Review for Operators and Linear systems for 18.03

Ross A. Lippert

March 10, 2004

If you are pressed for time, skip everything except the anatomy, vocabulary, common cases, and the last section.

1 Complex numbers and sinusoids

Things you should know already, but it doesn't hurt to review.

a, b, A, θ are all (related) real constants with $A \geq 0$.

Vocabulary: A is magnitude, modulus, absolute value, or amplitude. θ is phase, angle, or argument.

Anatomy of a complex number I: rectangular form: $z = a + ib$

Anatomy of a complex number II: polar form: $z = Ae^{i\theta}$

Conversion from $(A, \theta) \to (a, b)$,

$$A cos(\theta) = a$$
$$A sin(\theta) = b$$

use that triangle to go the other way.

Real part: $\mathcal{R}e\{a + ib\} = a$. Imaginary part: $\mathcal{I}m\{a + ib\} = b$. Real part is real. Imaginary part is **real**.

Anatomy of a sinusoid I: rectangular form

$$a cos(\omega t) + b sin(\omega t)$$

Anatomy of a sinusoid II: real part of complex exponential with rectangular coefficient

$$\mathcal{R}e\{(a + ib)e^{-i\omega t}\}$$

Anatomy of a sinusoid III: real part of complex exponential with polar coefficient

$$\mathcal{R}e\{Ae^{i\theta}e^{-i\omega t}\}$$

Vocabulary: here A is amplitude. θ is phase lag, or phase shift.

Anatomy of a sinusoid IV: polar form, amplitude A, phase θ

$$A cos(\omega t - \theta)$$

Note: If you don't always take the real part or you don't always use the $e^{-i\omega t}$ exponential (not $e^{i\omega t}$), the formulas change, the warranty is void.

Example,

$$\begin{aligned} 1 cos(11t) + \sqrt{3} sin(11t) &= \mathcal{R}e\{(1 + i\sqrt{3})e^{-i11t}\} \\ &= \mathcal{R}e\{2e^{i\frac{\pi}{3}}e^{-i11t}\} \\ &= \mathcal{R}e\{2e^{-i(11t-\frac{\pi}{3})}\} \\ &= 2\mathcal{R}e\{e^{-i(11t-\frac{\pi}{3})}\} \\ &= 2cos(11t - \frac{\pi}{3}) \end{aligned}$$

2 Operators

Why operators? Operator methods don't break down when two things are equal or something or another is zero. For LTI equations, they **always** get the general answer. Why not operators? They are usually **slower** than guessing and solving for parameters (the *shortcut* methods in section 5).

To try and keep things very clear, I will put a hat on top of all the operator variables like $\hat{A}$ instead of A.

The only operators of substantial interest to us are the linear differential operators. When I say *operator*, I really mean *linear operator*.

- $\hat{I}x(t) = x(t)$: the identity operator
- $\hat{D}x(t) = \dot{x}(t)$: differentiate.
- $C\hat{D}x(t) = C\dot{x}(t)$: differentiate and multiply by the constant C
- $f(t)\hat{D}x(t) = f(t)x(t)$: differentiate and multiply by $f(t)$
- $C\hat{I}x(t) = Cx(t)$: multiply by the constant C
- $f(t)\hat{I}x(t) = f(t)x(t)$: multiply by $f(t)$

Why is there $\hat{I}$? For disambiguation, really. $t^2\hat{I}$ is an operator (which multiplies by t^2). t^2 is a function of t.

When you have $\hat{A}\hat{B}x(t)$, it means do $\hat{B}$ to $x(t)$ first, then do $\hat{A}$ to that. $\hat{A}\hat{B}$ is also an operator. Often $\hat{A}\hat{B} \neq \hat{B}\hat{A}$, so **order is important**.

When you have $(\hat{A}+\hat{B})x(t)$ it means do $\hat{A}$ to $x(t)$ and do $\hat{B}$ to x(t) and add the results: $\hat{A}x(t)+\hat{B}x(t)$.

2.1 Operator identities

What an operator means is what it does.

$$\begin{aligned}
\hat{I}\hat{A} &= \hat{A} \\
\hat{A}\hat{I} &= \hat{A} \\
\hat{D}C\hat{I} = C\hat{I}\hat{D} &= C\hat{D} \\
\hat{D}f(t)\hat{I} &= f(t)\hat{D} + \dot{f}(t)\hat{I}
\end{aligned}$$

Where'd that last one come from?

