Solving Differential Equations with Long Division

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January 21, 2005

1 Introduction

This is a handout to describe the basics of the *long division* technique for solving differential equations. This technique evolved from my discussions with 18.03 students over the course of the year. While it won't solve every 18.03 problem, it will solve a good many.

For those of you who are familiar with 18.03, this technique can be considered a replacement for the *undertermined coefficients* technique in its capacities.

2 Background

2.1 What we need to know from calculus

In calculus, we learned about the derivative. Here is a short list of important facts about the derivative:

- $\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$
- $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
- $\frac{d}{dx}(f(x)g(x)) = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$
- $\frac{d}{dx}1 = 0$
- $\frac{d}{dx}x = 1$
- $\frac{d}{dx}e^{kx} = ke^{kx}$

These are all the facts we need to know to use $\frac{d}{dx}$ on any polynomial or any polynomial times an exponential. For example,

$$\frac{d}{dx}x^2 = \frac{d}{dx}(xx) = x\frac{d}{dx}x + x\frac{d}{dx}x = x \cdot 1 + x \cdot 1 = 2x$$

and

$$\frac{d}{dx}(3x^2+2x+4) = 3\frac{d}{dx}x^2 + 2\frac{d}{dx}x + 4\frac{d}{dx}1 = 6x + 2.$$

There are other properties of $\frac{d}{dx}$ not mentioned here, like the *chain rule*, but we will not need them.

2.2 What we need to know about integration

Integration was the first differential equation one solves. It asks, *for some* $g(x)$*, find a* $f(x)$ *such that* $\frac{d}{dx}f(x) = g(x)$. Solving this problem is easy if $g(x)$ is a polynomial,

$$\begin{aligned} \frac{d}{dx}f(x) &= x^3 \\ f(x) &= \frac{1}{4}x^4 \end{aligned}$$

or if $g(x)$ is an exponential,

$$\begin{aligned} \frac{d}{dx}f(x) &= e^{3x} \\ f(x) &= \frac{1}{3}e^{3x}. \end{aligned}$$

We quickly learned that since $\frac{d}{dx}C = C\frac{d}{dx}1 = 0$, that many different solutions were possible for a given integration problem,

$$\begin{aligned} \frac{d}{dx}f(x) &= x^3 \\ f(x) &= \frac{1}{4}x^4 \\ f(x) &= \frac{1}{4}x^4 + 1 \\ f(x) &= \frac{1}{4}x^4 + 10 \\ f(x) &= \frac{1}{4}x^4 + \pi \\ f(x) &= \frac{1}{4}x^4 + \sqrt{2} \end{aligned}$$

so to capture this idea we wrote $f(x) = \frac{1}{4}x^4 + C$ where C was some constant about which we had no information.

If we describe our differential equations with fractional notation, the integration problem is

$$\frac{x^3}{\frac{d}{dx}} = \frac{1}{4}x^4$$

or, more generally,

$$\frac{x^3}{\frac{d}{dx}} = \frac{1}{4}x^4 + C.$$

3 Simple differential equations with simple solutions

Complicating the integration problem:

$$\frac{d}{dx}f(x) + 3f(x) = 9x^3.$$

In other words, we are looking for a function which when 3 of it is added to its derivative, the result is $9x^3$.

In fractional notation, we might write

$$f(x) = \frac{9x^3}{3 + \frac{d}{dx}}.$$

Althought $\frac{9x^3}{3+\frac{d}{dx}}$ looks a bit alien, by thinking of it as a division, there is a straightforward way to get an answer,

$$\begin{array}{r|rrrr}
 & 3x^3 & -3x^2 & +2x & -\frac{2}{3} \\ \hline
3+\frac{d}{dx} & 9x^3 & & & \\
 & 9x^3 & +9x^2 & & \\ \cline{2-3}
 & & -9x^2 & & \\
 & & -9x^2 & -6x & \\ \cline{3-4}
 & & & 6x & \\
 & & & 6x & +2 \\ \cline{4-5}
 & & & & -2 \\
 & & & & -2 \\ \cline{5-5}
 & & & & 0
\end{array}$$

Is $3x^3 - 3x^2 + 2x - \frac{2}{3}$ really a function that if you add 3 of it to its derivative you get $9x^3$? This is easy to check,

$$\begin{aligned}
\frac{d}{dx}\left(3x^3 - 3x^2 + 2x - \frac{2}{3}\right) &= 9x^2 - 6x + 2 \\
3\left(3x^3 - 3x^2 + 2x - \frac{2}{3}\right) &= 9x^3 - 9x^2 + 6x - 2
\end{aligned}$$

and the *long division* process has worked.

