_V \) to compute the cohomology of mapping spaces is measured by the failure of the formation of mapping spaces to commute with profinite completion.