To show that two operators are the same: $\hat{A} = \hat{B}$, when $\hat{A}x(t) = \hat{B}x(t)$ for any $x(t)$. For example,

$$\begin{aligned}
\hat{D}t^2\hat{I}x(t) &= \hat{D}(t^2x(t)) \\
&= t^2\dot{x}(t) + 2tx(t) \\
&= t^2\hat{D}x(t) + 2t\hat{I}x(t) \\
&= (t^2\hat{D} + 2t\hat{I})x(t)
\end{aligned}$$

so $\hat{D}t^2\hat{I} = t^2\hat{D} + 2t\hat{I}$. note: $x(t)$ is just a placeholder (sometimes called a *test function*).

$\hat{D}t^2\hat{I} = t^2\hat{I}\hat{D} + 2t\hat{I}$ is an example where $\hat{D}t^2\hat{I}$ is different from $t^2\hat{I}\hat{D}$. Order was important in this case.

3 First order, linear differential equations

Just one identity,[1]

$$(\hat{D} + \dot{p}(t)\hat{I}) \quad = \quad e^{-pt}\hat{D}e^{p(t)}\hat{I}.$$

[1]verify this yourself, if you want, but memorize it — kind of like exponential shift $e^{p(t)}(\hat{D} + \dot{p}(t)\hat{I}) = \hat{D}e^{p(t)}\hat{I}$.

The left hand side: sum of operators. The right hand side: product of operators.

An Example:

$$\begin{aligned} t\dot{x}(t) + 3x(t) &= 2 \\ \dot{x}(t) + \frac{3}{t}x(t) &= \frac{2}{t} \\ (\hat{D} + \frac{3}{t}\hat{I})x(t) &= \frac{2}{t}. \end{aligned}$$

Using the identity,

$$\begin{aligned} (e^{-3ln(t)}\hat{D}e^{3ln(t)}\hat{I})x(t) &= \frac{2}{t} \\ (t^{-3}\hat{D}t^3\hat{I})x(t) &= \frac{2}{t} \\ t^{-3}\hat{D}(t^3x(t)) &= \frac{2}{t} \end{aligned}$$

Solve by peeling back one piece at a time (left to right)

$$\begin{aligned} \hat{D}(t^3x(t)) &= 2t^2 \\ t^3x(t) &= \frac{2}{3}t^3 + C \\ x(t) &= \frac{2}{3} + Ct^{-3} \end{aligned}$$

3.1 Common cases to remember

Exponential drive with constant coefficients.

r_1, r_2 are constants, not equal.

$$(\hat{D} - r_1\hat{I})x(t) = e^{r_2t}$$
$$x(t) = \frac{1}{r_2 - r_1}e^{r_2t} + Ce^{r_1t}$$

r is a constant.

$$\begin{aligned} (\hat{D} - r\hat{I})x(t) &= e^{rt} \\ x(t) &= te^{rt} + Ce^{rt} \end{aligned}$$

4 Higher order, linear, constant coefficients (LTI)

(Expressing diffeq operators as products of simple operators is a good way to go.)

$p(r)$ is a polynomial. We use a quadratic, $p(r) = r^2 + br + c$, but it can be any polynomial.

$$\begin{aligned} p(\hat{D}) &= g(t) \\ (\hat{D}^2 + b\hat{D} + k\hat{I})x(t) &= g(t) \end{aligned}$$

Anatomy of polynomials: they factor, $p(r) = (r - r_1)(r - r_2)$, and r_1, r_2 can be real or complex, different or the same.

Anatomy of LTI operators: they factor, $p(\hat{D}) = (\hat{D} - r_1\hat{I})(\hat{D} - r_2\hat{I})$.

$$
\begin{aligned}
(\hat{D} - r_1\hat{I})(\hat{D} - r_2\hat{I})x(t) &= g(t) \\
(e^{r_1 t}\hat{D}e^{-r_1 t}\hat{I})(e^{r_2 t}\hat{D}e^{-r_2 t}\hat{I})x(t) &= g(t)
\end{aligned}
$$

Order is not important because r_1, r_2 are constant,

$$(\hat{D} - r_1\hat{I})(\hat{D} - r_2\hat{I}) = (\hat{D} - r_2\hat{I})(\hat{D} - r_1\hat{I})$$