We can make up harder problems by taking more derivatives or making the right side more complicated

$$\frac{d}{dx}^2 f(x) + 2\frac{d}{dx}f(x) + 4f(x) = 4x^2 + x + 1.$$

The long division method can still give us a solution of $\frac{4x^2+x+1}{4+2\frac{d}{dx}+\frac{d}{dx}^2}$.

$$\begin{array}{r|rrr}
 & x^2 & -\frac{3}{4}x & +\frac{1}{8} \\ \hline
4+2\frac{d}{dx}+\frac{d}{dx}^2 & 4x^2 & +x & +1 \\
 & 4x^2 & +4x & +2 \\ \cline{2-4}
 & & -3x & -1 \\
 & & -3x & -\frac{3}{2} \\ \cline{3-4}
 & & & \frac{1}{2} \\
 & & & \frac{1}{2} \\ \cline{4-4}
 & & & 0
\end{array}$$

Again, the answer $x^2 - \frac{3}{4}x + \frac{1}{8}$ is not hard to check,

$$4\left(x^2 - \frac{3}{4}x + \frac{1}{8}\right) = 4x^2 - 3x + \frac{1}{2}$$

$$
\begin{aligned}
2\frac{d}{dx}\left(x^2-\frac{3}{4}x+\frac{1}{8}\right) &= 4x-\frac{3}{2}\\
\frac{d}{dx}^2\left(x^2-\frac{3}{4}x+\frac{1}{8}\right) &= 2
\end{aligned}
$$

which sums to $4x^2+x+1$ as it should.

3.1 Why does long division work?

Under certain common conditions, the long division technique above gets a solution. Those conditions are:

- The *numerator* is a polynomial in x
- The *denominator* is a polynomial in $\frac{d}{dx}$
- The *denominator* has a (non-zero) constant term

Under these conditions, long division will produce a polynomial in x which is a solution of the differential equation associated with that fraction.

Put in more mathemtical terms:

Theorem 3.1 (Long Division). *The differential equation*

$$a_n\frac{d}{dx}^n f(x)+a_{n-1}\frac{d}{dx}^{n-1} f(x)+\cdots+a_1\frac{d}{dx}f(x)+a_0f(x)=b_mx^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0$$

has a solution by the long division method whenever $a_0\neq 0$ and that solution is a polynomial of degree m or less.

Proof. We can see that if $m=0$, so the right hand side is b_0, then $f(x)=\frac{b_0}{a_0}$ is a solution (all its derivatives vanish since $f(x)$ is constant). The degree of $\frac{b_0}{a_0}$ is 0, because it is a constant, so the statement about degrees holds too.

More generally, at each step of the long division method, we approximate our solution $f(x)$ with $\frac{b_m}{a_0}x^m+R(x)$ where $R(x)$ is the rest. Plugging this into the differential equation gives

$$\frac{b_m}{a_0}\left(a_n\frac{d}{dx}^n x^m+\cdots+a_1\frac{d}{dx}x^m\right)+b_mx^m+a_n\frac{d}{dx}^n R(x)+\cdots+a_0R(x)=b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0$$

and moving terms from the left to the right (cancelling b_mx^m)

$$a_n\frac{d}{dx}^n R(x)+\cdots+a_0R(x)=b_{m-1}x^{m-1}+\cdots+b_0-\frac{b_m}{a_0}\left(a_n\frac{d}{dx}^n x^m+\cdots+a_1\frac{d}{dx}x^m\right)=r(x).$$

The operations on the left hand side are the same, only in terms of the new function $R(x)$ and the right hand side, $r(x)$, is a new polynomial of degree $m-1$ or less, corresponding to the remainder after the one step of long division. In fractions

$$\frac{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0}{a_0+a_1\frac{d}{dx}+\cdots+a_n\frac{d}{dx}^n}=\frac{b_m}{a_0}x^m+\frac{r(x)}{a_0+a_1\frac{d}{dx}+\cdots+a_n\frac{d}{dx}^n}$$

Thus, as we repeat, we reduce the degree of the right hand side with each step, until it is just a constant. □

4 Solving harder problems

The long division method itself only let's solve $\frac{N(x)}{D(\frac{d}{dx})}$ when $N(x)$ is a polynomial in x and $D(\frac{d}{dx})$ is a polynomial in $\frac{d}{dx}$ with $D(0) \neq 0$. There are some tricks we can use to extend the method beyond these obstacles.

4.1 The constant part of the denominator vanishes

An example is $\frac{6x^2+4x+1}{3\frac{d}{dx}+\frac{d}{dx}^2}$. This fraction stands for the differential equation

$$\frac{d}{dx}^2 f(x) + 3\frac{d}{dx} f(x) \quad = \quad 6x^2+4x+1.$$

That is the same as making up some intermediate function $g(x)$ and saying

$$\begin{aligned} \frac{d}{dx} f(x) + 3f(x) &= g(x) \\ \frac{d}{dx} g(x) &= 6x^2+4x+1 \end{aligned}$$

which is solved by

$$\begin{aligned} g(x) &= 2x^3+2x^2+x \\ \frac{d}{dx} f(x) + 3f(x) &= 2x^3+2x^2+x \\ f(x) &= \frac{2x^3+2x^2+x}{3+\frac{d}{dx}} \end{aligned}$$

which is a fraction we can solve with long division.

This could just as well be written compactly as

$$\frac{6x^2+4x+1}{3\frac{d}{dx}+\frac{d}{dx}^2} = \frac{\frac{6x^2+4x+1}{\frac{d}{dx}}}{3+\frac{d}{dx}} = \frac{2x^3+2x^2+x}{3+\frac{d}{dx}}$$

with

$$
\begin{array}{rllll}
 & \frac{2}{3}x^3 & & +\frac{1}{3}x & -\frac{1}{9} \\
\cline{2-5}
3+\frac{d}{dx}\Big| & 2x^3 & +2x^2 & x & \\
 & 2x^3 & +2x^2 & & \\
\cline{2-3}
 & & & x & \\
 & & & x & +\frac{1}{3} \\
\cline{4-5}
 & & & & -\frac{1}{3} \\
 & & & & -\frac{1}{3} \\
\cline{5-5}
 & & & & 0
\end{array}
$$

giving a solution.

Put in more mathemtical terms:

Lemma 4.1 (Integration step). *A solution to*

$$a_n \frac{d}{dx}^n f(x) + a_{n-1}\frac{d}{dx}^{n-1} f(x) + \cdots + a_1 \frac{d}{dx} f(x) = b_m x^{m-1} + b_{m-1}x^{m-2} + \cdots + b_1 x + b_0.$$

can be found by taking any solution of

$$a_n \frac{d}{dx}^{n-1} f(x) + a_{n-1}\frac{d}{dx}^{n-2} f(x) + \cdots + a_1 f(x) = \frac{1}{m+1} b_m x^{m+1} + \cdots + \frac{1}{2} b_1 x^2 + b_0 x$$

4.2 More complicated numerators: exponentials

The product rule tells us that

$$\frac{d}{dx}(e^{kx} f(x)) = e^{kx}\left(\frac{d}{dx} f(x) + k f(x)\right) = e^{kx}\left(\frac{d}{dx} + k\right) f(x).$$

This is true for higher derivatives as well,

$$\begin{aligned}
\frac{d}{dx}^2 (e^{kx} f(x)) &= \frac{d}{dx}\left(e^{kx}\left(\frac{d}{dx} f(x) + k f(x)\right)\right) \\
&= e^{kx}\left(\frac{d}{dx}\left(\frac{d}{dx} f(x) + k f(x)\right) + k\left(\frac{d}{dx} f(x) + k f(x)\right)\right) \\
&= e^{kx}\left(\frac{d}{dx}\left(\frac{d}{dx} + k\right) + k\left(\frac{d}{dx} + k\right)\right) f(x) \\
&= e^{kx}\left(\frac{d}{dx} + k\right)\left(\frac{d}{dx} + k\right) f(x) \\
&= e^{kx}\left(\frac{d}{dx} + k\right)^2 f(x).
\end{aligned}$$