Easy example, building on the common cases for first order linear,

$$
\begin{aligned}
(\hat{D}^2 + 4\hat{D} + 3\hat{I})x(t) &= sin(2t) \\
(\hat{D} + \hat{I})(\hat{D} + 3\hat{I})x(t) &= ie^{-2it} \\
(\hat{D} + 3\hat{I})x(t) &= \frac{1}{-2i+1}e^{-2it} + Ce^{-t} \\
x(t) &= \frac{1}{(-2i+3)(-2i+1)}e^{-2it} + \frac{C}{-1+3}e^{-t} + C_2e^{-3t} \\
x(t) &= \frac{1}{(-2i+3)(-2i+1)}e^{-2it} + C_1e^{-t} + C_2e^{-3t} \\
x(t) &= \frac{-1+8i}{65}e^{-2it} + C_1e^{-t} + C_2e^{-3t} \\
x(t) &= \frac{1}{65}(-cos(2t) + 8sin(2t)) + C_1e^{-t} + C_2e^{-3t}
\end{aligned}
$$

Icky example, but we can get through it

$$
\begin{aligned}
(\hat{D}^2 + 4\hat{D} + 3\hat{I})x(t) &= t^2 \\
(\hat{D} + \hat{I})(\hat{D} + 3\hat{I})x(t) &= t^2 \\
(e^{-t}\hat{D}e^{t}\hat{I})(e^{-3t}\hat{D}e^{3t}\hat{I})x(t) &= t^2 \\
e^{-t}\hat{D}e^{t}\hat{I}e^{-3t}\hat{D}e^{3t}\hat{I}x(t) &= t^2
\end{aligned}
$$

again, solve by peeling off one layer at a time, (and integrating by parts)

$$
\begin{aligned}
\hat{D}e^{t}\hat{I}e^{-3t}\hat{D}e^{3t}\hat{I}x(t) &= t^2e^t \\
e^{t}\hat{I}e^{-3t}\hat{D}e^{3t}\hat{I}x(t) &= (t^2 - 2t + 2)e^t + C \\
e^{-3t}\hat{D}e^{3t}\hat{I}x(t) &= (t^2 - 2t + 2) + Ce^t \\
\hat{D}e^{3t}\hat{I}x(t) &= (t^2 - 2t + 2)e^{3t} + Ce^{2t} \\
e^{3t}\hat{I}x(t) &= \frac{1}{27}(9t^2 - 24t + 26)e^{3t} + \frac{1}{2}Ce^{2t} + C_2 \\
x(t) &= \frac{1}{27}(9t^2 - 24t + 26) + \frac{1}{2}Ce^{-t} + C_2e^{-3t} \\
x(t) &= \frac{1}{27}(9t^2 - 24t + 26) + C_1e^{-t} + C_2e^{-3t}
\end{aligned}
$$

4.1 Common cases to remember

$p(r)$ is quadratic: $p(r) = r^2 + br + k$.

General homogeneous solution:

r_1, r_2 are constants, not equal.

$$
\begin{aligned}
(\hat{D} - r_1\hat{I})(\hat{D} - r_2\hat{I})x(t) &= 0 \\
x(t) &= C_1e^{r_1 t} + C_2e^{r_2 t}
\end{aligned}
$$

r is constant.

$$\begin{aligned}(\hat{D}-r\hat{I})(\hat{D}-r\hat{I})x(t) &= 0\\ x(t) &= C_1te^{rt}+C_2e^{rt}\end{aligned}$$

Classification of the general homogeneous solutions (depending on roots),

- 2 real: **overdamped:** $r_1, r_2 = -b/2 \pm \sqrt{(b/2)^2 - k}$

$$x(t) = C_1e^{r_1t} + C_2e^{r_2t}$$

- 1 real: **critically damped:** $r_1 = r_2 = -b/2$

$$x(t) = C_1te^{-\frac{b}{2}t} + C_2e^{-\frac{b}{2}t}$$

- 2 complex: **underdamped:** $r_1, r_2 = a \pm i\omega$, where $a = -b/2$ and $\omega = \sqrt{k-(b/2)^2}$

$$x(t) = C_1e^{-at}cos(\omega t) + C_2e^{-at}sin(\omega t)$$

Vocabulary: A homogeneous solution in the form e^{rt} is a *normal mode*. For example, $5e^{-t}cos(t)$ is not a normal mode, $e^{(-1-i)t}$ and $e^{(-1-i)t}$ are.