Every time e^{kx} moves left past a $\frac{d}{dx}$ it turns that $\frac{d}{dx}$ into $\frac{d}{dx} + k$. This fact is called the *exponential shift law.*

We can use this to do fractions with exponentials in the numerator,

$$\begin{aligned}
\frac{xe^{3x}}{1 + \frac{d}{dx}} &= e^{3x}\frac{x}{1 + \frac{d}{dx} + 3} = e^{3x}\frac{x}{4 + \frac{d}{dx}} \\
&= e^{3x}\left(\frac{1}{4}x - \frac{1}{16}\right)
\end{aligned}$$

where we used long division for $\frac{x}{4+\frac{d}{dx}} = \frac{1}{4}x - \frac{1}{16}$. We can check,

$$\begin{aligned}
\frac{d}{dx}\left(e^{3x}\left(\frac{1}{4}x - \frac{1}{16}\right)\right) &= e^{3x}\frac{1}{4} + 3e^{3x}\left(\frac{1}{4}x - \frac{1}{16}\right) \\
1 \cdot e^{3x}\left(\frac{1}{4}x - \frac{1}{16}\right) &= e^{3x}\left(\frac{1}{4}x - \frac{1}{16}\right).
\end{aligned}$$

Lemma 4.2 (Exponential shift step). *A solution to*

$$a_n \frac{d}{dx}^n f(x) + \cdots + a_1 \frac{d}{dx} f(x) + a_0 f(x) = e^{kx}(b_m x^{m-1} + b_{m-1}x^{m-2} + \cdots + b_1 x + b_0).$$

can be found by taking any solution of

$$a_n\left(\frac{d}{dx} + k\right)^n g(x) + \cdots + a_1\left(\frac{d}{dx} + k\right) g(x) + a_0 g(x) = b_m x^{m-1} + b_{m-1}x^{m-2} + \cdots + b_1 x + b_0.$$

and setting $f(x) = e^{kx} g(x)$.

4.3 More manipulations

Just as the fraction $\frac{a+b}{c}$ can be decomposed into $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$, so can the fractions from differential equations:

$$\frac{N_1(x) + N_2(x)}{D(\frac{d}{dx})} = \frac{N_1(x)}{D(\frac{d}{dx})} + \frac{N_2(x)}{D(\frac{d}{dx})}.$$

This is just stating that solutions to

$$D\left(\frac{d}{dx}\right) f(x) = N_1(x) + N_2(x)$$

can be obtained by taking solutions from the equations

$$\begin{aligned} D\left(\frac{d}{dx}\right) g(x) &= N_1(x) \\ D\left(\frac{d}{dx}\right) h(x) &= N_2(x) \end{aligned}$$

and adding them: $f(x) = g(x) + h(x)$.

And this is just a generalization of the integration step,

$$\frac{N(x)}{D_1(\frac{d}{dx})D_2(\frac{d}{dx})} = \frac{\frac{N(x)}{D_1(\frac{d}{dx})}}{D_2(\frac{d}{dx})}$$

which says merely that a solution of

$$D_1\left(\frac{d}{dx}\right)\left(D_2\left(\frac{d}{dx}\right)(f(x))\right) = N(x)$$

can be obtained by solving (in order)

$$\begin{aligned} D_1\left(\frac{d}{dx}\right) g(x) &= N(x) \\ D_2\left(\frac{d}{dx}\right) h(x) &= g(x). \end{aligned}$$

However, we won't have much need for this until we look at more advanced stuff.

5 Putting it all together

We now have three rules, integration, exponential shift, and long division, which combined give a powerful means to tackle approximately half of the differential equations one faces in a first class in differential equations.

To summarize, when faced with a differential equation $D(\frac{d}{dx})f(x) = h(x)$ and where $D(\frac{d}{dx})$ is some constant coefficient polynomial in $\frac{d}{dx}$ and $h(x)$ is some sum of polynomials, exponentials, and polynomials times exponentials, you can always find a sequence of steps and manipulations which will arrive at one solution.

Let's demonstrate with one complicated example. The differential equation is,

$$\frac{d}{dx}^2 f(x) + \frac{d}{dx}f(x) - 2f(x) = 1 + 3xe^{-x} + (x+1)e^{-2x}$$

which means our fraction is $\frac{1+3xe^{-x}+(x+1)e^{-2x}}{-2+\frac{d}{dx}+{\frac{d}{dx}}^2}$

$$\begin{aligned}
\frac{1+3xe^{-x}+(x+1)e^{-2x}}{-2+\frac{d}{dx}+{\frac{d}{dx}}^2} &= \frac{1}{-2+\frac{d}{dx}+{\frac{d}{dx}}^2}+\frac{3xe^{-x}}{-2+\frac{d}{dx}+{\frac{d}{dx}}^2}+\frac{(x+1)e^{-2x}}{-2+\frac{d}{dx}+{\frac{d}{dx}}^2} \\
&= \frac{1}{-2+\frac{d}{dx}+{\frac{d}{dx}}^2}+e^{-x}\frac{3x}{-2+(\frac{d}{dx}-1)+(\frac{d}{dx}-1)^2}+e^{-2x}\frac{x+1}{-2+(\frac{d}{dx}-2)+(\frac{d}{dx}-2)^2} \\
&= \frac{1}{-2+\frac{d}{dx}+{\frac{d}{dx}}^2}+e^{-x}\frac{3x}{-2-\frac{d}{dx}+{\frac{d}{dx}}^2}+e^{-2x}\frac{x+1}{-3\frac{d}{dx}+{\frac{d}{dx}}^2} \\
&= \frac{1}{-2+\frac{d}{dx}+{\frac{d}{dx}}^2}+e^{-x}\frac{3x}{-2-\frac{d}{dx}+{\frac{d}{dx}}^2}+e^{-2x}\frac{\frac{1}{2}x^2+x}{-3+\frac{d}{dx}}
\end{aligned}$$

Doing long divisions,

$$\begin{array}{r|l}
 & -\frac{1}{2} \\ \cline{2-2}
-2+\frac{d}{dx}+{\frac{d}{dx}}^2 & 1 \\
 & 1 \\ \cline{2-2}
 & 0
\end{array},
\quad
\begin{array}{r|ll}
 & -\frac{3}{2}x & +\frac{3}{4} \\ \cline{2-3}
-2-\frac{d}{dx}+{\frac{d}{dx}}^2 & 3x & \\
 & 3x & +\frac{3}{2} \\ \cline{2-3}
 & & -\frac{3}{2} \\
 & & -\frac{3}{2} \\ \cline{3-3}
 & & 0
\end{array},
\quad
\begin{array}{r|lll}
 & -\frac{1}{6}x^2 & -\frac{4}{9}x & -\frac{4}{27} \\ \cline{2-4}
-3+\frac{d}{dx} & \frac{1}{2}x^2 & +x & \\
 & \frac{1}{2}x^2 & -\frac{1}{3}x & \\ \cline{2-3}
 & & \frac{4}{3}x & \\
 & & \frac{4}{3}x & -\frac{4}{9} \\ \cline{3-4}
 & & & \frac{4}{9} \\
 & & & \frac{4}{9} \\ \cline{4-4}
 & & & 0
\end{array}$$

We arrive at a solution

$$-\frac{1}{2}+\left(-\frac{3}{2}x+\frac{3}{4}\right)e^{-x}+\left(-\frac{1}{6}x^2-\frac{4}{9}x-\frac{4}{27}\right)e^{-2x}.$$

6 General solutions

The techniques shown so far will produce a solution to an important class differential equations, but just as in calculus, having one answer is not always enough. While $f(x)=\frac{1}{2}x^2$ is a solution of $\frac{d}{dx}f(x)=x$ there are other solutions which all have the form $\frac{1}{2}x^2+C$ where C is some constant, i.e. "any two integrals differ by no more than a constant".