Exponential drive, general solutions:

r_1, r_2, r_3 are constants, none equal.

$$\begin{aligned}(\hat{D}-r_1\hat{I})(\hat{D}-r_2\hat{I})x(t) &= e^{r_3t}\\ x(t) &= \frac{1}{(r_3-r_1)(r_3-r_2)}e^{r_3t}+C_1e^{r_1t}+C_2e^{r_2t}\end{aligned}$$

r_1, r_2 are constants, not equal.

$$\begin{aligned}(\hat{D}-r_1\hat{I})(\hat{D}-r_1\hat{I})x(t) &= e^{r_2t}\\ x(t) &= \frac{1}{(r_2-r_1)^2}e^{r_2t}+C_1te^{r_1t}+C_2e^{r_1t}\end{aligned}$$

Vocabulary: the *normalized* homogeneous solutions are two homogeneous solutions, $x_1(t), x_2(1)$. $x_1(t)$ has C_1, C_2 set by $x_1(0) = 1, \dot{x}_1(0) = 0$, and $x_2(t)$ has C_1, C_2 set by $x_2(0) = 0, \dot{x}_2(0) = 1$. A particular solution, $x_p(t)$, is *normalized* when C_1, C_2 set so that $x_p(0) = 0, \dot{x}_p(0) = 0$. Definitions are analogous for higher (or lower) order equations.

4.2 Uncommon cases which are probably not important

$$\begin{aligned}(\hat{D}-r_1\hat{I})(\hat{D}-r_2\hat{I})x(t) &= e^{r_2t}\\ x(t) &= \frac{t}{r_2-r_1}e^{r_2t}+C_1e^{r_1t}+C_2e^{r_2t}\\ (\hat{D}-r_1\hat{I})(\hat{D}-r_1\hat{I})x(t) &= e^{r_2t}\\ x(t) &= \frac{1}{(r_2-r_1)^2}e^{r_2t}+C_1te^{r_1t}+C_2e^{r_1t}\end{aligned}$$

5 Shortcuts

Guessing the form of the answer and solving is less tedious than operator methods. They don't handle every case, but when they do work, they are faster. We know some abstract things about solutions to linear ODE's. This motivates some short cuts.

5.1 Basic shortcut

Anatomy of a linear ODE:

$$\hat{L}x(t) = g(t)$$

where $\hat{L}$ is *any* (linear) operator. The common cases in class have been $\hat{L} = \hat{D} - k(t)\hat{I}$, or $\hat{L} = \hat{D}^2 + b\hat{D} + k\hat{I}$.

Solutions to the homogeneous equation

$$\hat{L}x_h(t) = 0$$

are important.

Superposition I: If

$$\begin{aligned} \hat{L}x_h^{(1)}(t) &= 0 \\ \hat{L}x_h^{(2)}(t) &= 0 \end{aligned}$$

then for any constants c_1, c_2,

$$\hat{L}(c_1 x_h^{(1)}(t) + c_2 x_h^{(2)}(t)) = 0.$$

i.e. linear combinations of x_h's are also x_h's.

Superposition II: If

$$\begin{aligned} \hat{L}x^{(1)}(t) &= g(t) \\ \hat{L}x^{(2)}(t) &= g(t) \end{aligned}$$

then

$$\begin{aligned} \hat{L}(x^{(1)}(t) - x^{(2)}(t)) &= 0 \\ x^{(1)}(t) - x^{(2)}(t) &= x_h(t). \end{aligned}$$

i.e. two solutions differ (at most) by some homogenous solution.

Anatomy of the general solution: by Superposition II,

$$x(t) = x_p(t) + x_h(t)$$

$x_p(t)$ is one solution, $x_h(t)$ is the general homogeneous solution (it has all the C's).

Superposition III: If

$$\begin{aligned} \hat{L}x_1(t) &= g_1(t) \\ \hat{L}x_2(t) &= g_2(t) \end{aligned}$$

then for any constants c_1, c_2,

$$\hat{L}(c_1 x_1(t) + c_2 x_2(t)) = c_1 g_1(t) + c_2 g_2(t)$$

This helps find particular solutions for complicated $g(t)$'s.

The basic shortcut:

- Find the general homogeneous solution $x_h(t)$
- Find a particular solution $x_p(t)$.
  If $g(t) = c_1 g_1(t) + c_2 g_2(t)$, use superposition III:
  - find particulars $x_1(t), x_2(t)$
  - and put them together in $x_p(t) = c_1 g_1(t) + c_2 g_2(t)$
- General solution is $x(t) = x_p(t) + x_h(t)$

5.2 LTI shortcuts

Each of these facts can be derived with operators.

Generic homogeneous: If $p(r) = (r - r_1)(r - r_2)\cdots(r - r_n)$ with no two r_i equal,

$$\begin{aligned} p(\hat{D})x(t) &= 0 \\ x(t) &= C_1 e^{r_1 t} + C_2 e^{r_2 t} + \ldots \end{aligned}$$

is the *general* solution.