When we solve differential equations, one or several integrations occur, and getting the most general solution is a matter of keeping track of the Cs that come from them.

With the differential equations we are studying, this notion of a constant that solutions can differ by is replaced with something more general. Say $f_1(x)$ and $f_2(x)$ are both solutions of $\frac{N(x)}{D(\frac{d}{dx})}$, i.e.

$$\begin{aligned}
D\left(\frac{d}{dx}\right)f_1(x) &= N(x) \\
D\left(\frac{d}{dx}\right)f_2(x) &= N(x),
\end{aligned}$$

then, by subtracting the two equations, we have

$$D\left(\frac{d}{dx}\right)(f_1(x)-f_2(x))=0$$

showing that two solutions differ by no more than a $\frac{0}{D(\frac{d}{dx})}$ solution. When trying to find just one solution, not caring which, we have taken $\frac{0}{D(\frac{d}{dx})} = 0$. Finding all possible solutions of $\frac{N(x)}{D(\frac{d}{dx})}$ amounts to finding one solution of $\frac{N(x)}{D(\frac{d}{dx})}$ and then adding to it all solutions of $\frac{0}{D(\frac{d}{dx})}$.

Consider the example from integration,

$$\begin{aligned}\frac{d}{dx}f(x) &= x \\ f(x) &= \frac{1}{2}x^2 + \frac{0}{\frac{d}{dx}} \\ &= \frac{1}{2}x^2 + C\end{aligned}$$

since C is the most general solution of $\frac{0}{\frac{d}{dx}}$. This is what we already know about indefinite integrals.

For a more complicated $D\left(\frac{d}{dx}\right)$, we have to be more clever.

$$\begin{aligned}\frac{0}{2+\frac{d}{dx}} &= \frac{0e^{-2x}}{2+\frac{d}{dx}} \\ &= e^{-2x}\frac{0}{\frac{d}{dx}} \\ &= e^{-2x}C = Ce^{-2x}\end{aligned}$$

and we can check that $\frac{d}{dx}(Ce^{-2x}) + 2Ce^{-2x} = 0$. Thus

$$\begin{aligned}\frac{4x}{2+\frac{d}{dx}} &= 2x - 1 + \frac{0}{2+\frac{d}{dx}} \\ &= 2x - 1 + e^{-2x}\frac{0}{\frac{d}{dx}} \\ &= 2x - 1 + Ce^{-2x}\end{aligned}$$

is the general solution. It is no coincidence that we took 0 to be $0e^{-2x}$ in the above. If we had picked a different exponential, $0e^{-x}$, we would have gotten nowhere

$$\begin{aligned}\frac{0}{2+\frac{d}{dx}} &= \frac{0e^{-x}}{2+\frac{d}{dx}} \\ &= e^{-x}\frac{0}{1+\frac{d}{dx}}.\end{aligned}$$

Finding the right exponential shifts to make the $\frac{0}{\frac{d}{dx}}$s apparent requires a bit of trickery.

Lemma 6.1 (Roots of the denominator). *If k is some number such that $a_nk^n + \cdots + a_1k + a_0 = 0$, then*

$$\left(a_n\frac{d}{dx}^n + \cdots + a_1\frac{d}{dx} + a_0\right)e^{kx} = 0.$$

Conversely, the only solutions of

$$a_n\frac{d}{dx}^n f(x) + \cdots + a_1\frac{d}{dx}f(x) + a_0f(x) = 0$$

are of the form $f(x) = p(x)e^{kx}$ where k $D(k) = 0$ and $p(x)$ is a polynomial with degree at most m where m is the multiplicity of k in the factorization of D.

Proof. ${\frac{d}{dx}}^m e^{kx} = k^m e^{kx}$, so adding things up we have

$$\begin{aligned}\left(a_n \frac{d}{dx}^n + \cdots a_1 \frac{d}{dx} + a_0\right) e^{kx} &= \left(a_n k^n + \cdots a_1 k + a_0\right) e^{kx} \\ &= 0e^{kx}.\end{aligned}$$

The converse is beyond the scope of this discussion. ☐

In other words, to find all the solutions of $\frac{0}{D(\frac{d}{dx})}$ you need only consider those exponential shifts with e^{kx} where $D(k) = 0$.