Mutliple root homogenous: If $p(r) = (r - r_1)^{m_1}(r - r_2)^{m_2}\cdots(r - r_n)^{m_n}$ with no two r_i equal,

$$\begin{aligned} p(\hat{D})x(t) &= 0 \\ x(t) &= (C_1 + C_2 t + \ldots + C_{m_1} t^{m_1 - 1})e^{r_1 t} + (C_{m_1+1} + C_{m_1+2} t + \ldots + C_{m_1+m_2} t^{m_2 - 1})e^{r_2 t} + \ldots \end{aligned}$$

is the *general* solution. For example,

$$\begin{aligned} (\hat{D} + 3\hat{I})^2(\hat{D} + \hat{I})(\hat{D} + 2i)^3 x(t) &= 0 \\ x(t) &= (C_1 + C_2 t)e^{-3t} + C_3 e^{-t} + (C_4 + C_5 t + C_6 t^2)e^{-2it} \end{aligned}$$

Exponential response: If $p(s) \neq 0$,

$$\begin{aligned} p(\hat{D})x(t) &= e^{st} \\ x(t) &= \frac{1}{p(s)} e^{st} \end{aligned}$$

is a *particular* solution: the *exponential response*.

Undetermined coefficients (UC): If $p(0) \neq 0$, and $q(t)$ is any polynomial

$$p(\hat{D})x(t) = q(t)$$

there is a unique *particular* solution $x(t)$ which is a polynomial of the same degree as $q(t)$. For example,

$$p(\hat{D})x(t) = t^3 + 1$$

you can guess $x(t) = At^3 + Bt^2 + Ct + D$ and work out the constants A, B, C, D.

UC special case: If $p(0) = 0$, *UC can still help.* Since 0 is a root, we can factor out $\hat{D}$ from the operator

$$\begin{aligned} \hat{D}p(\hat{D})x(t) &= q(t) \\ p(\hat{D})x(t) &= \int q(t)dt \end{aligned}$$

and try **UC** again. For a particular solution, don't worry about the constant. For the general, keep track of it.

Exponential shift (ES): in operators, a couple different ways

$$\begin{aligned} e^{st}p(\hat{D}) &= p(\hat{D} - s\hat{I})e^{st}\hat{I} \\ p(\hat{D})e^{st}\hat{I} &= e^{st}p(\hat{D} + s\hat{I}) \end{aligned}$$

in practice,

$$\begin{aligned} p(\hat{D})x(t) &= g(t)e^{st} \\ p(\hat{D} + s\hat{I})(e^{-st}x(t)) &= g(t) \end{aligned}$$

then solve for $u(t) = e^{-st}x(t)$. **ES** doesn't give a solution. It just gives a way to rewrite (and hopefully simplify) the equation.

ES with UC: If $p(s) \neq 0$, and $q(t)$ is any polynomial

$$p(\hat{D})x(t) \quad = \quad q(t)e^{st}$$

there is a unique *particular* solution $x(t) = u(t)e^{st}$ where $u(t)$ is a polynomial of the same degree as $q(t)$, and it solves

$$p(\hat{D} + s\hat{I})u(t) \quad = \quad q(t).$$

UC and **EF** can **save you from the pain** of multiple ugly integration by parts (as in the "Icky example" previously).

Icky example, revisited: by **UC**, guess particular, $x_p(t) = At^2 + Bt + C$,

$$\begin{aligned} (\hat{D}^2 + 4\hat{D} + 3\hat{I})(At^2 + Bt + C) &= t^2 \\ 2A + 4(2At + B) + 3(At^2 + Bt + C) &= t^2 \\ 3At^2 + (8A + 3B)t + (2A + 4B + 3C) &= t^2 \end{aligned}$$

equating like coefficients,

$$\begin{aligned} 3A &= 1, \\ 8A + 3B &= 0, \\ 2A + 4B + 3C &= 0 \end{aligned}$$

so $A = \frac{1}{3}, B = -\frac{8}{9}, C = \frac{26}{27}$. The particular is $x_p(t) = \frac{1}{3}t^2 - \frac{8}{9}t + \frac{26}{27}$. The roots of $p(r) = r^2 + 4r + 3$ are $-1, -3$ so the general homogeneous solution is $x_h(t) = C_1e^{-t} + C_2e^{-3t}$, and the general solution is

$$x(t) = \frac{1}{3}t^2 - \frac{8}{9}t + \frac{26}{27} + C_1e^{-t} + C_2e^{-3t}.$$

Vocabulary: The homogeneous solution is sometimes called: the transient solution or the complementary solution. The particular solution from exponential response or ES combined with UC is sometimes called **the** particular solution (which is a bit silly), **the** periodic solution (also sometimes silly), or the sinusoidal response (assuming r_0 is imaginary).