For example, $k^2 + k - 2 = 0$ when k is either 1 or -2, so

$$\begin{aligned}\frac{0}{-2 + \frac{d}{dx} + \frac{d}{dx}^2} &= \frac{0e^{x}}{-2 + \frac{d}{dx} + \frac{d}{dx}^2} + \frac{0e^{-2x}}{-2 + \frac{d}{dx} + \frac{d}{dx}^2} \\ &= e^x \frac{0}{3\frac{d}{dx} + \frac{d}{dx}^2} + e^{-2x} \frac{0}{-3\frac{d}{dx} + \frac{d}{dx}^2} \\ &= e^x \frac{\frac{0}{\frac{d}{dx}}}{3 + \frac{d}{dx}} + e^{-2x} \frac{\frac{0}{\frac{d}{dx}}}{-3 + \frac{d}{dx}} \\ &= e^x \frac{C_1}{3 + \frac{d}{dx}} + e^{-2x} \frac{C_2}{-3 + \frac{d}{dx}} \\ &= e^x \frac{C_1}{3} - e^{-2x} \frac{C_2}{3} \\ &= C_1' e^x + C_2' e^{-2x}\end{aligned}$$

where C_1' and C_2' are constants which "suck up" the 3s and negative sign.

So what we have learned here is that to get general solutions, one has to find all the solutions of $\frac{0}{D(\frac{d}{dx})}$ which come from exponential shifts that cause a $\frac{0}{\frac{d}{dx}}$ (a.k.a. an unknown constant) to appear.

So the general solution to the large example problem is

$$-\frac{1}{2} + \left(-\frac{3}{2}x + \frac{3}{4}\right) e^{-x} + \left(-\frac{1}{6}x^2 - \frac{4}{9}x - \frac{4}{27}\right) e^{-2x} + C_1 e^x + C_2 e^{-2x}.$$

There are further simplifying tricks to finding all solutions of $\frac{0}{D(\frac{d}{dx})}$ but you will have to work those out on your own.

7 Problems

Here are some differential equations to try things out on.

These only require long division.

$$\begin{aligned} \frac{d}{dx}f(x) + 2f(x) &= 10x + 1 \\ \frac{d}{dx}^2 f(x) + \frac{d}{dx}f(x) - 2f(x) &= 6x^2 \end{aligned}$$

This requires long division and exponential shift.

$$\frac{d}{dx}f(x) + 4f(x) = xe^{-2x}$$

This requires all the tricks.

$$\frac{d}{dx}^2 f(x) + 2\frac{d}{dx}f(x) + f(x) = x + e^{-x}$$

After you find one solution, you might want to try finding the general solution (keeping track of the Cs). When the highest derivative is $\frac{d}{dx}^2$ there should be 2 of them and when the highest is $\frac{d}{dx}$ there should be 1.

7.1 Advanced

Show that if $D(\frac{d}{dx})$ is a polynomial in $\frac{d}{dx}$, and $D(k) \neq 0$, then

$$\frac{e^{kx}}{D\left(\frac{d}{dx}\right)}$$

has the solution $\frac{1}{D(k)}e^{kx}$. An easy way is to plug this into the differential equation. A more interesting way is to use long division. What solutions can you find if $D(k) = 0$?

The polynomial $k^2 + 3k + 2$ factors into $(k + 1)(k + 2)$. Show that

$$\frac{x^2 + x + 1}{2 + 3\frac{d}{dx} + \frac{d}{dx}^2} = \frac{x^2 + x + 1}{(1 + \frac{d}{dx})(2 + \frac{d}{dx})} = \frac{\frac{x^2+x+1}{1+\frac{d}{dx}}}{2 + \frac{d}{dx}},$$

that is, first solve $\frac{x^2+x+1}{1+\frac{d}{dx}}$ and then take the result and "divide" it by $\frac{d}{dx} + 2$ and check to see that it is a solution to $\frac{x^2+x+1}{2+3\frac{d}{dx}+\frac{d}{dx}^2}$.